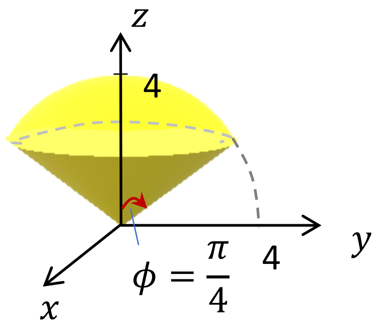
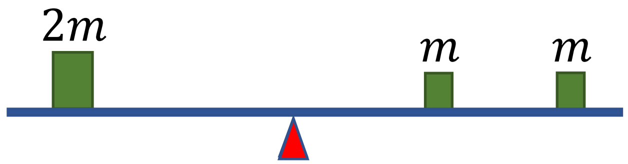
BEKG 2433
ENGINEERING MATHEMATICS 2

TRIPLE INTEGRAL IN SPHERICAL COORDINATES, MOMENT AND CENTRE OF GRAVITY





Ser Lee Loh
slloh@utem.edu.my

Lesson Outcomes

Upon completion of this lesson, students should be able to:

- evaluate triple integral in Spherical coordinates.
- compute moment and center of gravity of an object.

Spherical Coordinates

In spherical coordinates, a point P in $\mathbb{R}^3$ is represented by three coordinates (ρ, ϕ, θ).

$\Rightarrow$ ρ is the distance from the origin to P. $(0 \leq \rho \leq \infty)$

$\Rightarrow$ ϕ is the angle between the positive z-axis and the line OP. $(0 \leq \phi \leq \pi)$

$\Rightarrow$ θ is the same angle as in cylindrical coordinates; it measures rotation about the z-axis relative to the positive x-axis. $(0 \leq \theta \leq 2\pi)$

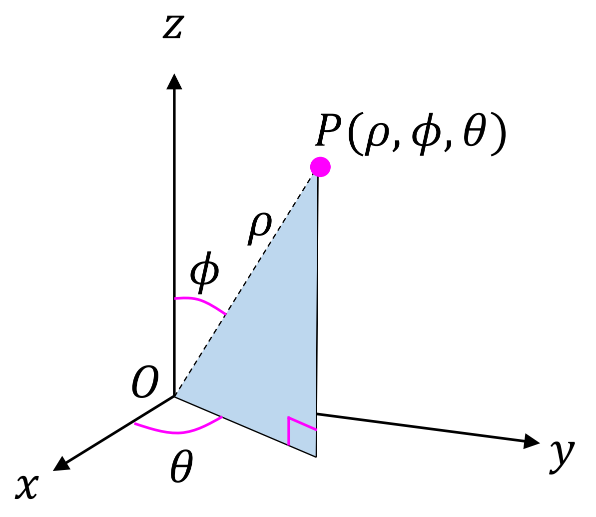


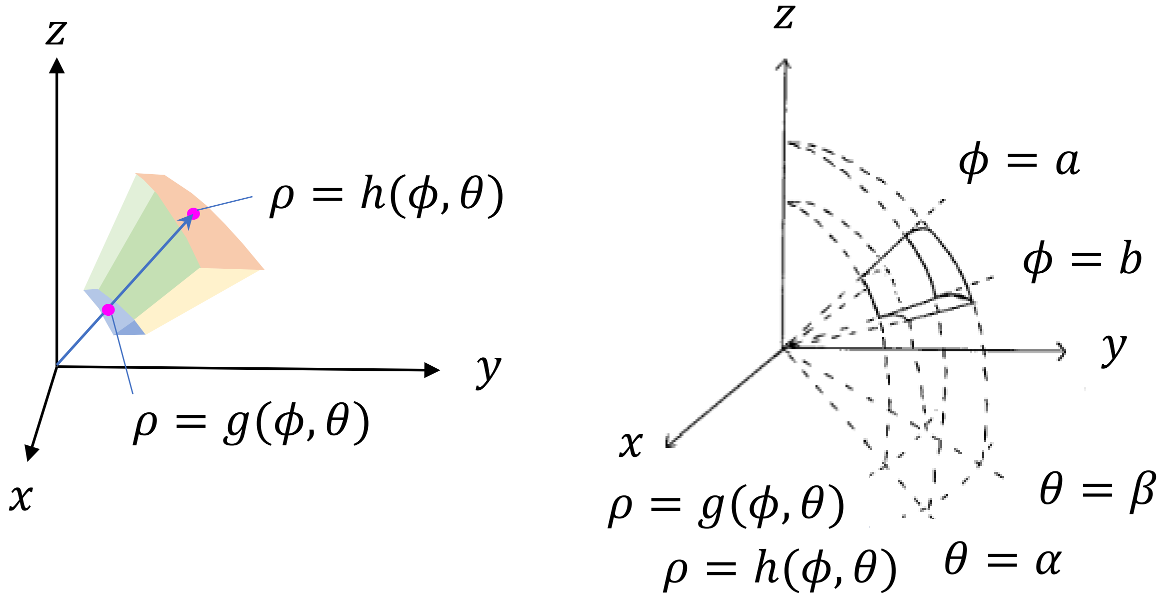
Figure: Spherical coordinates

Let f be continuous over the region

$$D = \{(\rho, \phi, \theta) \colon g(\phi, \theta) \leq \rho \leq h(\phi, \theta), a \leq \phi \leq b, \alpha \leq \theta \leq \beta\}.$$

Then f is integrable over D and the triple integral of f over D in spherical coordinates is

$$\iiint_D f(\rho, \phi, \theta)\, dV = \int_\alpha^\beta \int_a^b \int_{g(\phi,\theta)}^{h(\phi,\theta)} f(\rho, \phi, \theta)\, \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta.$$



$$x = \rho \sin\phi \cos\theta,$$

$$y = \rho \sin\phi \sin\theta,$$

$$z = \rho \cos\phi,$$

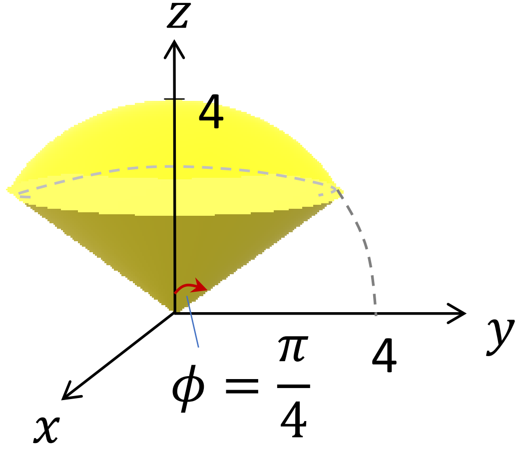
$$x^2 + y^2 + z^2 = \rho^2$$

Figure: Interval of spherical coordinates

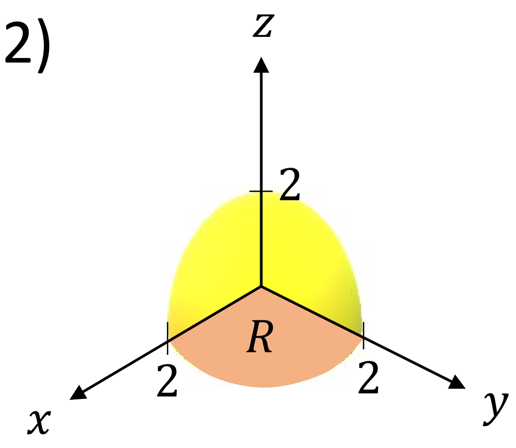
Example 9.1:

Identify the ranges of ρ, ϕ and θ for the following solids.

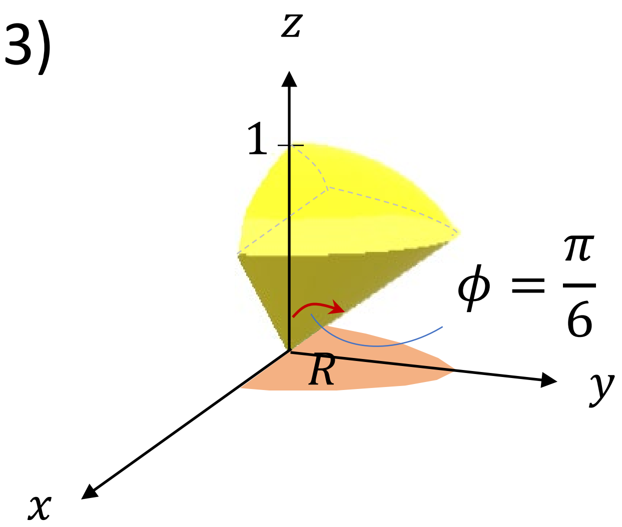
1)



2)



3)



Solutions:

1) $0 \le \rho \le 4, \quad 0 \le \phi \le \frac{\pi}{4}, \quad 0 \le \theta \le 2\pi$

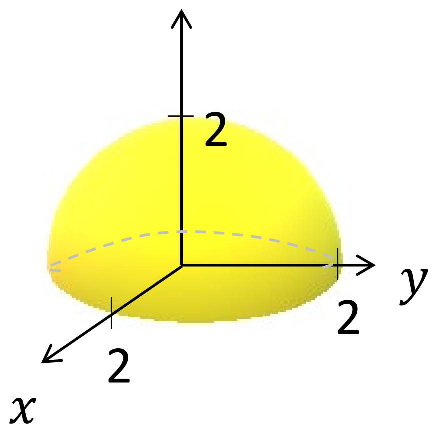
2) $0 \le \rho \le 2, \quad 0 \le \phi \le \frac{\pi}{2}, \quad 0 \le \theta \le \frac{\pi}{2}$

3) $0 \le \rho \le 1, \quad 0 \le \phi \le \frac{\pi}{6}, \quad 0 \le \theta \le \pi$

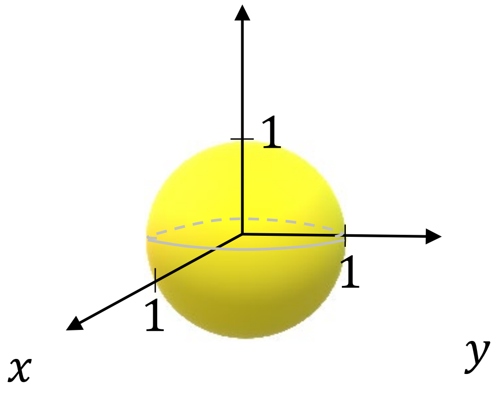
Exercise 9.1:

Sketch the 3D region defined by the given ranges.

1)

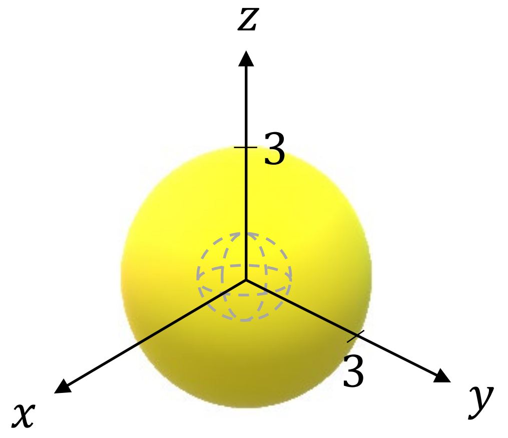


2)



A sphere with radius of 2

3)



A sphere with an empty sphere of radius 1 centred at the origin

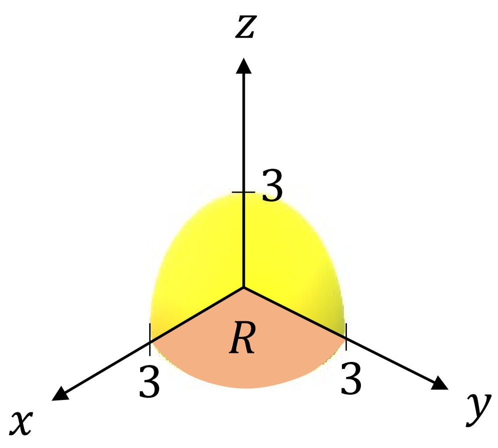
[Ans: $0 \le \rho \le 2,\ \ 0 \le \phi \le \frac{\pi}{2},\ \ 0 \le \theta \le 2\pi;$
$0 \le \rho \le 1,\ \ 0 \le \phi \le \pi,\ \ 0 \le \theta \le 2\pi;$
$1 \le \rho \le 3,\ \ 0 \le \phi \le \pi,\ \ 0 \le \theta \le 2\pi$]

Example 9.2:

Given D is the portion of a sphere of radius 3 in the first octant. Evaluate the following integral.

$$\iiint_D \sqrt{x^2+y^2+z^2}\,dV$$

Solution:



$$\iiint_D \sqrt{x^2+y^2+z^2}\,dV$$
$$= \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^3 (\sqrt{\rho^2})\,\rho^2 \sin\phi\,d\rho\,d\phi\,d\theta$$
$$= \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^3 \rho^3 \sin\phi\,d\rho\,d\phi\,d\theta$$
$$= \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \left[\frac{\rho^4}{4}\right]_0^3 \sin\phi\,d\phi\,d\theta$$
$$= \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \frac{81}{4}\sin\phi\;d\phi\,d\theta$$
$$= -\frac{81}{4}\int_0^{\frac{\pi}{2}}[\cos\phi]_0^{\frac{\pi}{2}}\,d\theta = \frac{81}{4}\int_0^{\frac{\pi}{2}} 1\,d\theta = \frac{81\pi}{8}$$

Example 9.3:

Evaluate the integral $\iiint_D 2(x^2+y^2+z^2)\,dV$ given D is the hemisphera.

Solution:

$$\iiint_D 2(x^2+y^2+z^2)\,dV$$

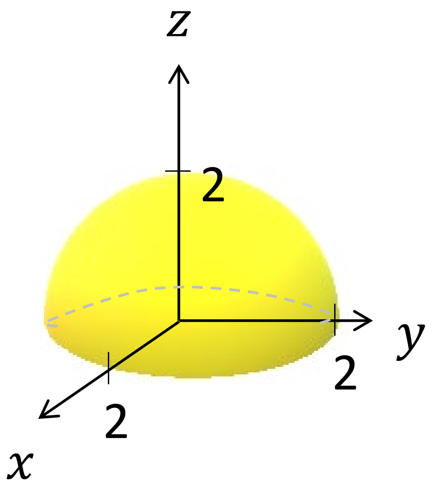
$$= \int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_0^{2}(2\,\rho^2)\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

$$= \int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_0^{2} 2\rho^4\sin\phi\,d\rho\,d\phi\,d\theta$$

$$= \int_0^{2\pi}\int_0^{\frac{\pi}{2}}\left[\frac{2\rho^5}{5}\right]_0^2\sin\phi\,d\phi\,d\theta$$

$$= \int_0^{2\pi}\int_0^{\frac{\pi}{2}}\frac{64}{5}\sin\phi\;d\phi\,d\theta$$

$$= -\frac{64}{5}\int_0^{2\pi}[\cos\phi]_0^{\frac{\pi}{2}}\,d\theta = \frac{64}{5}\int_0^{2\pi}1\,d\theta = \frac{128\pi}{5}$$



Example 9.4:

Evaluate the following integral using spherical coordinates.

$$\int_{-2}^{2}\int_{0}^{\sqrt{4-x^2}}\int_{0}^{\sqrt{4-x^2-y^2}} x^2+y^2+z^2 \quad dz\,dy\,dx$$

Solution:

$$\int_{-2}^{2}\int_{0}^{\sqrt{4-x^2}}\int_{0}^{\sqrt{4-x^2-y^2}} x^2+y^2+z^2 \quad dz\,dy\,dx$$

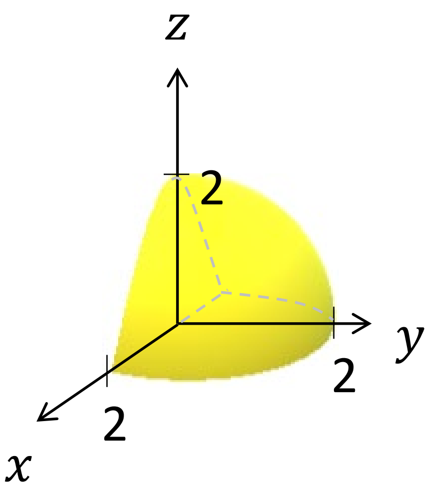
$$= \int_0^{\pi}\int_0^{\frac{\pi}{2}}\int_0^{2}(\rho^2)\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

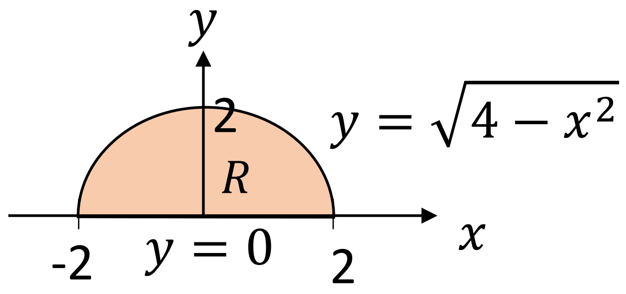
$$= \int_0^{\pi}\int_0^{\frac{\pi}{2}}\int_0^{2}\rho^4\sin\phi\,d\rho\,d\phi\,d\theta$$

$$= \int_0^{\pi}\int_0^{\frac{\pi}{2}}\left[\frac{\rho^5}{5}\right]_0^2\sin\phi\,d\phi\,d\theta$$

$$= \frac{32}{5}\int_0^{\pi}\int_0^{\frac{\pi}{2}}\sin\phi\,d\phi\,d\theta$$

$$= -\frac{32}{5}\int_0^{\pi}[\cos\phi]_0^{\frac{\pi}{2}}\,d\theta = \frac{32}{5}\int_0^{\pi}1\,d\theta = \frac{32\pi}{5}$$





Exercise 9.2:

1) Sketch the solid and carry out the integration.

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{2}\rho^2\sin\phi\ \ d\rho\ d\phi d\theta$$

What is the value of this integration representing?

[Ans: $\frac{32\pi}{3}$]

2) Evaluate the following integral using spherical coordinates.

$$\int_0^{1}\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}}\frac{1}{x^2+y^2+z^2}\ \ dz\ dy\ dx$$

[Ans: $\frac{\pi}{2}$]

3) Given Q is the unit ball: $x^2+y^2+z^2\leq 1$. Evaluate the following integral.

$$\iiint_Q\ \cos\left(x^2+y^2+z^2\right)^{\frac{3}{2}}\ dV$$

[Ans: $\frac{4\pi}{3}\sin 1$]

Exercise 9.2:

4) Given Q is bounded by the hemisphere $z = -\sqrt{9 - x^2 - y^2}$ and the xy-plane. Evaluate the following integral.

$$\iiint_Q \sqrt{x^2 + y^2 + z^2}\; dV$$

[Ans: $\frac{81\pi}{2}$]

5) Evaluate $\int_0^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} \sqrt{x^2 + y^2 + z^2}\; dz\, dy\, dx.$

[Ans: $\frac{\pi}{2}$]

6) Find the volume of the solid region D that lies inside the cone $\phi = \frac{\pi}{6}$ and inside the sphere $\rho = 4$.

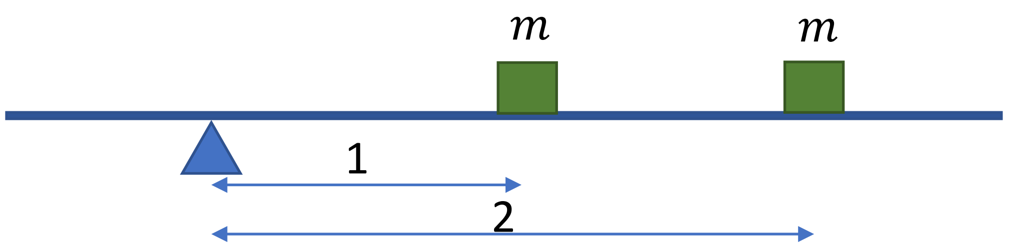
[Ans: $\frac{64\pi}{3}(2 - \sqrt{3})$]

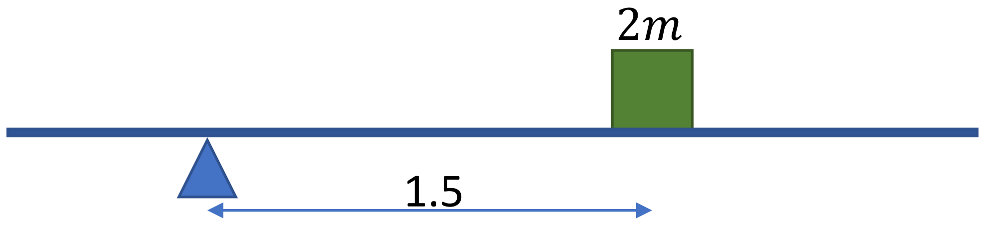
Moment and Centre of Gravity

Introduction

What is **moment**? => **Turning effect** of a force

The single object of mass $2m$ has the same turning effect as the two objects each of mass m:





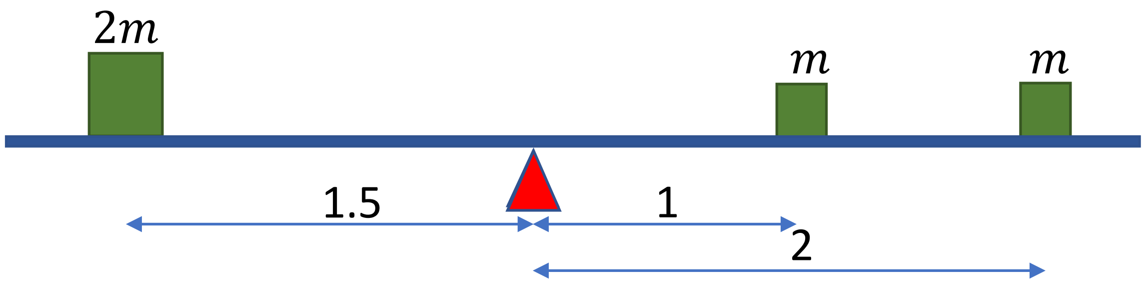
Moment $= (1)(m) + (2)(m) = 3m$

Moment $= (1.5)(2m) = 3m$

Hence, if we have the following situation, the system is in balance:

Moment $= (1.5)(2m) = 3m$

Moment $= (1)(m) + (2)(m) = 3m$

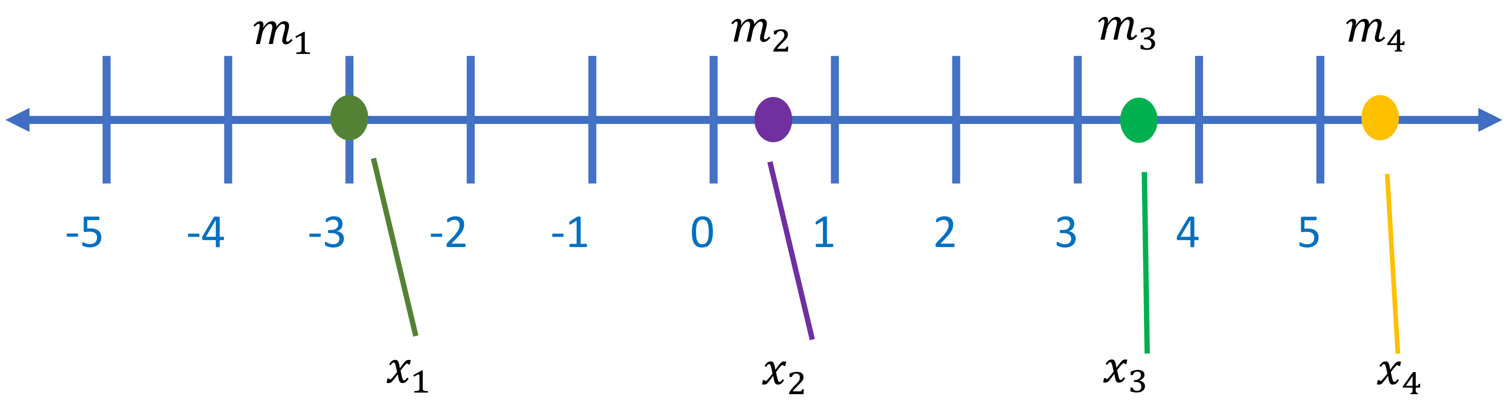


Moment and Centre of Gravity for Lamina

Discrete Case

One-Dimensional:

Given four mass, m_i, tabulated in one-dimensional at location x_i, respectively, as follows:



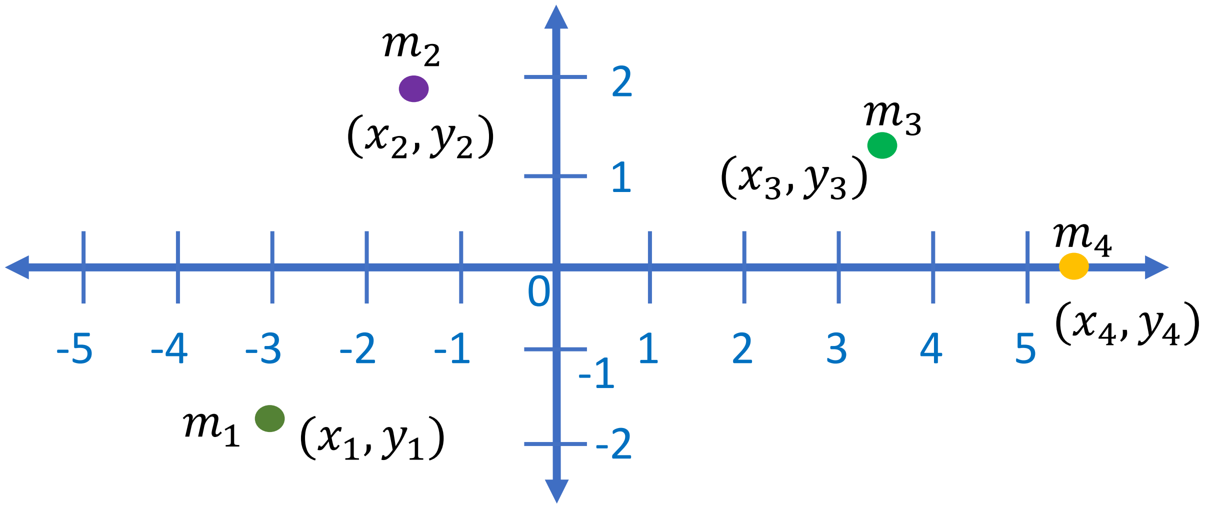
$$\text{center of mass} = \bar{x} = \frac{M}{m} = \frac{m_1x_1 + m_2x_2 + m_3x_3 + m_4x_4}{m_1 + m_2 + m_3 + m_4}$$

In general:

$$\bar{x} = \frac{Moment\ of\ mass}{mass} = \frac{M}{m} = \frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i}$$

Two-Dimensional:

Given four mass, m_i, tabulated at point (x_i, y_i), respectively, as follows:



$$\bar{x} = \frac{M}{m} = \frac{m_1x_1+m_2x_2+m_3x_3+m_4x_4}{m_1+m_2+m_3+m_4}, \qquad \bar{y} = \frac{M}{m} = \frac{m_1y_1+m_2y_2+m_3y_3+m_4y_4}{m_1+m_2+m_3+m_4}$$

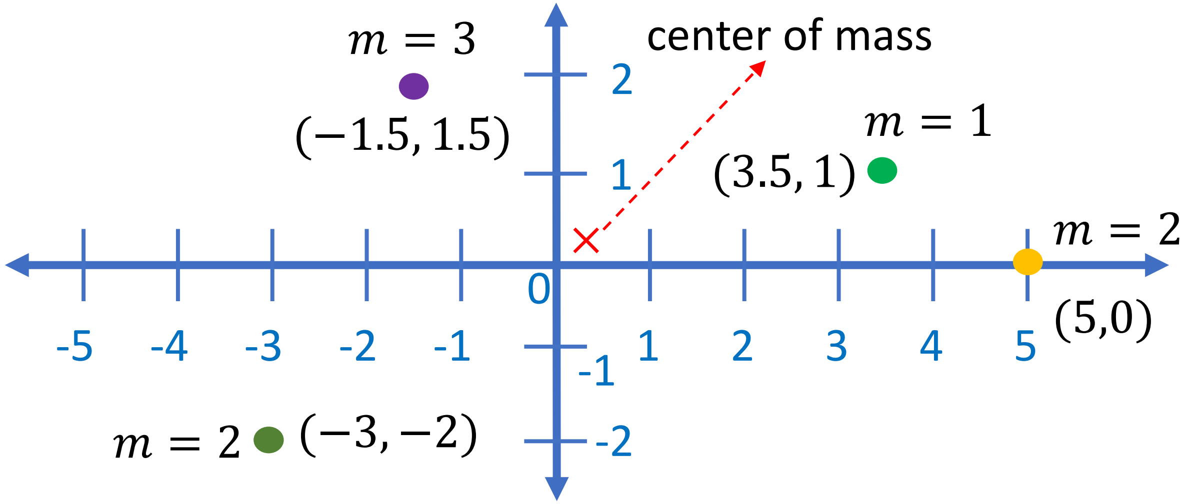
In general:

$$\bar{x} = \frac{Moment}{mass} = \frac{\sum_{i=1}^{n} m_i x_i}{\sum_{i=1}^{n} m_i}, \qquad \bar{y} = \frac{Moment}{mass} = \frac{\sum_{i=1}^{n} m_i y_i}{\sum_{i=1}^{n} m_i}$$

Center of mass/gravity = $(\bar{x}, \bar{y})$

Exercise 9.3:

Given



Find the center of mass of the system.

[Ans: (0.375, 0.1875)]

Continuous Case

One-Dimensional:

Given a thin rod or wire on the interval $[a, b]$ with a density $\sigma(x)$, the center of mass is located at the point

$$\bar{x} = \frac{M}{m}$$

where total moment M and total mass m are

$$M = \int_a^b x\sigma(x)\,dx \quad \text{and} \quad m = \int_a^b \sigma(x)\,dx$$

Two-Dimensional:

Given a region R with an area **density $\boldsymbol{\sigma(x,y)}$**, the center of mass is $(\bar{x},\bar{y})$:

$$\bar{x} = \frac{M_y}{m} = \frac{\iint_R x\sigma(x,y)\,dA}{\iint_R \sigma(x,y)\,dA}$$

$$\bar{y} = \frac{M_x}{m} = \frac{\iint_R y\sigma(x,y)\,dA}{\iint_R \sigma(x,y)\,dA}$$

If $\sigma(x,y)$ is a constant, the centre of mass is called **centroid**.

* M_i involves distances from i-axis and density = mass per unit area

Example 9.5:

Suppose a thin uniform 3-m rod is made of silver whose density in kg/m is $\sigma(x) = 1.5$, where $0 \leq x \leq 3$. Find the centroid of the rod.

Solution:

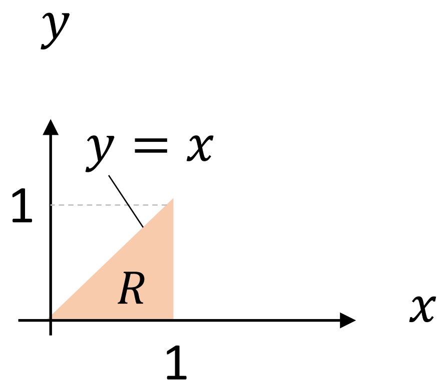
$$\bar{x} = \frac{M}{m} = \frac{\int_a^b x\sigma(x)\,dx}{\int_a^b \sigma(x)\,dx}$$

Since the density is a constant, where they can be cancelled out by each other,

$$\bar{x} = \frac{\int_a^b x\,dx}{\int_a^b 1\,dx} = \frac{\int_0^3 x\,dx}{\int_0^3 1\,dx} = \frac{\left[\frac{x^2}{2}\right]_0^3}{[x]_0^3} = \frac{\frac{9}{2}}{3} = \frac{3}{2} \text{ meter}$$

Example 9.6:

Find the center of mass of the following region R with density $\sigma(x,y) = x + 1$.



Solution:

$$M_y = \iint_R x\sigma(x,y)\,dA = \int_0^1 \int_0^x x(x+1)\,dy\,dx = \frac{7}{12}$$

$$M_x = \iint_R y\sigma(x,y)\,dA = \int_0^1 \int_0^x y(x+1)\,dy\,dx = \frac{7}{24}$$

$$m = \iint_R \sigma(x,y)\,dA = \int_0^1 \int_0^x (x+1)\,dy\,dx = \frac{5}{6}$$

Hence,

$$\bar{x} = \frac{M_y}{m} = \frac{\frac{7}{12}}{\frac{5}{6}} = \frac{7}{10}, \qquad \bar{y} = \frac{M_x}{m} = \frac{\frac{7}{24}}{\frac{5}{6}} = \frac{7}{20}$$

Moment of Inertia for Lamina in 2D

About x-axis: $I_x = \iint_R \; y^2\sigma(x,y)dA$

About y-axis: $I_y = \iint_R \; x^2\sigma(x,y)dA$

About the origin: $I_o = \iint_R (x^2+y^2)\sigma(x,y)dA$
$= I_x + I_y$

Example 9.7:

Given the rectangular plate in first quadrant bounded by $x = 4$ and $y = 1$ with $\sigma(x, y) = x^2$. Find the moment of inertia about the y-axis.

Solution:

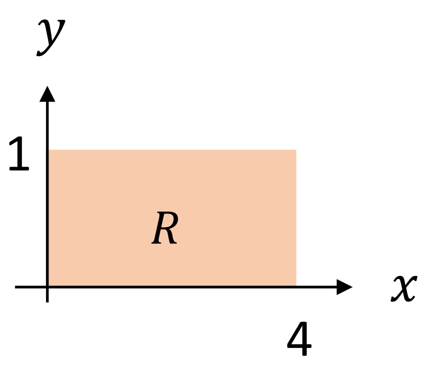
$$I_y = \iint_R x^2 \sigma(x, y) dA$$

$$= \int_0^4 \int_0^1 x^2 (x^2)\, dy\, dx$$

$$= \int_0^4 \int_0^1 x^4\, dy\, dx$$

$$= \int_0^4 [x^4 y]_0^1\, dx$$

$$= \int_0^4 x^4\, dx = \left[\frac{x^5}{5}\right]_0^4 = 204.8$$



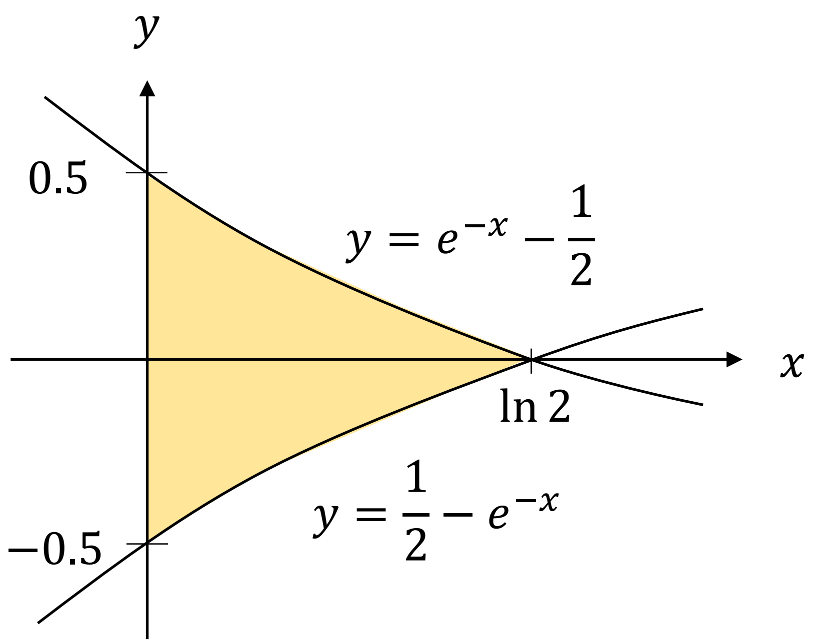
Exercise 9.4:

1) Suppose a thin 2-m bar is made of an alloy whose density in kg/m is $\sigma(x) = 1 + x^2$, where $0 \leq x \leq 2$. Find the center of mass of the bar.

[Ans: 9/7 meter]

2) Find the centroid of the following shaded region.

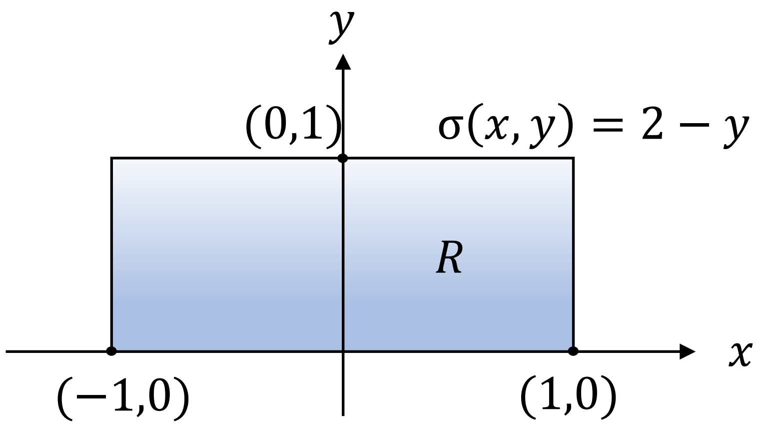
[Ans: (0.217, 0)]



Exercise 9.4:

3) Find the center of mass of the following shaded region.

[Ans: (0, 4/9)]



What is the center of mass if the density is changed into $\sigma(x, y) = x + y$?

[Ans: (2/3, 2/3)]

4) Given the triangular plate in first quadrant bounded by $x + y = 4$ with $\sigma(x, y) = y$. Find the moment of inertia about the x-axis.

[Ans: 256/5]

Moment and Centre of Gravity for Solid in 3D
First Moment Formulas for Solid in 3D

Given a solid D with a **density $\sigma(x, y, z)$**,

Mass, $m = \iiint_D \sigma(x, y, z)\, dV$

Moment about xy-plane: $M_{xy} = \iiint_D z\sigma(x, y, z)\, dV$

Moment about xz-plane: $M_{xz} = \iiint_D y\sigma(x, y, z)\, dV$

Moment about yz-plane: $M_{yz} = \iiint_D x\sigma(x, y, z)\, dV$

Center of Mass:

$$\bar{x} = \frac{M_{yz}}{m}, \qquad \bar{y} = \frac{M_{xz}}{m}, \qquad \bar{z} = \frac{M_{xy}}{m}$$

* M_{ij} involves distances from ij-plane

Example 9.8:

Find the center of mass of the solid D with the density $\sigma(x,y,z) = x + z$, where

$$D = \{(x,y,z): 0 \le x,y,z \le 1\}$$

Solution:

$$m = \iiint_D \sigma(x,y,z)\, dV = \int_0^1 \int_0^1 \int_0^1 x + z\, dzdydx = 1$$

$$M_{xy} = \iiint_D z\sigma(x,y,z)\, dV = \int_0^1 \int_0^1 \int_0^1 z(x+z)\, dzdydx = \frac{7}{12}$$

$$M_{xz} = \iiint_D y\sigma(x,y,z)\, dV = \int_0^1 \int_0^1 \int_0^1 y(x+z)\, dzdydx = \frac{1}{2}$$

$$M_{yz} = \iiint_D x\sigma(x,y,z)\, dV = \int_0^1 \int_0^1 \int_0^1 x(x+z)\, dzdydx = \frac{7}{12}$$

Center of Mass:

$$\bar{x} = \frac{M_{yz}}{m} = \frac{7}{12}, \qquad \bar{y} = \frac{M_{xz}}{m} = \frac{1}{2}, \qquad \bar{z} = \frac{M_{xy}}{m} = \frac{7}{12}$$

Exercise 9.5:

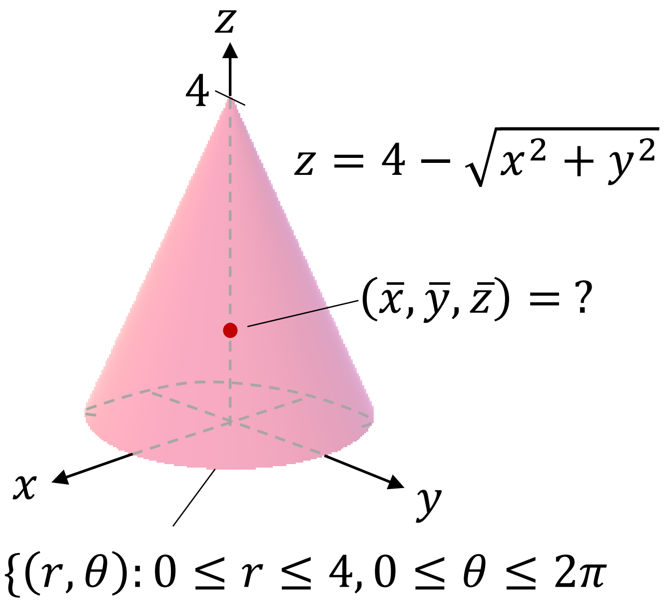
1) Find the mass of the solid cylinder

$$D = \{(r, \theta, z): 0 \leq r \leq 3, 0 \leq \theta \leq 2\pi, 0 \leq z \leq 2\},$$

with density $\sigma(r, \theta, z) = 5e^{-r^2}$.

[Ans: $10\pi(1 - e^{-9})$]

2) Find the centroid of mass of the following solid cone.



[Ans: $(0,0,1)$]

Moment of Inertia for Solid in 3D

Moment of Inertia is also known as Second Moment of Mass

Continuous Case:

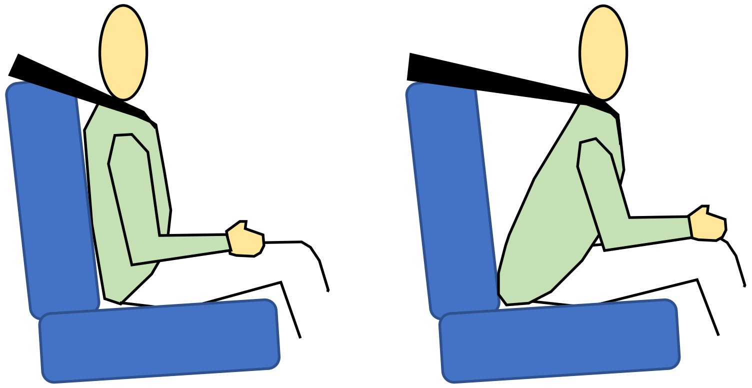
About x−axis: $I_x = \iiint_D (y^2 + z^2)\sigma(x, y, z)\, dV$

About y−axis: $I_y = \iiint_D (x^2 + z^2)\sigma(x, y, z)\, dV$

About z−axis: $I_z = \iiint_D (x^2 + y^2)\sigma(x, y, z)\, dV$

***What is the Inertia??**

Tendency of an object to **remain at its stage** or **to resist the change in motion**.



Due to inertia, passenger feels jerk when brake is applied

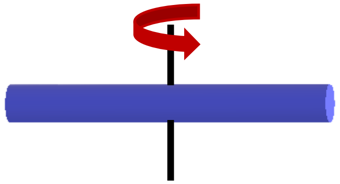
What is the Moment of Inertia??

Measurement of an object's **resistance to changes** in a **rotation direction**

Less force is needed to twist a rod when the mass concentrated around spin axis

Smaller moment of inertia

More force is needed to twirl a rod when the mass distributed away from the spin axis



Larger moment of inertia

An object with larger **moment of inertia** needs more force to change its movement from current stage / direction.

<u>Real Life Application:</u>
A **skater** can spin faster by reducing their average radius (holding arms at chest) to decrease their **moment of inertia.**

Example 9.9:

Find the moment of inertia about the x-axis of cylindrical solid bounded by $x^2 + y^2 = 1$, $z = 0$ and $z = 2$, with $\sigma(x, y, z) = z$. Next, find the moment of inertia about the z-axis and y-axis .

Solution:

$$I_x = \iiint_D (y^2 + z^2)\sigma(x, y, z)\, dV = \int_0^{2\pi}\int_0^1\int_0^2 (r^2 \sin^2\theta + z^2) z\, dz\, r dr d\theta = \frac{9\pi}{2}$$

$$I_z = \iiint_D (x^2 + y^2)\sigma(x, y, z)\, dV = \int_0^{2\pi}\int_0^1\int_0^2 (r^2) z\, dz\, r dr d\theta = \pi$$

$$I_y = \iiint_D (x^2 + z^2)\sigma(x, y, z)\, dV = \int_0^{2\pi}\int_0^1\int_0^2 (r^2 \cos^2\theta + z^2) z\, dz\, r dr d\theta = \frac{9\pi}{2}$$

Exercise 9.6:

Find the moment of inertia about the z-axis of cylindrical solid bounded by $x^2 + y^2 = 4$, $z = 0$ and $z = 3$, with $\sigma(x, y, z) = 5$. Next, find the moment of inertia about the x-axis.

[Ans: 120π, 240π]

Reference

1) W. Briggs, L. Cochran, B. Gillett and E. Schulz. (2018). Calculus: Early Transcendentals, Pearson.

2) Y. Mohammad Yusof, S. Baharun and R. Abdul Rahman. (2012). Multivariable Calculus for Independent Learners, Pearson Revised Second Edition.

THANK YOU