CALCULUS FOR TECHNOLOGY (BETU 1023)

WEEK 2

DIFFERENTIATION

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LEARNING OUTCOMES

By the end of this topic, students are able to:

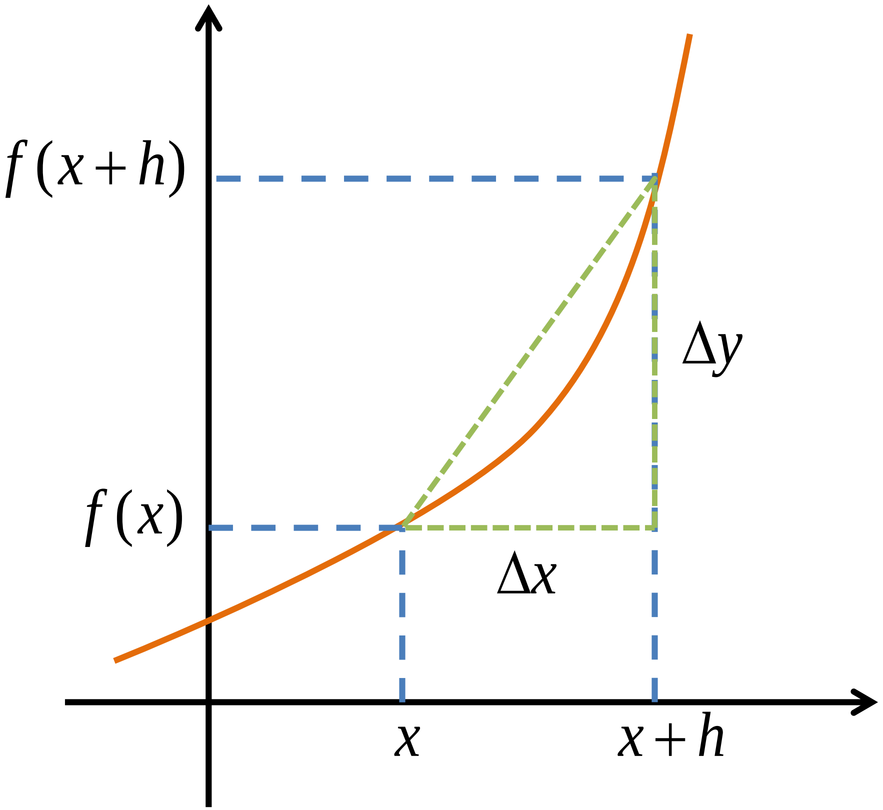
- Define differentiation from geometrical meaning.
- Understand the definition and rules of differentiation.
- Able to solve higher order differentiation.

Concept of Differentiation

Baldwin Street, Dunedin (NZ) was known as the world's steepest residential street. It was even got recognition from *Guiness Book of Records*. This is because not only the road were tilted, but the house built there was also have the same pattern. Our main attention was how much sloppy was the area.

This problem can be turned into a graphical situation, to make the analysis become easier.

Geometrical Meaning of Differentiation



We will use the formula of gradient from a straight line equation, where

$$\text{Gradient}, m = \frac{\Delta y}{\Delta x}$$

h is the distance from a point, then we will have

$$m = \frac{f(x+h) - f(x)}{(x+h) - x} = \frac{f(x+h) - f(x)}{h}$$

To find the value of *m*, take limit as small as possible (*x* approaches 0), the equation will become

$$m = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

This was known as our **First Principle of Differentiation**.

Definition & Rules of Differentiation

DEFINITION

- Differentiation is a mathematical tool used to study rates of change.
- Differentiation is the process of finding a derivative.
- Common notations for derivative are:

$$y' \quad ; \quad f'(x) \quad ; \quad \frac{dy}{dx}$$

FIRST PRINCIPLE

Basic concept of Differentiation:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

STEPS IN FIRST PRINCIPLE

- Write an expression for $f(x)$
- Write an expression for $f(x+h)$
- Substitute $f(x)$ and $f(x+h)$ into the formula
- Simplify the expression.
- Evaluate $$\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$$

EXAMPLE

Use the first principle to find the derivative of the following functions

(a) $f(x) = 2x + 5$

(b) $f(x) = x^2 + 2x$

(c) $f(x) = \sqrt{x}$

Solution: (a)

$$f(x) = 2x + 5$$

Write an expression for $f(x)$

$$f(x+h) = 2(x+h) + 5$$

Write an expression for $f(x+h)$

$$f'(x) = \lim_{h \to 0} \frac{[2(x+h)+5] - [2x+5]}{h}$$

Substitute $f(x)$ and $f(x+h)$

$$f'(x) = \lim_{h \to 0} \frac{2\cancel{h}}{\cancel{h}}$$

Simplify

$$= \lim_{h \to 0} 2 = 2$$

Evaluate & Answer

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Solution: (b)

$$f(x) = x^2 + 2x$$

$$f(x+h) = (x+h)^2 + 2(x+h)$$

$$f'(x) = \lim_{h\to 0} \frac{\left[(x+h)^2 + 2(x+h)\right] - \left[x^2 + 2x\right]}{h}$$

$$= \lim_{h\to 0} \frac{(x^2 + 2hx + h^2 + 2x + 2h) - (x^2 + 2x)}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{h}(2x + h + 2)}{\cancel{h}}$$

$$= 2x + 2$$

Solution: (c)

$$f(x) = \sqrt{x} \quad ; \quad f(x+h) = \sqrt{x+h}$$

$$f'(x) = \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

$$= \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \times \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}}$$

$$= \lim_{h\to 0} \frac{(x+h) - x}{h(\sqrt{x+h} + \sqrt{x})}$$

$$= \lim_{h\to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}}$$

$$= \frac{1}{2\sqrt{x}}$$

Try Yourself 1

Use the first principle to find the derivative, *f* ’(*x*) of the following functions.

(a) $f(x) = 5\sqrt{x}$

(b) $f(x) = \frac{2}{x^2} + x$

(c) $f(x) = \frac{1-x}{2+x}$

ANSWER :

(a) $$f'(x) = \frac{5}{2\sqrt{x}}$$

(b) $$f'(x) = -\frac{4}{x^3} + 1$$

(c) $$f'(x) = -\frac{3}{(2+x)^2}$$

RULES OF DIFFERENTIATION

1. Derivative of a constant

If *c* is any real numbers, then

$$\frac{d}{dx}[c] = 0$$

2. The Power Rule

If *n* is a positive integer, then

$$\frac{d}{dx}[x^n] = nx^{n-1}$$

3. Constant Multiple Rule

If *c* is any real numbers, and *f(x)* is a function, then

$$\frac{d}{dx}[cf(x)] = c\frac{d}{dx}[f(x)]$$

4. Sum and Difference Rules

If *f* and *g* are differentiable by *x*, then

$$\frac{d}{dx}[f(x)+g(x)] = \frac{d}{dx}[f(x)] + \frac{d}{dx}[g(x)]$$

$$\frac{d}{dx}[f(x)-g(x)] = \frac{d}{dx}[f(x)] - \frac{d}{dx}[g(x)]$$

5. Derivative of Trigonometric Functions

$$\frac{d}{dx}[\sin x] = \cos x \qquad \frac{d}{dx}[\cot x] = -\csc^2 x$$

$$\frac{d}{dx}[\cos x] = -\sin x \qquad \frac{d}{dx}[\sec x] = \sec x \tan x$$

$$\frac{d}{dx}[\tan x] = \sec^2 x \qquad \frac{d}{dx}[\csc x] = -\csc x \cot x$$

6. Derivative of Exponential Functions

$$\frac{d}{dx}[b^x] = b^x \ln b$$

$$\frac{d}{dx}[e^x] = e^x$$

7. Derivative of Logarithmic Functions

$$\frac{d}{dx}[\ln x] = \frac{1}{x}, \qquad x > 0$$

$$\frac{d}{dx}[\log_b x] = \frac{1}{x \ln b}, \qquad x > 0$$

8. Product Rule

To find derivative of product of a function $y = u(x)v(x)'$

$$\frac{dy}{dx} = vu' + uv'$$

9. Quotient Rule

To find derivative of fraction

function $y = \frac{u(x)}{v(x)}$,

$$\frac{dy}{dx} = \frac{vu' - uv'}{v^2}$$

EXAMPLE

Differentiate the following functions.

(a) $y = (4x+1)^2(3x)$

(b) $y = x\sin x$

(c) $y = \dfrac{e^t}{t^2}$

Solution: (a)

$$\text{(a) } y=(4x+1)^2(3x)$$

Use product rule since this is the product of two function

$$u=(4x+1)^2 \qquad v=3x$$

$$u'=2(4x+1)\frac{d}{dx}[4x+1] \qquad v'=3$$

$$=2(4x+1)(4)$$

$$=8(4x+1)$$

Let u and v

$$\frac{dy}{dx}=vu'+uv'$$

Use formula for product rule

$$=(3x)[8(4x+1)]+(4x+1)^2(3)$$

Substitute into formula, then simplify

$$=24x(4x+1)+3(4x+1)^2$$

Solution: (b)

(b) $y = x\sin x$ → Use product rule since this is the product of two function

$$u = x \qquad v = \sin x$$
$$u' = 1 \qquad v' = \cos x$$

→ Let *u* and *v*

$$\frac{dy}{dx} = vu' + uv'$$

→ Use formula for product rule

$$= (\sin x)(1) + x(\cos x)$$

→ Substitute into formula, then simplify

$$= \sin x + \cos x$$

Solution: (c)

$$\text{(c)} \quad y = \frac{e^t}{t^2}$$

Use quotient rule since this is a fraction function

$$u = e^t \qquad v = t^2$$

$$u' = e^t \qquad v' = 2t$$

Let u and v

$$\frac{dy}{dx} = \frac{vu' - uv'}{v^2}$$

Use formula for quotient rule

$$= \frac{(t^2)(e^t) - (e^t)(2t)}{(t^2)^2}$$

Substitute into formula, then simplify

$$= \frac{te^t[t-2]}{t^4}$$

$$= \frac{e^t[t-2]}{t^3}$$

Try Yourself 2

Find y' of the following functions.

(a) $y = x^3 \sin 3x$

(b) $y = e^{2t} \ln 5t$

(c) $y = x^2 \tan x$

(d) $y = e^{5x}(3x+1)$

(e) $y = \dfrac{\sin 3x}{x+1}$

(f) $y = \dfrac{\ln x}{e^{2x}}$

(g) $y = \dfrac{e^{2t}}{t^2+1}$

ANSWER:

(a) $$\frac{dy}{dt} = e^{2t}\left(\frac{1}{t} + 2\ln 5t\right)$$

(b) $$\frac{dy}{dx} = 3x^2(x\cos 3x + \sin 3x)$$

(c) $$\frac{dy}{dx} = x(x\sec^2 x + 2\tan x)$$

(d) $$\frac{dy}{dx} = e^{5x}(8 + 15x)$$

(e) $$\frac{dy}{dx} = \frac{3(x+1)\cos 3x - \sin 3x}{(x+1)^2}$$

(f) $$\frac{dy}{dx} = \frac{\frac{1}{x} - 2\ln x}{e^{2x}}$$

(g) $$\frac{dy}{dt} = \frac{2e^{2t}(t^2 - t + 1)}{(t^2+1)^2}$$

Try Yourself 3

Find y' of the following functions.

(a) $y = 4x^3 - 5x^2$

(b) $y = 3\sin 5t + 2e^{4t}$

(c) $y = \sin 4t + 3\cos 2t - t$

(d) $y = \tan 3t - 2$

(e) $y = 2e^{3t} + 17 - 4\sin 2t$

(f) $y = \dfrac{1}{t^3} + \dfrac{\cos 5t}{2}$

(g) $y = \dfrac{2x^3}{3} + \dfrac{e^{4x}}{2}$

(h) $y = \tan(-e^{2x})$

ANSWER :

(a) $$\frac{dy}{dx} = 12x^2 - 10x$$

(b) $$\frac{dy}{dt} = 15\cos 5t + 8e^{4t}$$

(c) $$\frac{dy}{dt} = 4\cos 4t - 6\sin 2t - 1$$

(d) $$\frac{dy}{dt} = 3\sec^2 3t$$

(e) $$\frac{dy}{dt} = 6e^{3t} - 8\cos 2t$$

(f) $$\frac{dy}{dt} = -\frac{3}{t^4} - \frac{5\sin 5t}{2}$$

(g) $$\frac{dy}{dx} = 2x^2 + 2e^{4x}$$

(h) $$\frac{dy}{dx} = -2e^{2x}\sec^2(-e^{2x})$$

Higher Order Derivative

If $y = f(x)$, then

- first derivative : $\dfrac{dy}{dx} = f'(x)$

- second derivative : $\dfrac{d^2y}{dx^2} = f''(x)$

- third derivative : $\dfrac{d^3y}{dx^3} = f'''(x)$

then

- n^{th} derivative of y : $\dfrac{d^ny}{dx^n} = f^n(x)$

EXAMPLE

Find $\frac{d^2y}{dx^2}$ for each of the following.

(a) $y = 8x^4 + 7x^3$

(b) $y = 5(3x+1)^6$

(c) $y = 2e^{-2x} + 3e^{3x}$

Solution: (a)

(a) $y = 8x^4 + 7x^3$

$$\frac{dy}{dx} = 8[4x^3] + 7[3x^2]$$

$$= 32x^3 + 21x^2$$

Differentiate with respect to x , simplify the answer obtained

$$\frac{d^2y}{dx^2} = 32[3x^2] + 21[2x]$$

$$= 96x^2 + 42x$$

Differentiate again with respect to x , simplify the answer obtained

Solution: (a)

(b) $y = 5(3x+1)^6$

Differentiate with respect to x ,
simplify the answer obtained

$$\frac{dy}{dx} = 5\frac{d}{dx}[(3x+1)^6]$$

$$= 5[6(3x+1)^5]\frac{d}{dx}[3x+1]$$

$$= 30(3x+1)^5(3)$$

$$= 90(3x+1)^5$$

Differentiate again with respect to x ,
simplify the answer obtained

$$\frac{d^2y}{dx^2} = 90\frac{d}{dx}[(3x+1)^5]$$

$$= 90[5(3x+1)^4]\frac{d}{dx}[3x+1]$$

$$= 450(3x+1)^4(3)$$

$$= 1350(3x+1)^4$$

Solution: (a)

(d) $y = 2e^{-2x} + 3e^{3x}$

Differentiate with respect to x , simplify the answer obtained

$$\frac{dy}{dx} = 2\frac{d}{dx}[e^{-2x}] + 3\frac{d}{dx}[e^{3x}]$$

$$= 2[e^{-2x}]\frac{d}{dx}[-2x] + 3[e^{3x}]\frac{d}{dx}[3x]$$

$$= 2[e^{-2x}][-2] + 3[e^{3x}][3]$$

$$= -4e^{-2x} + 9e^{3x}$$

Differentiate again with respect to x , simplify the answer obtained

$$\frac{d^2y}{dx^2} = -4\frac{d}{dx}[e^{-2x}] + 9\frac{d}{dx}[e^{3x}]$$

$$= -4[e^{-2x}]\frac{d}{dx}[-2x] + 9[e^{3x}]\frac{d}{dx}[3x]$$

$$= -4[e^{-2x}][-2] + 9[e^{3x}][3]$$

$$= 8e^{-2x} + 27e^{3x}$$

Try Yourself 4

Find $\dfrac{d^2y}{dx^2}$ for each of the following.

(a) $y = 3x^3 - 2x^2 + x + 2$

(b) $y = (x^2 + 3)^2$

(c) $y = \dfrac{2x^3}{3} + \dfrac{e^{4x}}{2}$

(d) $y = 3\sin 5t + 2e^{4t}$

(e) $y = \dfrac{1}{t^3} + \dfrac{\cos 5t}{2}$

(f) $y = (4x - 5)^6$

ANSWER :

(a) $$\frac{dy}{dx} = 9x^2 - 4x + 1$$

(b) $$\frac{dy}{dx} = 4x(x^2 + 3)$$

(c) $$\frac{dy}{dx} = 2x^2 + 2e^{4x}$$

(d) $$\frac{dy}{dx} = 15\cos 5t + 8e^{4t}$$

(e) $$\frac{dy}{dx} = -\frac{3}{t^4} - \frac{5\sin 5t}{2}$$

(f) $$\frac{dy}{dx} = 24(4x-5)^5$$

REFERENCES

- James, S. (2012). *Calculus* (7th ed.). Cengage Learning.
- Bivens, I.C., Stephen, D., & Howard, A. (2012). Calculus Early Transcedentals (10th ed.). John Willey & Sons Inc.