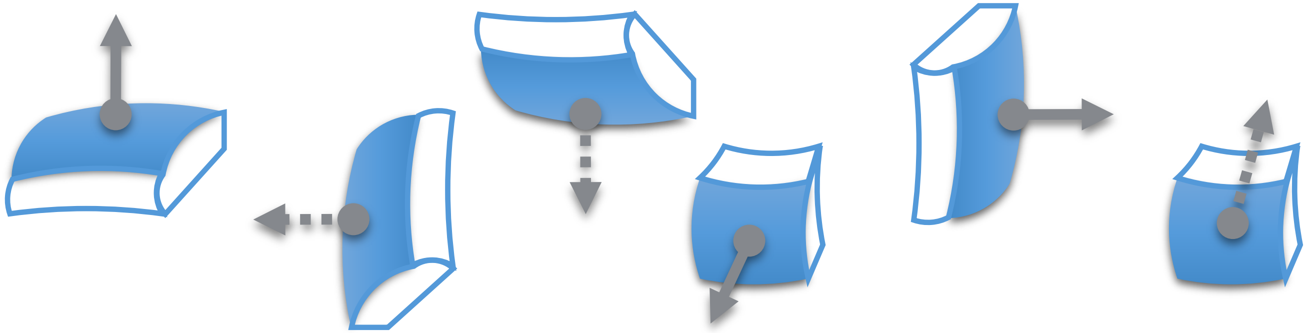
BEKG 2433
ENGINEERING MATHEMATICS 2

PARAMETRIC SURFACES AND ITS AREAS
SURFACE INTEGRALS



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Lesson Outcomes

Upon completion of this lesson, students should be able to:

- write the parametric surfaces
- evaluate surface integrals.
- solve the flux of the vector field.

Parametric Surfaces

A position vector on one dimensional space from some interval $a < t < b$, can be written as parameterized of some open region containing curve in three dimensional spaces as

$$\boldsymbol{r}(t) = x(t)\boldsymbol{i} + y(t)\boldsymbol{j} + z(t)\boldsymbol{k}.$$

Let S be a surface and let R be its projection on the two dimensional plane, where the parametric equations for any points (u, v) in the surfaces, can be written explicitly as

$$x = x(u, v), y = y(u, v), z = z(u, v).$$

Then the parametric surfaces or parametric representation is given by

$$\boldsymbol{r}(u, v) = x(u, v)\boldsymbol{i} + y(u, v)\boldsymbol{j} + z(u, v)\boldsymbol{k}.$$

The resulting set of vectors will be the position vectors for the points of the surface S.

Surfaces Integrals

For evaluating a surface integral from the given parameterization we will need the vector partial derivatives, $\boldsymbol{r}_u \times \boldsymbol{r}_v$ where

$$\boldsymbol{r}_u(u,v) = \frac{\partial x(u,v)}{\partial u}\boldsymbol{i} + \frac{\partial y(u,v)}{\partial u}\boldsymbol{j} + \frac{\partial z(u,v)}{\partial u}\boldsymbol{k}$$

and $\boldsymbol{r}_v(u,v) = \frac{\partial x(u,v)}{\partial v}\boldsymbol{i} + \frac{\partial y(u,v)}{\partial v}\boldsymbol{j} + \frac{\partial z(u,v)}{\partial v}\boldsymbol{k}$

Let S be a surface with equation $z = f(x,y)$ and let R be its projection on the xy −plane. If $z = f(u,v)$ and the parametric representation can be written as

$$\boldsymbol{r}(u,v) = x(u,v)\boldsymbol{i} + y(u,v)\boldsymbol{j} + f(u,v)\boldsymbol{k}.$$

Thus, the surface integral on S, is given by

$$\iint_S g(x,y,z)\,dS = \iint_R g\big(\boldsymbol{r}(u,v)\big)\|\boldsymbol{r}_u \times \boldsymbol{r}_v\|\,dA$$

where $\|\boldsymbol{r}_u \times \boldsymbol{r}_v\| = \sqrt{\left(\frac{\partial f}{\partial u}\right)^2 + \left(\frac{\partial f}{\partial v}\right)^2 + 1}$

Example 12.1:

Write the parametric representations for each of the following surfaces

1. The cylindrical solid bounded by $x^2 + z^2 = 49$ the xz −plane and plane $y = 2$.
2. The sphere $x^2 + y^2 + z^2 = 1$.
3. The elliptic paraboloid $x = 2y^2 + 4z^2 - 9$.

Solution:

1. Let parametric representation $x = 7\cos\theta$, $y = y$ and $z = 7\sin\theta$ for $0 \le y \le 2$ and $0 \le \theta \le 2\pi$, such that $\boldsymbol{r}(y,\theta) = 7\cos\theta\,\boldsymbol{i} + y\boldsymbol{j} + 7\sin\theta\,\boldsymbol{k}$
2. Let the parametric representation $x = \sin\phi\cos\theta$, $y = \sin\phi\sin\theta$ and $z = \cos\phi$ for $0 \le \phi \le \pi$ and $0 \le \theta \le 2\pi$, such that $\boldsymbol{r}(\phi,\theta) = \sin\phi\cos\theta\,\boldsymbol{i} + \sin\phi\sin\theta\,\boldsymbol{j} + \cos\phi\,\boldsymbol{k}$.
3. Since $x = 2y^2 + 4z^2 - 9$, then $\boldsymbol{r}(y,z) = 2y^2 + 4z^2 - 9\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$.

Example 12.2:

Evaluate the surface integral $\iint_S x^2\, dS$ where S is the unit sphere $x^2 + y^2 + z^2 = a^2$ when $a = 1$.

Solution:

Let the parametric representation $x = \sin\phi\cos\theta$, $y = \sin\phi\sin\theta$ and $z = \cos\phi$ for $0 \le \phi \le \pi$ and $0 \le \theta \le 2\pi$, such that

$\boldsymbol{r}(\phi,\theta) = \sin\phi\cos\theta\,\boldsymbol{i} + \sin\phi\sin\theta\,\boldsymbol{j} + \cos\phi\,\boldsymbol{k}$.

From, $\boldsymbol{r}_\phi = \cos\phi\cos\theta\,\boldsymbol{i} + \cos\phi\sin\theta\,\boldsymbol{j} + \sin\phi\,\boldsymbol{k}$ and $\boldsymbol{r}_\theta = -\sin\phi\sin\theta\,\boldsymbol{i} + \sin\phi\cos\theta\,\boldsymbol{j}$, we can obtain

$$\boldsymbol{r}_\phi \times \boldsymbol{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\phi\cos\theta & \cos\phi\sin\theta & \sin\phi \\ -\sin\phi\sin\theta & \sin\phi\cos\theta & 0 \end{vmatrix}$$

$$= (0 - \sin^2\phi\cos\theta)\mathbf{i} - (0 + \sin^2\phi\sin\theta)\mathbf{j} + (\cos\phi\sin\phi\cos^2\phi + \cos\phi\sin\phi\sin^2\phi)\mathbf{k}$$

$$= -\sin^2\phi\cos\theta\,\boldsymbol{i} - \sin^2\phi\sin\theta\,\boldsymbol{j} + \cos\phi\sin\phi\,\boldsymbol{k}$$

Solution:

Hence, $\boldsymbol{r}_\phi \times \boldsymbol{r}_\theta = -\sin^2\phi\cos\theta\,\boldsymbol{i} - \sin^2\phi\sin\theta\,\boldsymbol{j} + \cos\phi\sin\phi\,\boldsymbol{k}$ and

$$\left\|\boldsymbol{r}_\phi \times \boldsymbol{r}_\theta\right\| = \sqrt{(-\sin^2\phi\cos\theta)^2+(-\sin^2\phi\sin\theta)^2+(\cos\phi\sin\phi)^2}$$
$$= \sqrt{\sin^4\phi(\cos\theta)^2 + \sin^4\phi(\sin\theta)^2 + \cos^2\phi\sin^2\phi}$$
$$= \sqrt{\sin^2\phi \quad (\sin^2\phi + \cos^2\phi)} = \sin\phi\,.$$

$$\iint_S x^2\,dS = \iint_R g(\boldsymbol{r}(\phi,\theta))\left\|\boldsymbol{r}_\phi \times \boldsymbol{r}_\theta\right\| dA = \iint_R (\sin\phi\cos\theta)^2\left\|\boldsymbol{r}_\phi \times \boldsymbol{r}_\theta\right\| dA$$

$$= \int_0^{2\pi}\int_0^{\pi} (\sin\phi\cos\theta)^2\sin\phi\,d\phi\,d\theta = \frac{1}{2}\int_0^{2\pi}(1+\cos 2\theta)\left[-\cos\phi + \frac{1}{3}\cos^3\phi\right]_0^{\pi} d\theta$$

$$= \frac{2\pi}{3}\int_0^{2\pi} 1+\cos 2\theta\,d\theta = \frac{2\pi}{3}\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{2\pi} = \frac{4\pi}{3}$$

Surfaces Integrals

Let S be a surface with equation $z = f(x, y)$ and let R be its projection on the xy −plane. If f has continuous first partial derivatives on R and $g(x, y, z)$ is continuous on S, then

$$\iint_S g(x,y,z)\, dS = \iint_S g(x,y,f(x,y))\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\, dA$$

Example 12.3:

Evaluate the surface integral $\iint_S xz\, dS$ where S is part of the plane $x + y + z = 1$ that lies in the first octant.

Solution:

Rewrite the S plane $z = f(x, y) = 1 - y - x$ in the first octant. Hence, $\frac{\partial z}{\partial x} = -1$ and $\frac{\partial z}{\partial y} = -1$.

Solution:

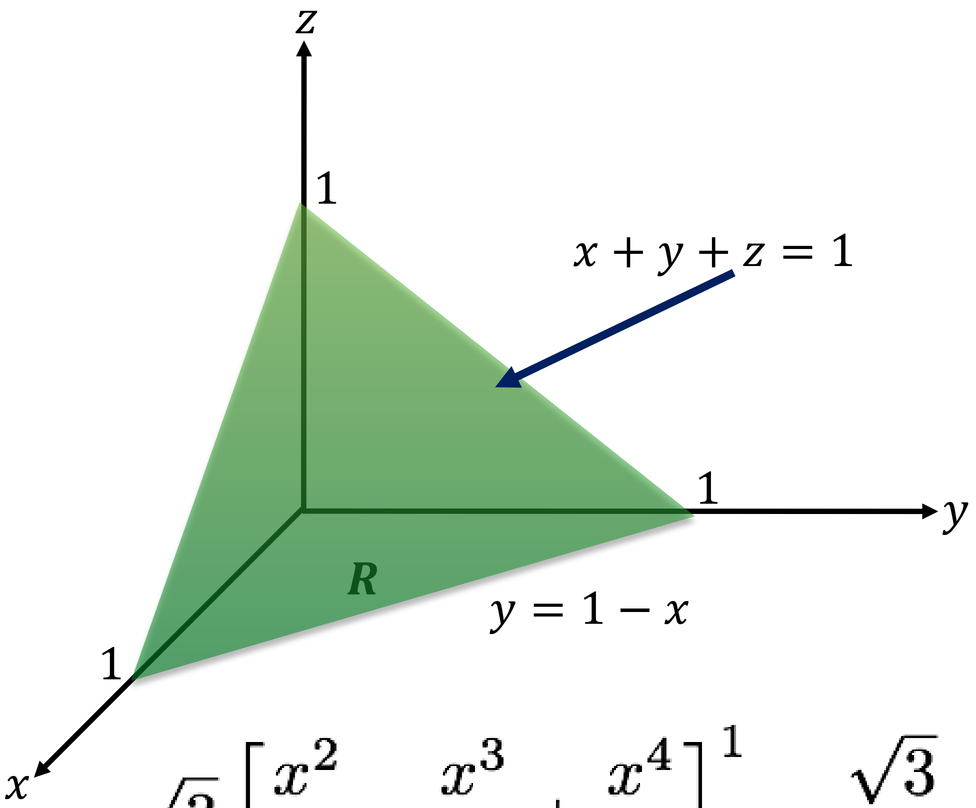
$$\iint_S xz\,dS = \iint_R x(1-y-x)\sqrt{(-1)^2+(-1)^2+1}\,dA$$

$$= \sqrt{3}\int_0^1\int_0^{1-x}(x-xy-x^2)\,dy\,dx$$

$$= \sqrt{3}\int_0^1\left[xy-\frac{xy^2}{2}-x^2y\right]_0^{1-x}dx$$

$$= \sqrt{3}\int_0^1\left(\frac{x}{2}-x^2+\frac{x^3}{2}\right)dx$$

$$= \sqrt{3}\left[\frac{x^2}{4}-\frac{x^3}{3}+\frac{x^4}{8}\right]_0^1 = \frac{\sqrt{3}}{24}$$



$$= \sqrt{3}\left[\frac{x^2}{4}-\frac{x^3}{3}+\frac{x^4}{8}\right]_0^1 = \frac{\sqrt{3}}{24}$$

Surfaces Integrals

Example 12.4:

Evaluate the surface integral $\iint_S x\, dS$ where S the surface $z = x^2 + 4y$, plane $0 \leq x \leq 1$ and plane $0 \leq y \leq 2$.

Solution:

S is the surface $z = x^2 + 4y$. Hence, $\frac{\partial z}{\partial x} = 2x$ and $\frac{\partial z}{\partial y} = 4$.

$$\iint_S x\, dS = \iint_S x\sqrt{(2x)^2 + (4)^2 + 1}\, dA = \int_0^1 \int_0^2 x\sqrt{17 + 4x^2} dy\, dx$$

$$\int_0^1 2x\sqrt{17 + 4x^2} dx = \int_{17}^{21} \frac{1}{4}\sqrt{u} du = \frac{1\left(21\sqrt{21} - 17\sqrt{17}\right)}{6}$$

Parametric Surfaces: Extended Theorem

Let S be a surface with equation $y = f(x, z)$ and let R be its projection on the xz −plane. If f has continuous first partial derivatives on R and $f(x, y, z)$ is continuous on S, then

$$\iint_S g(x, y, z)\, dS = \iint_S g(x, f(x, z), z)\sqrt{\left(\frac{\partial y}{\partial x}\right)^2 + \left(\frac{\partial y}{\partial z}\right)^2 + 1}\, dA$$

Let S be a surface with equation $x = f(y, z)$ and let R be its projection on the yz −plane. If f has continuous first partial derivatives on R and $f(x, y, z)$ is continuous on S, then

$$\iint_S g(x, y, z)\, dS = \iint_S g(f(y, z), y, z)\sqrt{\left(\frac{\partial x}{\partial y}\right)^2 + \left(\frac{\partial x}{\partial z}\right)^2 + 1}\, dA$$

Alternative Solution Example 12.3:

The equation of plane can be written as $y = f(x,z) = 1 - x - z$ and $g(x,y,z) = xz$.

Hence, $\dfrac{\partial y}{\partial x} = -1$ and $\dfrac{\partial y}{\partial z} = -1$.

$$\iint_S xz\,dS = \iint_R xz\sqrt{(-1)^2 + (-1)^2 + 1}\,dA$$

$$= \sqrt{3}\int_0^1 \int_0^{1-x} xz\,dz\,dx = \sqrt{3}\int_0^1 \left[\frac{xz^2}{2}\right]_0^{1-x} dx$$

$$= \frac{\sqrt{3}}{2}\int_0^1 (x - 2x^2 + x^3)dx$$

$$= \frac{\sqrt{3}}{2}\left[\frac{x^2}{2} - \frac{2x^3}{3} + \frac{x^4}{4}\right]_0^1 = \frac{\sqrt{3}}{24}$$

Example 12.5:

Evaluate the surface integral

$$\iint_S y^2 z^2 \, dS$$

where S is part of the cone $z = \sqrt{x^2 + y^2}$, between the plane $z = 2$ and $z = 1$.

Solution:

The equation of plane can be written as $z = \sqrt{x^2 + y^2}$ and $g(x, y, z) = y^2 z^2$.

Hence,

$$\frac{\partial z}{\partial x} = \frac{x}{\sqrt{x^2+y^2}}, \frac{\partial z}{\partial y} = \frac{y}{\sqrt{x^2+y^2}} \text{ and} \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1} = \sqrt{\left(\frac{x}{\sqrt{x^2+y^2}}\right)^2 + \left(\frac{y}{\sqrt{x^2+y^2}}\right)^2 + 1} = \sqrt{3}.$$

Solution:

$$\iint_S y^2 z^2\, dS = \sqrt{3}\iint_R y^2(x^2+y^2)\, dA$$

Let R be the projection of the cone on xy −plane.
Based on the region, R, convert to polar coordinate,

$$\sqrt{3}\iint_R y^2(x^2+y^2)\, dA = \sqrt{3}\int_0^{2\pi}\int_1^2 \sin^2\theta\, r^2 r dr\, d\theta$$

$$= \sqrt{3}\left[\frac{r^4}{4}\right]_1^2 \int_0^{2\pi} \frac{1-\cos 2\theta}{2}\, d\theta = \frac{5}{4}\sqrt{3}\pi$$

Areas Surface Integrals

Integrating function, $g(x, y, z)$, over some surface, S in R^3. The region S lie above some region R that lies in the xy −plane.

Theorem : Suppose g is defined and continuous on surface S. The surface integral of $g(x, y, z)$ over S is denoted by:

$$\iint_S g(x, y, z)\, dS$$

Remarks: If $g(x, y, z) = 1$, it becomes surface area, A of S

$$A = \iint_S (1)\, dS$$

Example 12.6:

Evaluate the surface area , A where S is part of the cone $z=\sqrt{x^2+y^2}$, between the plane $z=2$ and $z=1$.

Solution:

The equation of plane can be written as $z=\sqrt{x^2+y^2}$ and $g(x,y,z)=1$.
Hence,

$$\frac{\partial z}{\partial x}=\frac{x}{\sqrt{x^2+y^2}},\ \frac{\partial z}{\partial y}=\frac{y}{\sqrt{x^2+y^2}} \text{ and } \sqrt{\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2+1}=\sqrt{\left(\frac{x}{\sqrt{x^2+y^2}}\right)^2+\left(\frac{y}{\sqrt{x^2+y^2}}\right)^2+1}=\sqrt{3}.$$

Hence,

$$A=\iint_S dS=\iint_R \sqrt{3}\,dA=\sqrt{3}\int_1^2\int_0^{2\pi} r\,d\theta\,dr=3\sqrt{3}\pi$$

Exercise 12.1:

1. Evaluate the surface integral $\iint_S 3(y+2z-2)\,dS$ where S is part of the plane $2x+3y+6z=12$ that lies in the first octant.
2. Evaluate the surface integral $\iint_S 3xy\,dS$ over the surface S of the sphere $x^2+y^2+z^2=1$ in the positive octant.
3. Evaluate the surface integral $\iint_S x\,dS$ where S the surface $y=x^2+4z$, and plane $0\le x\le 2, 0\le z\le 2$.

[Ans: 1. 24 2. $\frac{3}{8}$ 3. $\frac{33\sqrt{33}-17\sqrt{17}}{6}$]

Surface Integrals Involving Vector Valued Functions

If $\boldsymbol{F}(x, y, z) = f(x, y, z)\mathbf{i} + g(x, y, z)\mathbf{j} + h(x, y, z)\mathbf{k}$, then has continuous components on the oriented surface, S and given that $\boldsymbol{n} = \boldsymbol{n}(x, y, z)$ is the **Unit Normal Vector** of the orientation at the point (x, y, z), then

$$\iint_S \boldsymbol{F} \cdot d\boldsymbol{S} = \iint_S \boldsymbol{F} \cdot \boldsymbol{n}\, dS$$

is called the *flux integral* of $\boldsymbol{F}$ over S or the *surface integral of* $\boldsymbol{F} \cdot \boldsymbol{n}$ *over* S or the *surface integral of the normal component of* $\boldsymbol{F}$ *over* S.

Unit Normal Vector for surface functions

To compute the unit normal, $\boldsymbol{n}$, to a surface S given by an equation $z = f(x, y)$.
First, rewrite the equation of a surface S as,

$$z - f(x, y) = G(x, y, z),$$

which let the surface for the function

$$G(x, y, z) = z - f(x, y).$$

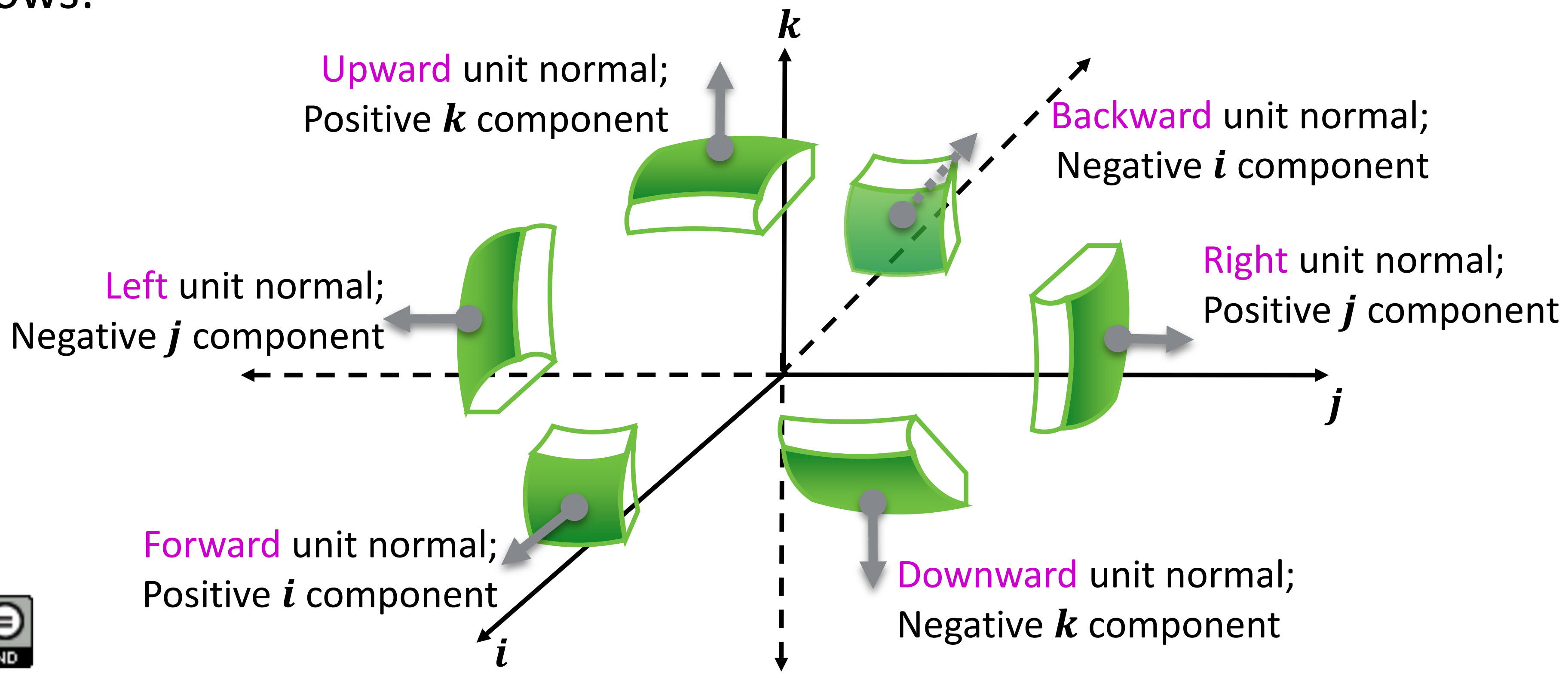
If $G(x, y, z) = 0,$ then **Unit Normal Vector** to S at the point (x, y, z) is

$$\boldsymbol{n} = \frac{\nabla G}{\|\nabla G\|}$$

where the gradient $\nabla G = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right)\left(z - f(x, y)\right) = -\frac{\partial f}{\partial x}\mathbf{i} - \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial z}{\partial z}\mathbf{k}$ and $\|\nabla G\| = \sqrt{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 + 1}$, such that the Vector Differential Operator, *Del*, $\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle$.

Surface Integrals Involving Vector Valued Functions: Terminology unit normal vector

Terminology of surface integrals in 6 direction within 3 axis can be described mathematically, as follows:



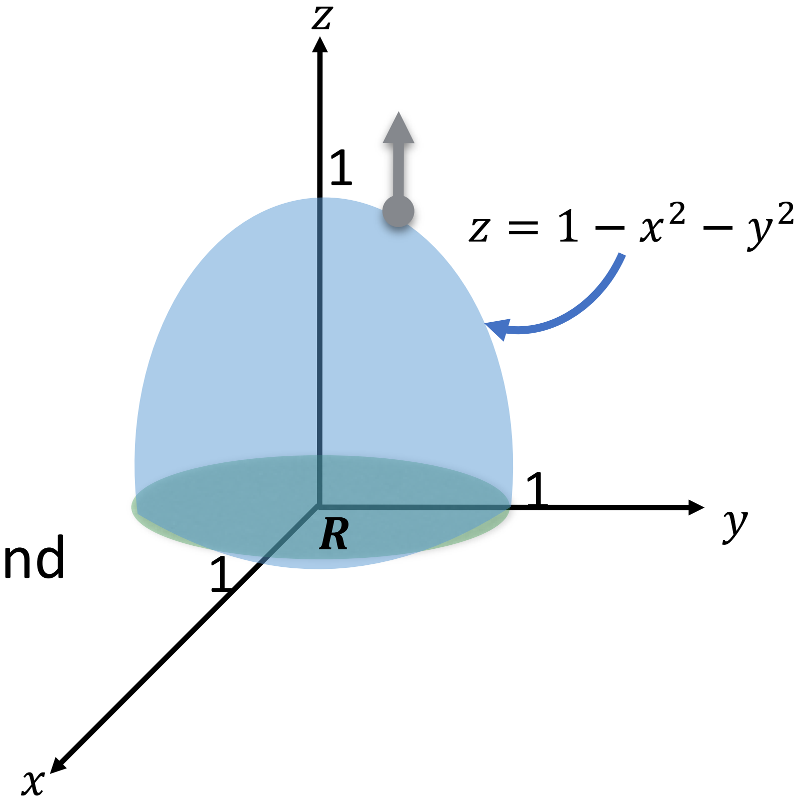
Surface Integrals Involving Vector Valued Functions:

Example 12.7:

Suppose that S is the portion of the surface $z = 1 - x^2 - y^2$ above the xy −plane. Let S be oriented by upward normal and given that

$$\boldsymbol{F}(x, y, z) = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$$

Evaluate $\iint_S \boldsymbol{F} \cdot \boldsymbol{n}\, dS$



Solution:

Rewrite $G(x, y, z) = z + x^2 + y^2 - 1$,

$\nabla G = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right)(z + x^2 + y^2 - 1) = 2x\mathbf{i} + 2y\mathbf{j} + \mathbf{k}$ and

$$\|\nabla G\| = \sqrt{(2x)^2 + (2y)^2 + 1}.$$

Solution:

Compute $\boldsymbol{n} = \frac{\nabla G}{\|\nabla G\|}$. Hence,

$$\iint_S \boldsymbol{F}\cdot\boldsymbol{n}\,dS = \iint_R \boldsymbol{F}\cdot\frac{2x\mathbf{i}+2y\mathbf{j}+\mathbf{k}}{\sqrt{(2x)^2+(2y)^2+1}}\sqrt{(2x)^2+(2y)^2+1}\,dA$$

$$= \iint_R \boldsymbol{F}\cdot(2x\mathbf{i}+2y\mathbf{j}+\mathbf{k})\,dA = \iint_R (x\mathbf{i}+y\mathbf{j}+z\mathbf{k})\cdot(2x\mathbf{i}+2y\mathbf{j}+\mathbf{k})\,dA$$

$$= \iint_R 2x^2+2y^2+z\,dA = \iint_R 2x^2+2y^2+1-x^2-y^2\,dA$$

$$= \iint_R 1+x^2+y^2\,dA = \int_0^1\int_0^{2\pi}(1+r^2)r\,d\theta\,dr = \frac{3}{2}\pi$$

Evaluating Surface Integrals

If the surface S is given by an equation of the form $z = z(x, y)$ where z has continuous first partial derivatives and if R is the projection of the surface on the xy −plane, Hence

$$\iint_S \boldsymbol{F}\cdot\boldsymbol{n}\,dS = \iint_R \boldsymbol{F}\cdot\left(-\frac{\partial}{\partial x}\mathbf{i} - \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right)dA$$

if S is oriented by **upward normal.**

$$\iint_S \boldsymbol{F}\cdot\boldsymbol{n}\,dS = \iint_R \boldsymbol{F}\cdot\left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} - \frac{\partial}{\partial z}\mathbf{k}\right)dA$$

if S is oriented by **downward normal.**

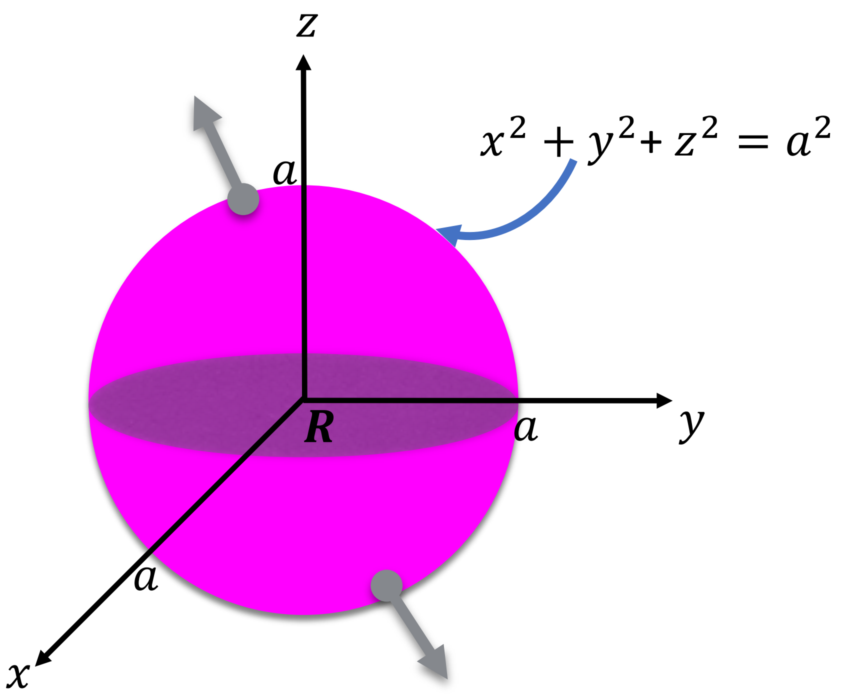
Evaluating Surface Integrals

Example 12.8:

Let S be the sphere $x^2 + y^2 + z^2 = a^2$ oriented by outward normal and given that

$$\boldsymbol{F}(x, y, z) = z\boldsymbol{k}$$

Evaluate $\iint_S \boldsymbol{F} \cdot \boldsymbol{n}\, dS$.



Solution:

On the upper hemisphere the outward unit normal vector is upward while the lower hemisphere it is downwards unit normal.

Solution:

Since different formula for these normals apply on the two hemispheres, the surface integrals:

$$\iint_S \boldsymbol{F}\cdot\boldsymbol{n}\,dS = \iint_{S_1} \boldsymbol{F}\cdot\boldsymbol{n}\,dS + \iint_{S_2} \boldsymbol{F}\cdot\boldsymbol{n}\,dS$$

where S_1 and S_2 is the upper and lower hemisphere, respectively.
Rewrite sphere expression $x^2+y^2+z^2=a^2$ to obtain

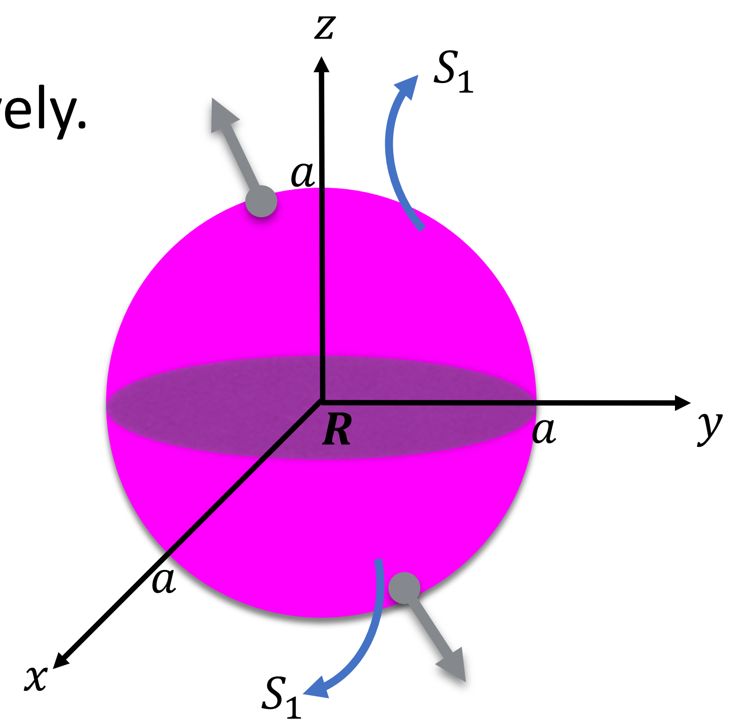
$$z = \pm\sqrt{a^2-x^2-y^2}$$

which can be obtained into upper hemisphere expression

$$z = \sqrt{a^2-x^2-y^2},$$

and lower hemisphere expression

$$z = -\sqrt{a^2-x^2-y^2}.$$



Solution: For upper hemisphere expression

$$G(x,y,z) = z - \sqrt{a^2 - x^2 - y^2},$$

$$\nabla G = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right)\left(z - \sqrt{a^2 - x^2 - y^2}\right) = \frac{x}{\sqrt{a^2 - x^2 - y^2}}\mathbf{i} + \frac{y}{\sqrt{a^2 - x^2 - y^2}}\mathbf{j} + \mathbf{k}.$$

$$\iint_{S_1} \boldsymbol{F} \cdot \boldsymbol{n}\, dS = \iint_R (z\mathbf{k}) \cdot \left(\frac{x}{\sqrt{a^2 - x^2 - y^2}}\mathbf{i} + \frac{y}{\sqrt{a^2 - x^2 - y^2}}\mathbf{j} + \mathbf{k}\right) dA$$

$$= \iint_R z\, dA = \iint_R \sqrt{a^2 - x^2 - y^2}\, dA = \int_0^a \int_0^{2\pi} r\sqrt{a^2 - r^2}\, d\theta\, dr$$

$$= \int_0^a r\sqrt{a^2 - r^2}\,[\theta]_0^{2\pi}\, dr = 2\pi\left[-\frac{1}{3}\left(\sqrt{a^2 - r^2}\right)^3\right]_0^a = \frac{2\pi a^3}{3}$$

Solution: For lower hemisphere expression

$$G(x,y,z) = z + \sqrt{a^2 - x^2 - y^2},$$

$$\nabla G = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right)\left(z + \sqrt{a^2 - x^2 - y^2}\right) = \frac{x}{\sqrt{a^2 - x^2 - y^2}}\mathbf{i} + \frac{y}{\sqrt{a^2 - x^2 - y^2}}\mathbf{j} - \mathbf{k}.$$

$$\iint_{S_2} \boldsymbol{F} \cdot \boldsymbol{n}\, dS = \iint_R -z\, dA = \iint_R \sqrt{a^2 - x^2 - y^2}\, dA = \frac{2\pi a^3}{3}$$

Hence, the surface integrals:

$$\iint_S \boldsymbol{F} \cdot \boldsymbol{n}\, dS = \iint_{S_1} \boldsymbol{F} \cdot \boldsymbol{n}\, dS + \iint_{S_2} \boldsymbol{F} \cdot \boldsymbol{n}\, dS$$

$$= \frac{2\pi a^3}{3} + \frac{2\pi a^3}{3} = \frac{4\pi a^3}{3}$$

Example 12.9:

Given that $\boldsymbol{F}(x,y,z) = z^2\boldsymbol{k}$ is the upper hemisphere given by $z = \sqrt{1-x^2-y^2}$ oriented by upward unit normal vector. Evaluate $\iint_R \boldsymbol{F}\cdot\boldsymbol{n}\,dS$.

Solution:

Apply upper hemisphere expression

$$G(x,y,z) = z - \sqrt{1-x^2-y^2},$$

$$\iint_S \boldsymbol{F}\cdot\boldsymbol{n}\,dS = \iint_R (z^2\mathbf{k})\cdot\left(\frac{x}{\sqrt{1-x^2-y^2}}\mathbf{i} + \frac{y}{\sqrt{1-x^2-y^2}}\mathbf{j} + \mathbf{k}\right)dA$$

$$= \iint_R z^2\,dA = \iint_R 1-x^2-y^2\,dA = \int_0^1\int_0^{2\pi} r(1-r^2)d\theta\,dr = \frac{3}{2}\pi$$

Exercise 12.2:

1. Evaluate $\iint_S \boldsymbol{F}\cdot\boldsymbol{n}\,dS$ where $\boldsymbol{F}(x,y,z) = 2xy\boldsymbol{i} + yz^2\boldsymbol{j} + xz\boldsymbol{k}$ and S is the surface of the plane $0 \le y \le 2, 0 \le z \le 3$ and $0 \le x \le 1$ in the first octant.
2. Let S be the parabolic cylinder $y^2 = 8x$ in the first octant bounded by the planes $y = 4$ and $z = 6$ given that $\boldsymbol{F}(x,y,z) = 2y\boldsymbol{i} - z\boldsymbol{j} + x^2\boldsymbol{k}$. Evaluate $\iint_S \boldsymbol{F}\cdot\boldsymbol{n}\,dS$.
3. Evaluate $\iint_S \boldsymbol{F}\cdot\boldsymbol{n}\,dS$ where $\boldsymbol{F}(x,y,z) = (x + y^2)\boldsymbol{i} - 2x\boldsymbol{j} + 2yz\boldsymbol{k}$ and S be in the first octant for the part of the plane $2x + y + 2z - 6 = 0$.

[Ans: 1. 33   2. 132 3. 81]

Flux

If $\boldsymbol{F}$ represents velocity of the fluid at any point on a closed surface S, then surface integral $\iint_S \boldsymbol{F} \cdot \boldsymbol{n}\, dS$ represents the flux of $\boldsymbol{F}$ over S, that is a volume of the fluid flowing out from per unit time.

Remarks: If $\iint_S \boldsymbol{F} \cdot \boldsymbol{n}\, dS = 0$, then $\boldsymbol{F}$ is called a solenoidal vector point function.

Example 12.10:

Find the flux of $\boldsymbol{F}(x, y, z) = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$ over the surface of the sphere $x^2 + y^2 + z^2 = 1$ in the positive octant.

Solution: I'm

Flux $= \iint_S \boldsymbol{F} \cdot \boldsymbol{n}\, dS$

Solution: Rewrite $G(x, y, z) = x^2 + y^2 + z^2 - 1$,

$$\boldsymbol{n} = \frac{\nabla G}{\|\nabla G\|} = \frac{2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}}{\sqrt{4x^2 + 4y^2 + 4z^2}} = \frac{2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}}{\sqrt{4(x^2 + y^2 + z^2)}} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

Use dA as projection on xy −plane of $dS = \frac{dxdy}{\|\boldsymbol{n}\cdot\mathbf{k}\|}$ where $\|\boldsymbol{n} \cdot \mathbf{k}\| = z$.

$$\iint_S \boldsymbol{F} \cdot \boldsymbol{n}\, dS = \iint_R \boldsymbol{F} \cdot (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \frac{dxdy}{\|\boldsymbol{n} \cdot \mathbf{k}\|}$$

$$= \iint_R (yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}) \cdot (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) \frac{dxdy}{z}$$

$$= \iint_R 3xyz \frac{dxdy}{z} = \iint_R 3xy\, dxdy = 3 \int_0^1 \int_0^{\pi/2} r \sin\theta \cdot r\cos\theta \cdot r d\theta\, dr$$

$$= \frac{3}{2} \int_0^1 \int_0^{\pi/2} \sin 2\theta\, r^3 d\theta\, dr = \frac{3}{8}$$

Exercise 12.3:

1. Find the flux of $\boldsymbol{F}(x, y, z) = 3y\mathbf{i} + 2z\mathbf{j} + x^2yz\mathbf{k}$ over the surface of $y^2 = 5x$ in the positive octant bounded by the planes $x = 3$ and $z = 4$.
2. Find the flux of $\boldsymbol{F}(x, y, z) = 18z\mathbf{i} - 12\mathbf{j} + 3y\mathbf{k}$ and surface is the part of the plane $2x + 3y + 6z = 12$ in the first octant.
3. Evaluate $\iint_S \nabla \times \boldsymbol{F} \cdot \boldsymbol{n}\, dS$ where $\boldsymbol{F}(x, y, z) = y^2\mathbf{i} + y\mathbf{j} - xz\mathbf{k}$ over surface, S of the upper half of the sphere $x^2 + y^2 + z^2 = 21$.

[Ans: 1. -42 2. 24 3. 0]

Reference

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THANK YOU