BEKG 2433
ENGINEERING MATHEMATICS 2

STOKES’ THEOREM

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Lesson Outcomes

Upon completion of this lesson, students should be able to:

- Solve surface integrals and line integral using Stoke’s Theorem

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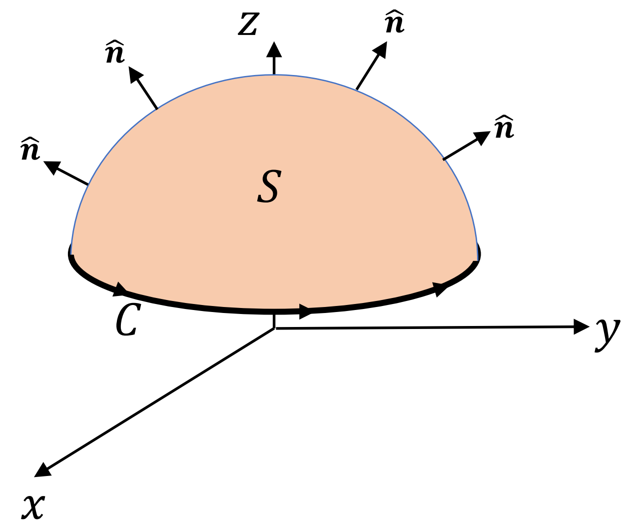
The surface S and the curve C

Around the edge of this surface we have a curve C. This curve is called the boundary curve. The orientation of the surface S will induce the positive orientation of C.



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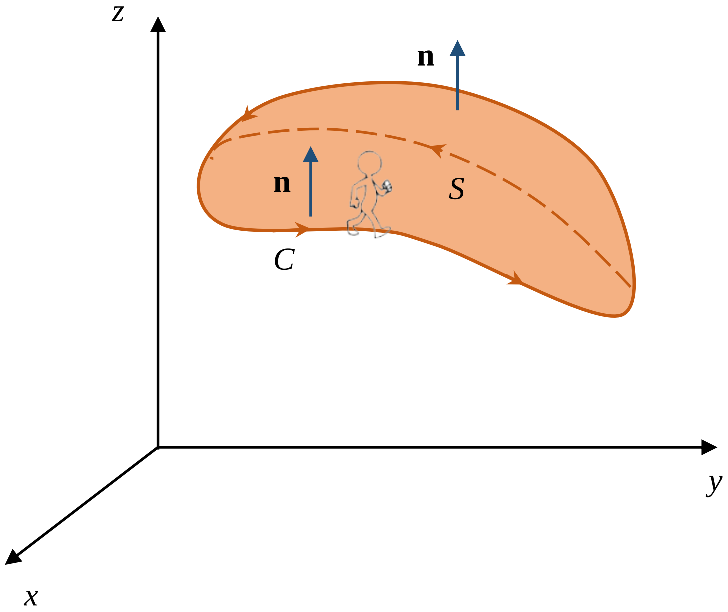
Oriented surface with unit normal vector $\mathbf{n}$.

To get the positive orientation of C think of yourself as walking along the curve. While you are walking along the curve if your head is pointing in the same direction as the unit normal vectors while the surface is on the left then you are walking in the positive direction on C.



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Stokes’ Theorem

A theorem that is a higher dimensional version of Green’s Theorem. In Green’s Theorem we relate a line integral to a double integral over some region (surface integral).

The history of Stokes' Theorem

- The theorem is named after the Irish mathematical physicist Sir George Stokes (1819–1903).
- What we call Stokes' Theorem was actually discovered by the Sir William Thomson, a Scottish physicist also known as Lord Kevin (1824–1907).
- Stokes learned of it in a letter from Thomson in 1850

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The idea behind Stokes' Theorem

- The integral of the normal component of $\boldsymbol{\nabla} \times \vec{\mathbf{F}}$ over the surface S equals the line integral of the vector field around C, the boundary of S.

- We note that the particular shape of S is unimportant.

- Thus the surface integral over S is determined entirely by the line integral along its boundary.

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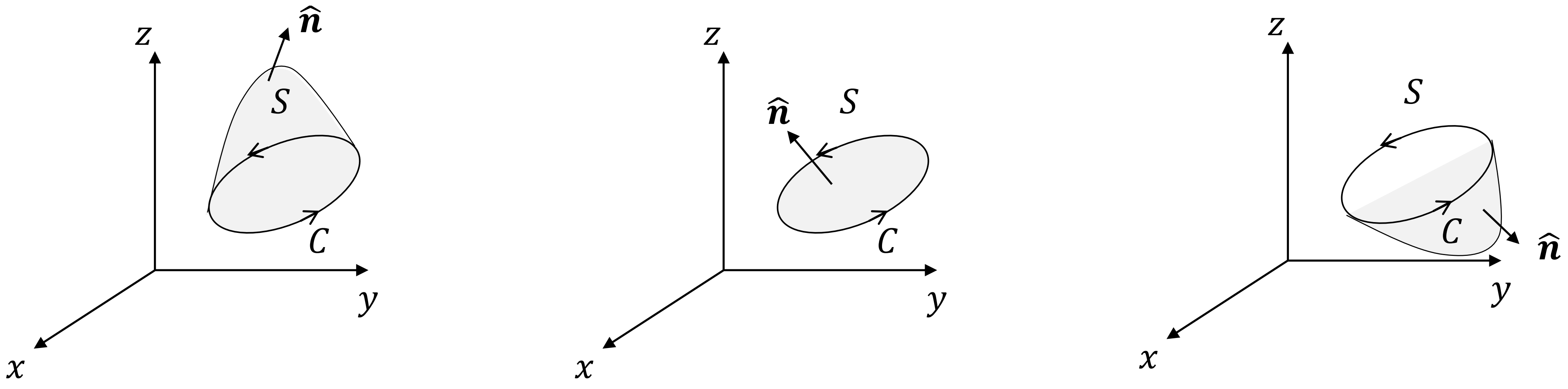
The idea behind Stokes' Theorem

- For example, the surface integrals of $(\boldsymbol{\nabla} \times \vec{\mathbf{F}}) \cdot \vec{\boldsymbol{n}}$ over each of surfaces which has has the same boundary C are the same.



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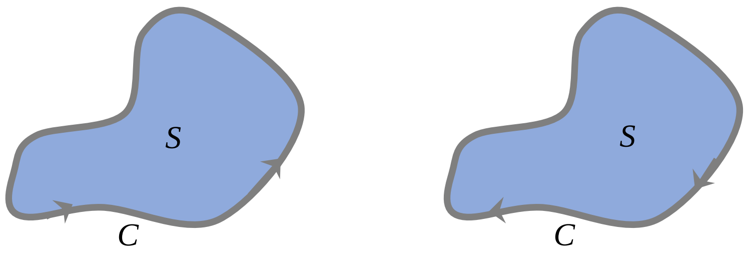
Stokes' Theorem – Definition

Let S be an oriented smooth surface that is bounded by a simple, closed, smooth boundary curve C with positive orientation. Also let $\vec{\mathbf{F}}$ be a vector field then,

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\boldsymbol{r}} = \iint_S curl\vec{\mathbf{F}} \cdot d\vec{S} = \iint_S \left(\nabla \times \vec{\mathbf{F}}\right) \cdot \vec{\mathbf{n}} dS$$

$$\vec{\mathbf{F}}(x, y, z) = f(x, y, z)\mathbf{i} + g(x, y, z)\mathbf{j} + h(x, y, z)\mathbf{k}$$



Positive direction of C Negative direction of C

The curve C bounded by the surface S.

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Stokes' Theorem – Surface Integral

- Recall that a surface integral of a vector field is the integral of the component of the vector field perpendicular to the surface.
- Stokes' Theorem says the surface integral of $curl\vec{\mathbf{F}}$ over a surface S $\left(\iint_S curl\vec{\mathbf{F}} \cdot d\vec{S}\right)$ is the circulation of $\vec{\mathbf{F}}$ around the boundary of the surface $\left(\int_C \vec{\mathbf{F}} \cdot d\vec{\boldsymbol{r}}\right)$.
- Once we have Stokes' Theorem, we can see that the surface integral $curl\vec{\mathbf{F}}=\nabla \times \vec{\mathbf{F}}$ is a special integral. The integral cannot change if we can change the surface S to any surface as long as the boundary of S is still the curve C.

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Example 13.1:

If S is the portion of the paraboloid $z = 1 - x^2 - y^2$ for which $z \geq 0$ and $\mathbf{F}(x, y, z) = z\mathbf{i} + 2x\mathbf{j} + 4y\mathbf{k}$

Solution:

First, determine the outward normal vector (upward normal)

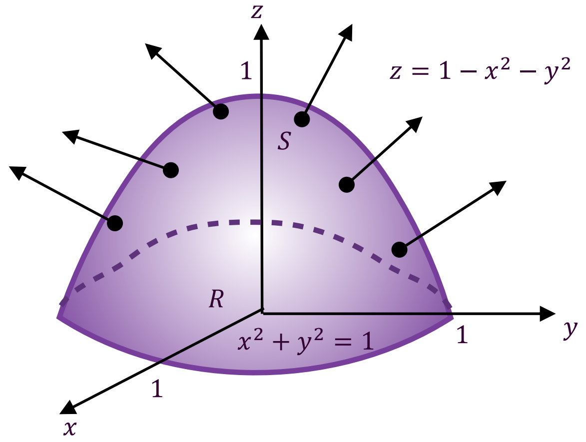
$$z = 1 - x^2 - y^2.$$

Thus, the potential

$$\phi(x, y, z) = z - (1 - x^2 - y^2)$$

Then, determine the gradient of the potential

$$\vec{\mathbf{n}}dS = \nabla\phi = \left[\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right] \cdot \left(z - (1 - x^2 - y^2)\right)$$
$$= 2x\mathbf{i} + 2y\mathbf{j} + \mathbf{k}$$



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Solution:

Next, determine the curl of $\mathbf{F}$ by

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ z & 2x & 4y \end{vmatrix} = 4\mathbf{i} + \mathbf{j} + 2\mathbf{k}$$

Use the Stoke's Theorem

$$\int_C \vec{\mathbf{F}} \cdot d\vec{\boldsymbol{r}} = \iint_S \left(\nabla \times \vec{\mathbf{F}}\right) \cdot \vec{\mathbf{n}}\, dS$$

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Solution:

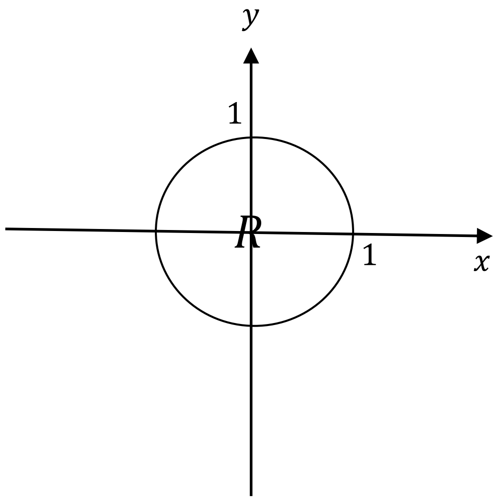
$$\int_C \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = \iint_S (\nabla \times \vec{\mathbf{F}}) \cdot \vec{\mathbf{n}} dS = \iint_R (4i + j + 2k) \cdot (2xi + 2yj + k) dA$$

$$= \iint_R (8x + 2y + 2) dA = \int_0^{2\pi} \int_0^1 (8r\cos\theta + 2r\sin\theta + 2) r dr d\theta$$

$$= \int_0^{2\pi} \left[\frac{8r^3}{3}\cos\theta + \frac{2r^3}{3}\sin\theta + r^2\right]_0^1 d\theta$$

$$= \int_0^{2\pi} \left(\frac{8}{3}\cos\theta + \frac{2}{3}\sin\theta + 1\right) d\theta$$

$$= \left[\frac{8}{3}\sin\theta - \frac{2}{3}\cos\theta + \theta\right]_0^{2\pi} = -\frac{2}{3} + 2\pi + \frac{2}{3} = -2\pi$$



The region R is a circle radius 1

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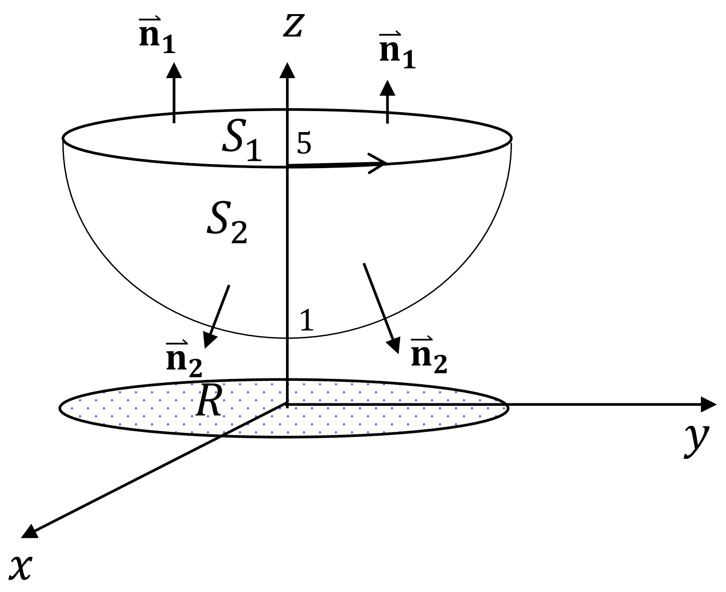
Example 13.2:

Verify Stokes' Theorem if $\vec{\mathbf{F}}$ is the vector field $\vec{\mathbf{F}} = 4y\mathbf{i} - 2x\mathbf{j} - 2z^2\mathbf{k}$ and S is the portion of the paraboloid $z = 1 + x^2 + y^2$ bounded by planes $z = 1$ and $z = 5$. Take $\vec{\mathbf{n}}$ to be the outward unit normal vector to S.

Solution:



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Solution:

The curve C is a circle $x^2 + y^2 = 4$ on $z = 5$, which has parametrization $x = 2\cos t$, $y = 2\sin t$, $z = 5$ $(0 \le t \le 2\pi)$ and so

$$dx = -2\sin t\,dt, dy = 2\cos t\,dt, dz = 0.$$

Thus the line integral

$$\oint_C \vec{\mathbf{F}} \cdot dr = \oint_C (4y\mathbf{i} - 2x\mathbf{j} - 2z^2\mathbf{k}) \cdot (dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k})$$

$$= \oint_C 4ydx - 2xdy - 2z^2dz$$

$$= \int_0^{2\pi} 4(2\sin t)(-2\sin t) - 4\cos t\ (2\cos t)dt$$

$$= -8\int_0^{2\pi} 2\sin^2 t + \cos^2 t\,dt$$

$$= -8\int_0^{2\pi} \sin^2 t + 1dt$$

$$= -4\int_0^{2\pi} 3 - \cos 2t\,dt = -4\left[3t - \frac{\sin 2t}{2}\right]_0^{2\pi} = -24\pi$$

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Solution:

$S_1: z = 5$ (point upwards towards z-positive)

$$\vec{\mathbf{n}} = \nabla\phi = \mathbf{k}$$

$$\vec{\mathbf{F}} \cdot (\nabla\phi) = (4y\mathbf{i} - 2x\mathbf{j} - 2z^2\mathbf{k}) \cdot (\mathbf{k}) = -2z^2$$

$$\iint_{S_1} \vec{\mathbf{F}} \cdot d\vec{\boldsymbol{S}} = \iint_{S_1} \vec{\mathbf{F}} \cdot \vec{\mathbf{n}} dS = \iint_{R_1} \vec{\mathbf{F}} \cdot (\nabla\phi) dA = \iint_{R_1} -2z^2 dA$$

$$= \int_0^{2\pi} \int_0^1 -10r dr d\theta = \int_0^{2\pi} -5r^2 \sin\theta \,|_0^1 d\theta$$

$$= -5\cos\theta \,|_0^{2\pi} = 0$$

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Solution:

$S_2: z = 1 + x^2 + y^2$. The unit normal vector $\vec{\mathbf{n}}_2$ is oriented downwards, however C is traversed in the counterclockwise direction viewed from above and thus corresponds to taking inward normal $\vec{\mathbf{n}}_2$ to the parabola (which has a positive z-component).

$$z = 1 + x^2 + y^2; \quad \phi(x, y, z) = z - (1 + x^2 + y^2)$$

$$\vec{\mathbf{n}} = -\boldsymbol{\nabla}\phi = -\left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right)\left(z - (1 + x^2 + y^2)\right) = 2x\mathbf{i} + 2y\mathbf{j} - \mathbf{k}$$

$$\nabla \times \vec{\mathbf{F}} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 4y & -2x & -2z^2 \end{vmatrix} = -6\mathbf{k}$$

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Solution:

$$\nabla \times \vec{\mathbf{F}} \cdot (\nabla \phi) = (6\mathbf{k}) \cdot (2x\mathbf{i} + 2y\mathbf{j} - \mathbf{k}) = -6$$

$$\iint_{S_2} \nabla \times \vec{\mathbf{F}} \cdot \vec{\mathbf{n}}_2 dS = -6 \iint_{R_2} dA = \underbrace{-6(\pi(2)^2)}_{\substack{Area\ of\ a\ circle \\ r=2}} = -24\pi$$

Thus,

$$\iint_S \vec{\mathbf{F}} \cdot d\vec{\boldsymbol{S}} = \iint_{S_1} \vec{\mathbf{F}} \cdot \vec{\mathbf{n}} dS + \iint_{S_2} \vec{\mathbf{F}} \cdot \vec{\mathbf{n}} dS = 0 - 24\pi = -24\pi$$

Therefore, it verifies the Stokes' theorem.

$$\oint_C \vec{\mathbf{F}} \cdot dr = \iint_S \vec{\mathbf{F}} \cdot d\vec{\boldsymbol{S}}$$

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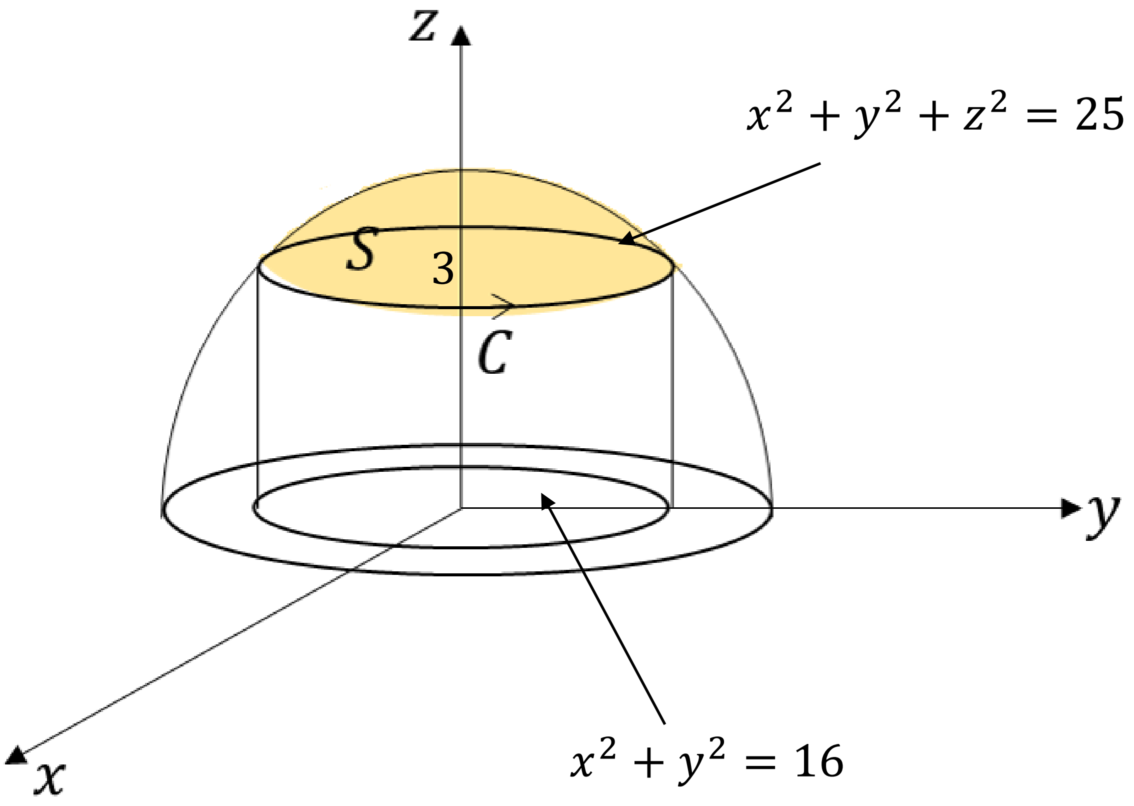
Example 13.3 (Calculating a surface Integral):

Calculate surface integral $\iint_S \; curl\vec{\mathbf{F}} \cdot d\vec{S}$, where $\vec{\mathbf{F}} = z\mathbf{i} + x\mathbf{j} + y\mathbf{k}$ and S is the part of the sphere $x^2 + y^2 + z^2 = 25$ inside the cylinder $x^2 + y^2 = 16$ above the xy −plane.

Solution:



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Solution:

The curve C can be parametrized as

$$\vec{\mathbf{r}}(t) = 4\cos t\,\mathbf{i} + 4\sin t\,\mathbf{j} + 3\mathbf{k}$$
$$\vec{\mathbf{r}}'(t) = -\sin t\,\mathbf{i} + \cos t\,\mathbf{j}$$

Note that: the curve C touches both surfaces at $z = 3$

Thus,

Converting the $\vec{\mathbf{F}} = -y\mathbf{i} + x\mathbf{j} + z\mathbf{k}$ as a function of t, we obtain

$\vec{\mathbf{F}}\big(\vec{\mathbf{r}}(t)\big) = -4\sin t\,\mathbf{i} + 4\cos t\,\mathbf{j} + 3\mathbf{k}$

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Solution:

Using Stokes' Theorem,

$$\iint_S curl\vec{\mathbf{F}} \cdot d\vec{S} = \int_C \vec{\mathbf{F}} \cdot d\vec{\boldsymbol{r}} = \int_0^{2\pi} \vec{\mathbf{F}}\big(\vec{\mathbf{r}}(t)\big) \cdot \vec{\mathbf{r}}'(t)dt$$

$$= \int_0^{2\pi} (-4\sin t\,\mathbf{i} + 4\cos t\,\mathbf{j} + 3\mathbf{k}) \cdot (-\sin t\,\mathbf{i} + \cos t\,\mathbf{j})\,dt$$

$$= \int_0^{2\pi} (4\sin^2 t + 4\cos^2 t)dt = 4(2\pi) = 8\pi$$

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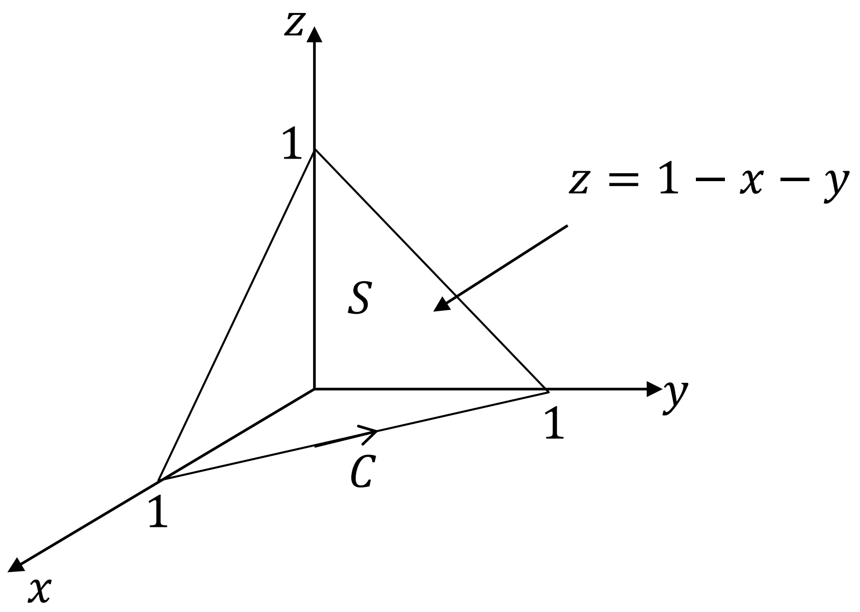
Example 13.4 (Calculating a line Integral):

Evaluate the line integral $\oint_C \vec{\mathbf{F}} \cdot dr$ when $\vec{\mathbf{F}} = z\mathbf{i} - y\mathbf{j} + xy\mathbf{k}$ and C is the triangle defined by $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$.

Solution:

Plane S: $z = 1 - x - y$



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Example 13.4 (Calculating a line Integral):

Evaluate the line integral $\oint_C \vec{\mathbf{F}} \cdot dr$ when $\vec{\mathbf{F}} = z\mathbf{i} - y\mathbf{j} + xy\mathbf{k}$ and C is the triangle defined by $(1,0,0)$, $(0,1,0)$ and $(0,0,1)$.

Solution:

Plane S: $z = 1 - x - y$

Thus, the potential

$$\phi(x, y, z) = z - (1 - x - y)$$

Then, determine the gradient of the potential

$$\vec{\mathbf{n}}dS = \nabla\phi = \left[\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right] \cdot \big(z - (1 - x - y)\big)$$

$$= x\mathbf{i} + y\mathbf{j} + \mathbf{k}$$

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Solution:

$$\nabla \times \vec{\mathbf{F}} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ z & -y & xy \end{vmatrix} = x\mathbf{i} - (x-1)\mathbf{j}$$

$$\iint_S (\nabla \times \vec{\mathbf{F}}) \cdot \vec{\mathbf{n}}\,dS = \iint_R (x\mathbf{i} - (x-1)\mathbf{j}) \cdot (x\mathbf{i} + y\mathbf{j} + \mathbf{k})\,dA$$

$$\int_0^1 \int_0^{1-x} x^2 - y(x-1)\,dy\,dx = \int_0^1 \left[x^2 y - \frac{xy^2}{2} + \frac{y^2}{2}\right]_0^{1-x} dx$$

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Solution:

$$\int_0^1 \int_0^{1-x} x^2 - y(x-1)\,dy\,dx = \int_0^1 \left[x^2 y - \frac{xy^2}{2} + \frac{y^2}{2}\right]_0^{1-x} dx$$

$$\int_0^1 x^2(1-x) - \frac{x(1-x)^2}{2} + \frac{(1-x)^2}{2} dx$$

$$= \left[\frac{x^3}{3} - \frac{x^4}{4} - \frac{x^2}{4} - \frac{x^3}{6} + \frac{x^4}{8} + \frac{(1-x)^3}{-6}\right]_0^1 = -\frac{1}{24}$$

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Exercise 13.1:

By using Stoke's theorem, evaluate $\oint_C \mathbf{F}\cdot d\mathbf{r}$ for the vector $F = xz\mathbf{i} + xy\mathbf{j} + 3yz\mathbf{k}$ and the curve C which is the perimeter of a closed triangle in the first octant with vertices (3,0,0), (0,3,0), and (0,0,6) in the counterclockwise direction, when viewed from the positive z-axis.

[Ans:$67\frac{1}{2}$]

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Exercise 13.2:

By using Stoke's theorem, evaluate $\oint_C \mathbf{F} \cdot d\mathbf{r}$ for the vector $F = xz\mathbf{i} + xy^2\mathbf{j} + 3xz\mathbf{k}$ and the curve C which is the intersection of the plane $z + x = 3$ and the cylinder $x^2 + y^2 = 4$, in the counterclockwise direction, when viewed from the positive z-axis.

[Ans:4π]

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Exercise 13.3:

By using Stoke's theorem, evaluate $\iint_S \; curl\vec{\mathbf{F}} \cdot d\vec{S}$ for the vector $F = y\mathbf{i} + x\mathbf{j} + yx^3\mathbf{k}$ and S is hemisphere of radius 4, in the counterclockwise direction, when viewed from the positive z-axis.

[Ans:-32π]

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Reference

1) R. Ranom, I.W. Jamaludin, N. A. Razak, N.I.A Apandi, Engineering Mathematics, UTeM Press, 2021.

2) Abd Wahid Md. Raji, Ismail Kamis, Mohd Nor Mohamad & Ong Chee TiongAdvanced Calculus for Science and Engineering Students, UTM Press, 2021.

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