CALCULUS FOR TECHNOLOGY (BETU 1023)

WEEK 3

DIFFERENTIATION

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- DIFFERENTIATION OF LOGARITHMIC AND EXPONENTIAL FUNCTIONS

LEARNING OUTCOMES

At the end of this topic, students should be able to:

- Differentiate the function by using chain rule
- Differentiate the logarithmic functions
- Differentiate the exponential functions

THE CHAIN RULE??

- When dealing with composite functions such as $y=\sqrt{1-2x^2}$, the power rule of differentiation alone is not sufficient because composite function by their nature of being "functions of other function", tend to be complicated. This is where the Chain Rule could be useful.
- Let

$$F(x)=f(g(x)) \text{ then } F'(x)=f'(g(x)).\frac{d}{dx}(g(x))$$

- In other words, we would need to differentiate the outside function f and multiply the result with the derivative of the inside function $g'(x)$.

EXAMPLE 1

Differentiate $y = \sqrt{1 - 2x^2}$

SOLUTION 1

$$y = \sqrt{1-2x^2} = (1-2x^2)^{\frac{1}{2}}$$

Let $u = 1-2x^2$, then $y = u^{\frac{1}{2}}$. From here we can get $\frac{du}{dx} = -4x$ and $\frac{dy}{du} = \frac{1}{2}u^{-\frac{1}{2}} = \frac{1}{2\sqrt{u}}$

What we need here is not $\frac{du}{dx}$ or $\frac{dy}{du}$ although they will prove useful shortly.

We really need $\frac{dy}{dx}$. This is where the Chain Rule steps.

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= \frac{1}{2\sqrt{u}}(-4x)$$

$$= -\frac{2x}{\sqrt{1-2x^2}}$$

Chain Rule in Other Way

- Is there a shorter way of solving the composite function and still use the Chain Rule.
- By using question from Example 1 then the solution is

$$y = \sqrt{1-2x^2} = \left(1-2x^2\right)^{\frac{1}{2}}$$

$$\frac{dy}{dx} = \frac{1}{2}\left(1-2x^2\right)^{-\frac{1}{2}} \frac{d}{dx}\left(1-2x^2\right)$$

$$= \frac{1}{2}\left(1-2x^2\right)^{-\frac{1}{2}}\left(-4x\right)$$

$$= -\frac{2x}{\sqrt{1-2x^2}}$$

The Chain Rule $\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx}$ has been retained throughout this solution.

EXAMPLE 2

Differentiate $y = \sqrt[3]{1 - x^2 + x^3}$ by using Chain Rule.

SOLUTION 2

$$y = \sqrt[3]{1 - x^2 + x^3} = \left(1 - x^2 + x^3\right)^{\frac{1}{3}}$$

Let $u = 1 - x^2 + x^3$, then $y = u^{\frac{1}{3}}$. The derivative of this functions are

$$\frac{du}{dx} = -2x + 3x^2, \quad \frac{dy}{du} = \frac{1}{3}u^{-\frac{2}{3}}$$

By using Chain Rule

$$\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx} = \frac{1}{3}u^{-\frac{2}{3}}\left(-2x + 3x^2\right)$$

$$= \frac{\left(-2x + 3x^2\right)}{3}\left(1 - x^2 + x^3\right)^{-\frac{2}{3}}$$

SOLUTION 2

A shorter version of the Chain Rule is recommended and is as follows

$$y=\sqrt[3]{1-x^2+x^3}=\left(1-x^2+x^3\right)^{\frac{1}{3}}$$

Then,

$$\frac{dy}{dx}=\frac{1}{3}\left(1-x^2+x^3\right)^{-\frac{2}{3}}\frac{d}{dx}\left(1-x^2+x^3\right)$$

$$=\frac{1}{3}\left(1-x^2+x^3\right)^{-\frac{2}{3}}\left(-2x+3x^2\right)$$

$$=\frac{\left(-2x+3x^2\right)}{3}\left(1-x^2+x^3\right)^{-\frac{2}{3}}$$

EXAMPLE 3

By using Chain Rule, find the derivative of $f(x) = \left(\frac{1-x}{1+x}\right)^4$.

SOLUTION 3

A shorter version of the Chain Rule applied to this question

$$f(x)=\left(\frac{1-x}{1+x}\right)^4$$

$$f'(x)=4\left(\frac{1-x}{1+x}\right)^3\frac{d}{dx}\left(\frac{1-x}{1+x}\right)$$

$$=4\left(\frac{1-x}{1+x}\right)^3\left[\frac{(1+x)(-1)-(1-x)(1)}{(1+x)^2}\right]$$

Applied the quotient rule to differentiate this part.

$$=4\left(\frac{1-x}{1+x}\right)^3\left[\frac{-2}{(1+x)^2}\right]$$

$$=-\frac{8(1-x)^3}{(1+x)^5}$$

Chain Rule in Trigonometric Functions

- The Chain Rule is applicable to all functions including trigonometric functions.
- A quick summary of trigonometric functions would be appropriate at this moment.

  - $$\frac{d}{dx}(\sin x) = \cos x$$
  - $$\frac{d}{dx}(\cos x) = -\sin x$$
  - $$\frac{d}{dx}(\tan x) = \sec^2 x$$
  - $$\frac{d}{dx}(\sec x) = \sec x \tan x$$

EXAMPLE 4

Find the derivative of the following functions

a) $y = \csc x$

b) $y = \sqrt{\tan\left(x^5\right)}$

c) $y = \cos^3\left(\pi\sqrt{x}\right)$

SOLUTION 4

a) Let $y = \csc x = \dfrac{1}{\sin x}$

$$= (\sin x)^{-1}$$

then

$$\frac{dy}{dx} = -1(\sin x)^{-2}.\frac{d}{dx}(\sin x)$$

$$= -\frac{1}{\sin^2 x}.\cos x$$

$$= -\frac{1}{\sin x}.\frac{\cos x}{\sin x}$$

$$= -\csc x \cot x$$

SOLUTION 4

b) Let $y=\sqrt{\tan\left(x^5\right)}=\left(\tan\left(x^5\right)\right)^{\frac{1}{2}}$

then

$$\frac{dy}{dx}=\frac{1}{2}\left(\tan\left(x^5\right)\right)^{-\frac{1}{2}}.\frac{d}{dx}\left(\tan\left(x^5\right)\right)$$

$$=\frac{1}{2\sqrt{\tan\left(x^5\right)}}.\sec^2\left(x^5\right).\frac{d}{dx}\left(x^5\right)$$

$$=\frac{5x^4\sec^2\left(x^5\right)}{2\sqrt{\tan\left(x^5\right)}}$$

SOLUTION 4

c) $y = \cos^3\left(\pi\sqrt{x}\right)$

Let $u = \pi\sqrt{x}$, then $y = \cos^3 u$. Therefore $\frac{du}{dx} = \frac{\pi}{2\sqrt{x}}$ and $\frac{dy}{du} = 3\cos^2 u(-\sin u)$

hence,

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= -3\cos^2 u \sin u\left(\frac{\pi}{2\sqrt{x}}\right)$$

$$= \frac{-3\pi}{2\sqrt{x}}\cos^2\left(\pi\sqrt{x}\right)\sin\left(\pi\sqrt{x}\right)$$

Differentiation of Logarithmic Functions

The most useful logarithm is the natural logarithm, that is logarithm to the base exponential often denoted by

$$y = \ln x \text{ or } y = \log_e x \text{ for } x > 0$$

If $\log_e x = y$, then $x = e^y$

Since $y = \log_e x$ correspond to $x = e^y$, then $\frac{dx}{dy} = e^y.1 = e^y$. Therefore $\frac{dy}{dx} = \frac{1}{e^y} = \frac{1}{x}$.

For $y = \log_e x = \ln x$, then $\frac{d}{dx}(\ln x) = \frac{1}{x}$.

EXAMPLE 5

Differentiate $y = \ln\left(1 - 2x^5\right)$ by using Chain Rule.

SOLUTION 5

Let $u = 1 - 2x^5$, then $y = \ln u$. Therefore $\frac{du}{dx} = -10x^4$ and $\frac{dy}{du} = \frac{1}{u}$.

By using Chain Rule:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

$$= \frac{1}{u}\left(-10x^4\right)$$

$$= \frac{-10x^4}{1 - 2x^5}$$

EXAMPLE 6

Find the derivative of $y = \ln\left(1 - \cos^2 x\right)$.

SOLUTION 6

Let $u = 1 - \cos^2 x$, then $y = \ln u$. Therefore $\frac{du}{dx} = -2\cos x(-\sin x)$ and $\frac{dy}{du} = \frac{1}{u}$.

By using Chain Rule:

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$
$$= \frac{1}{u}(2\cos x \sin x)$$
$$= \frac{2\cos x \sin x}{1 - \cos^2 x}$$

SOLUTION 6

You can shortest the step as follows:

$$y = \ln\left(1 - \cos^2 x\right)$$

$$\frac{dy}{dx} = \frac{1}{1-\cos^2 x} \cdot \frac{d}{dx}\left(1 - \cos^2 x\right)$$

$$= \frac{1}{1-\cos^2 x}\left(-2\cos x\left(-\sin x\right)\right)$$

$$= \frac{2\cos x \sin x}{1-\cos^2 x}$$

EXAMPLE 7

Find the derivative of the following functions:

a) $y = \ln\left(\frac{2x-5}{x^2-1}\right)$

b) $y = \frac{(x+1)^{3/2}(x-4)^{5/2}}{(5x+3)^{2/3}}$

SOLUTION 7

a) Differentiation y using quotient rule would be a tedious affair. It would be much easier if we use the law of logarithm as follows:

$$y = \ln\left(\frac{2x-5}{x^2-1}\right) = \ln(2x-5) - \ln(x^2-1)$$

$$\frac{dy}{dx} = \frac{1}{2x-5}(2) - \frac{1}{x^2-1}(2x)$$

$$= \frac{2}{2x-5} - \frac{2x}{x^2-1}$$

SOLUTION 7

b) To differentiate this function $y=\dfrac{(x+1)^{3/2}(x-4)^{5/2}}{(5x+3)^{2/3}}$, we taking logarithm (to base exponential) of both sides.

$$\ln y=\ln\left[\frac{(x+1)^{3/2}(x-4)^{5/2}}{(5x+3)^{2/3}}\right]$$

$$\ln y=\frac{3}{2}\ln(x+1)+\frac{5}{2}\ln(x-4)-\frac{2}{3}\ln(5x+3)$$

$$\frac{1}{y}\frac{dy}{dx}=\frac{3}{2(x+1)}+\frac{5}{2(x-4)}-\frac{2(5)}{3(5x+3)}$$

$$\frac{dy}{dx}=y\left[\frac{3}{2(x+1)}+\frac{5}{2(x-4)}-\frac{10}{3(5x+3)}\right]$$

$$\frac{dy}{dx}=\frac{(x+1)^{3/2}(x-4)^{5/2}}{(5x+3)^{2/3}}\left[\frac{3}{2(x+1)}+\frac{5}{2(x-4)}-\frac{10}{3(5x+3)}\right]$$

Differentiation of Exponential Functions

Generally exponential function can be written as $y = a^x$ where a is a positive constant. The number a is called the base of the exponential function. There is a particularly unique base with the value $a \approx 2.718$. This base $a \approx 2.718$ is given the symbol e by mathematicians and the exponential function can simply be written as $y = e^x$.

Derivative of Exponential Functions

- Let $y = e^x$ and to differentiate this function requires only two simple steps.
  - First step, copy the given function $y = e^x$
  - Second step, cover up the base e and differentiate the exponent (or power) with respect to x. Now multiply your answer to step 1 with your answer to step two and you would obtain the derivative of $y = e^x$.
- To summaries the steps are:
  - $y = e^x$
  - $\frac{d}{dx}(x) = 1$

$$\frac{dy}{dx} = e^x . \frac{d}{dx}(x)$$

$$= e^x(1)$$

$$= e^x$$

EXAMPLE 8

Find the derivative of the following functions:

a) $y = e^{\sqrt{x}}$

b) $y = e^{-x} + e^{2\pi}$

c) $y = e^{-\pi x} \cos\left(\pi x^2\right)$

SOLUTION 8

a) $y = e^{\sqrt{x}}$

$$\frac{dy}{dx} = e^{\sqrt{x}} . \frac{d}{dx}\left(\sqrt{x}\right)$$

$$= e^{\sqrt{x}}\left(\frac{1}{2}x^{-\frac{1}{2}}\right)$$

$$= \frac{e^{\sqrt{x}}}{2\sqrt{x}}$$

b) $y = e^{-x} + e^{2\pi}$

$$\frac{dy}{dx} = e^{-x} . \frac{d}{dx}(-x) + 0$$

$$= e^{-x}(-1)$$

$$= -e^{-x}$$

SOLUTION 8

c) $y = e^{-\pi x}\cos\left(\pi x^2\right)$

$$\frac{dy}{dx} = e^{-\pi x}.\frac{d}{dx}\left(\cos\left(\pi x^2\right)\right) + \cos\left(\pi x^2\right).\frac{d}{dx}\left(e^{-\pi x}\right)$$

$$= e^{-\pi x}\left(-2\pi x\sin\left(\pi x^2\right)\right) + \cos\left(\pi x^2\right)\left(-\pi x e^{-\pi x}\right)$$

$$= -2\pi x e^{-\pi x}\sin\left(\pi x^2\right) - \pi x e^{-\pi x}\cos\left(\pi x^2\right)$$

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TRY IT YOURSELF 1

Differentiate the following function by using Chain Rule, $\frac{dy}{dx}=\frac{dy}{du}.\frac{du}{dx}$.

a) $y=(1-\pi x^2)^9$

b) $y=(5x-1)^3(3x^3-2x^2+x+1)^4$

c) $y=\cot x$

Solution

a) $$\frac{dy}{dx} = -18\pi x\left(1-\pi x^2\right)^8$$

b) $$\frac{dy}{dx} = (5x-1)^2\left(3x^3-2x^2+x+1\right)\left[4(5x-1)\left(3x^3-2x^2+x+1\right)^2\left(9x^2-4x+1\right)+15\right]$$

c) $$\frac{dy}{dx} = -\csc^2 x$$

TRY IT YOURSELF 2

Differentiate the following function.

a) $y = \sqrt[3]{\ln 2x}$

b) $y = x^{\tan x}$

c) $y = e^{\sec x}$

d) $y = e^{-2x} \cos 4x$

e) $y = \dfrac{(3x+1)e^{2x}}{x+1}$

Solution

a) $$\frac{dy}{dx}=\frac{1}{3x(\ln 2x)^{\frac{2}{3}}}$$

b) $$\frac{dy}{dx}=x^{\tan x}\left(\frac{\tan x}{x}+\ln x\sec^2 x\right)$$

c) $$\frac{dy}{dx}=\sec x\tan xe^{\sec x}$$

d) $$\frac{dy}{dx}=-4e^{-2x}\sin 4x-2e^{-2x}\cos 4x$$

e) $$\frac{dy}{dx}=\left(\frac{(3x+1)e^{2x}}{x+1}\right)\left(\frac{3}{3x+1}+2-\frac{1}{x+1}\right)$$

REFERENCES

- James, S. (2012). *Calculus* (7th ed.). Cengage Learning.
- Bivens, I.C., Stephen, D., & Howard, A. (2012). Calculus Early Transcedentals (10th ed.). John Willey & Sons Inc.