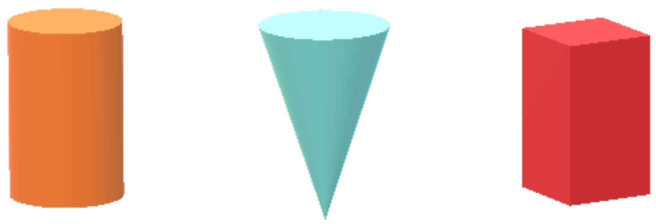
BEKG 2433
ENGINEERING MATHEMATICS 2

TRIPLE INTEGRAL IN CARTESIAN AND CYLINDRICAL COORDINATES



Ser Lee Loh
slloh@utem.edu.my

Lesson Outcomes

Upon completion of this lesson, students should be able to:

- evaluate triple integral over non-rectangular regions in Cartesian coordinates.
- evaluate triple integral in Cylindrical Coordinates.

Volume (Triple Integrals)

Triple integral can be used to solve for the volume of a three-dimensional solid, D. The formula of volume is

$$V = \iiint\limits_{D} f(x, y, z)\, dV \quad \text{where} \quad f(x, y, z) = 1$$

We may integrate in any of the orders; hence we will analyze the solid D to choose the order that is the easiest in terms of computation of the integral.

There are three different possibilities for a 3-Dimensional region.

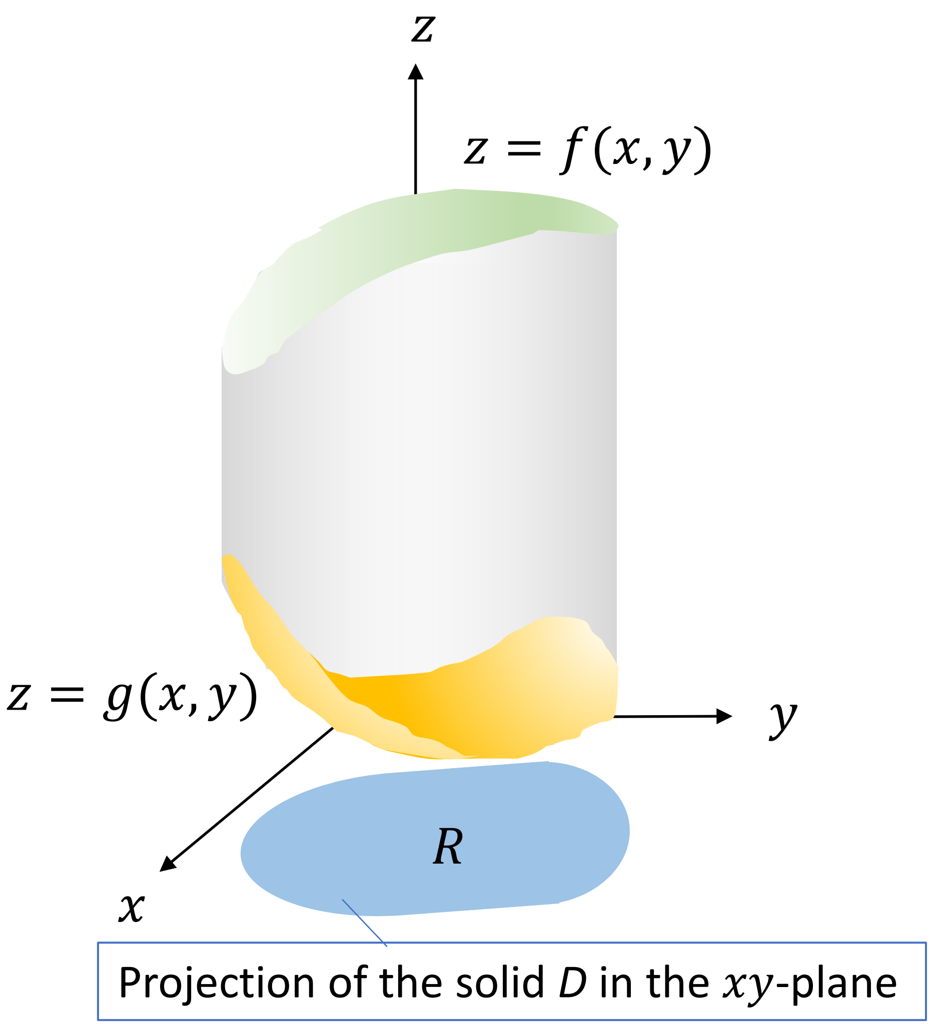
1st possibility:

The solid D as shown in the figure is defined as

$$D = \{(x, y, z): (x, y) \in R, g(x, y) \leq z \leq f(x, y)\}.$$

Hence,

$$\text{Volume} = \iint_R \int_{g(x,y)}^{f(x,y)} 1 \, dz \, dA$$



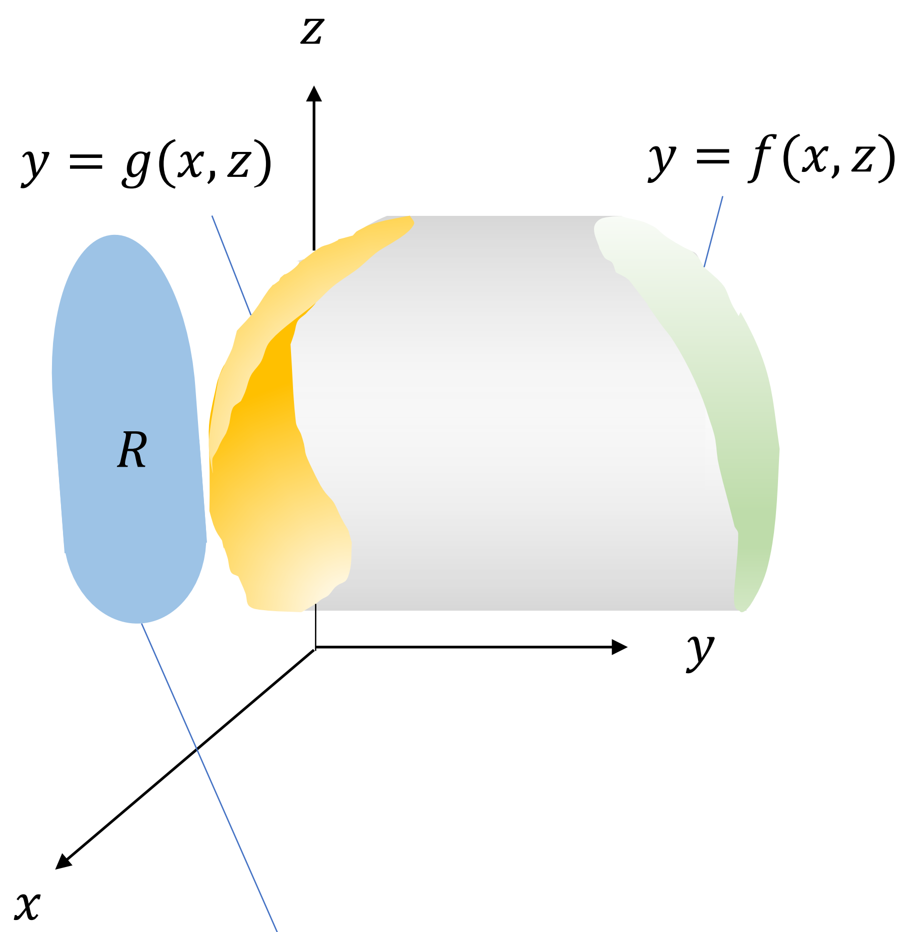
2nd possibility:

The solid D as shown in the figure is defined as

$$D = \{(x, y, z): (x, z) \in R, g(x, z) \leq y \leq f(x, z)\}.$$

Hence,

$$\text{Volume} = \iint_R \int_{g(x,z)}^{f(x,z)} 1\, dy\, dA$$



Projection of the solid D in the xz-plane

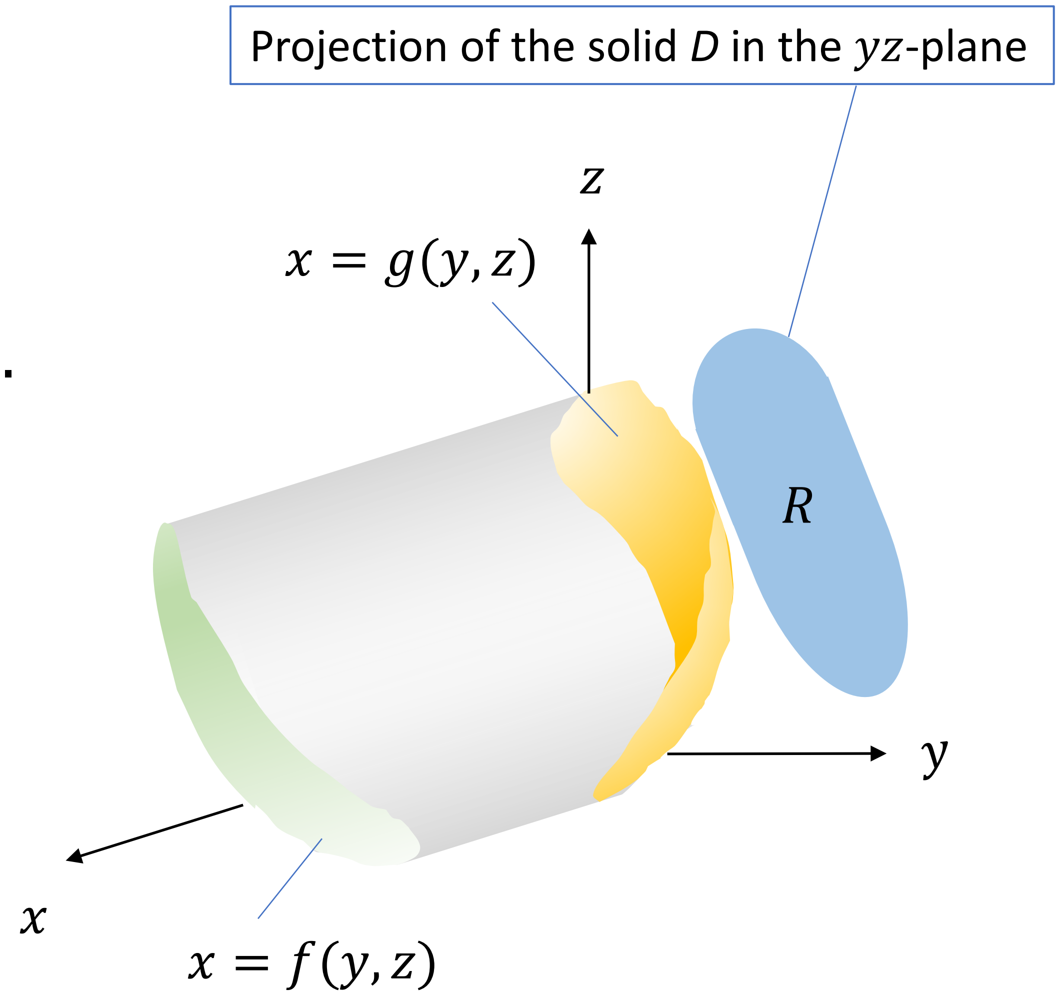
3rd possibility:

The solid D as shown in the figure is defined as

$$D = \{(x, y, z): (y, z) \in R, g(y, z) \leq x \leq f(y, z)\}.$$

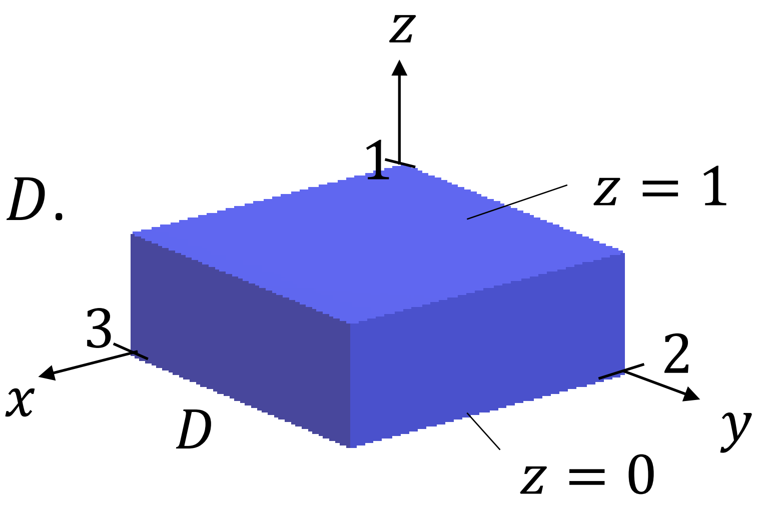
Hence,

$$\text{Volume} = \iint_R \int_{g(y,z)}^{f(y,z)} 1\, dx\, dA$$



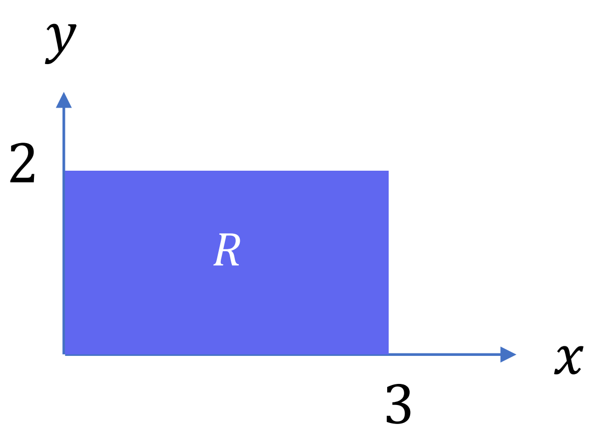
Example 8.1:

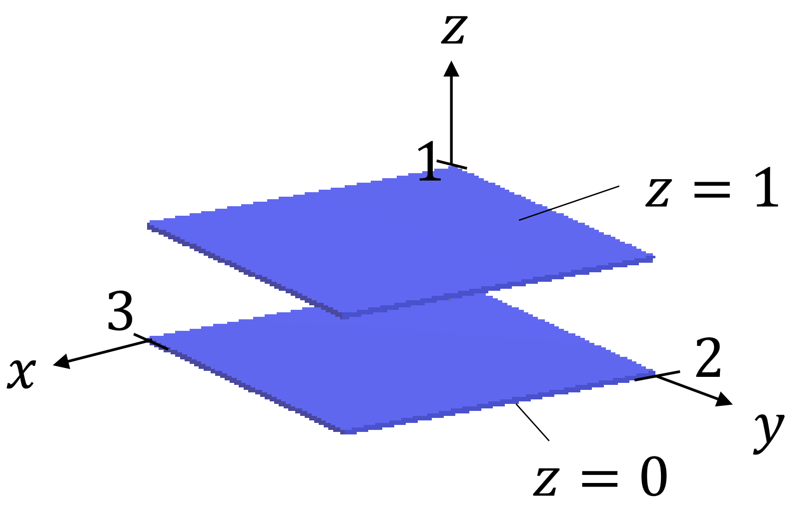
Find all the three possible integrals for the volume of the solid D.



Solution:

The first possible integral has the region lies in xy-plane and the solid is bounded by planes $z = 0$ and $z = 1$:

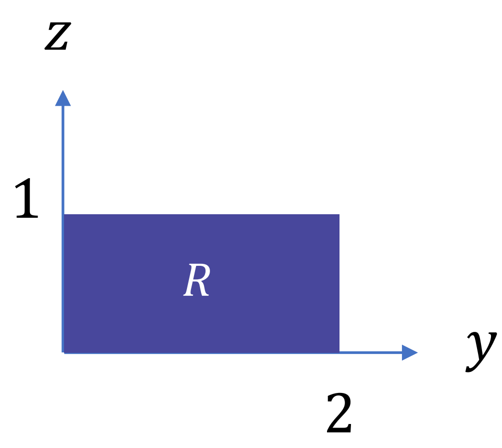


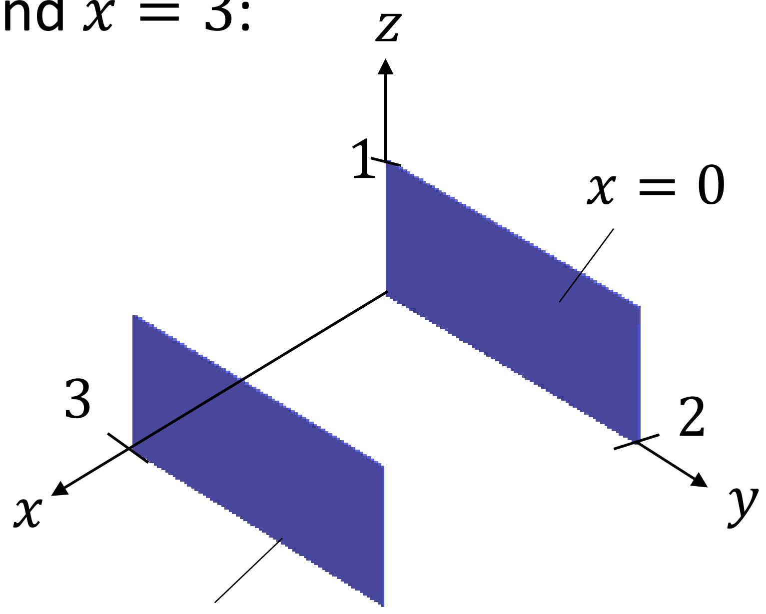


$$\text{Volume} = \int_0^3 \int_0^2 \int_0^1 1\, dz\,dy\,dx$$

Solution:

The second possible integral has the region lies in yz-plane and the solid is bounded by planes $x = 0$ and $x = 3$:

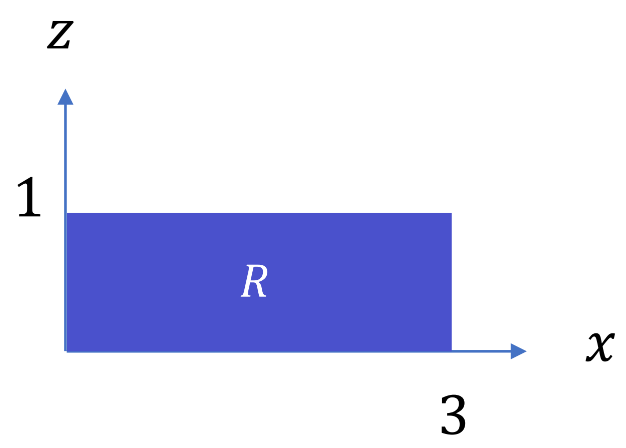


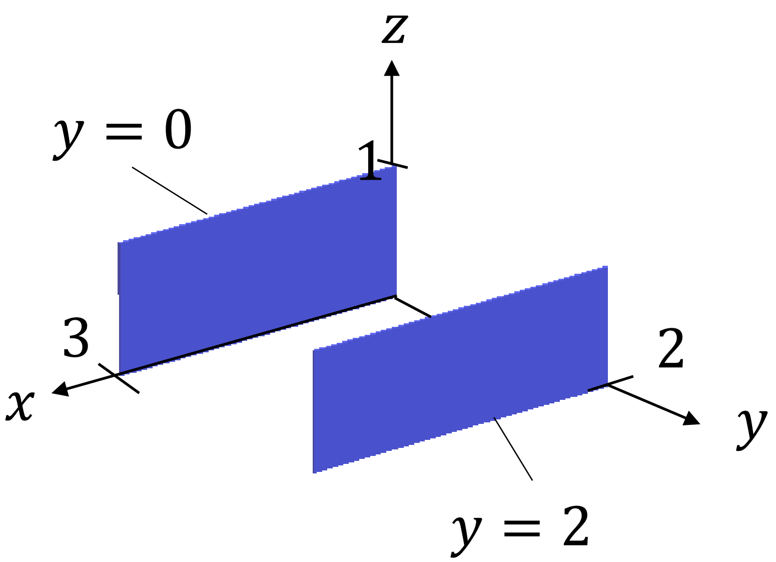


$$\text{Volume} = \int_0^2 \int_0^1 \int_0^3 1 \, dx\,dz\,dy$$

Solution:

The third possible integral has the region lies in xz-plane and the solid is bounded by planes $y = 0$ and $y = 2$:

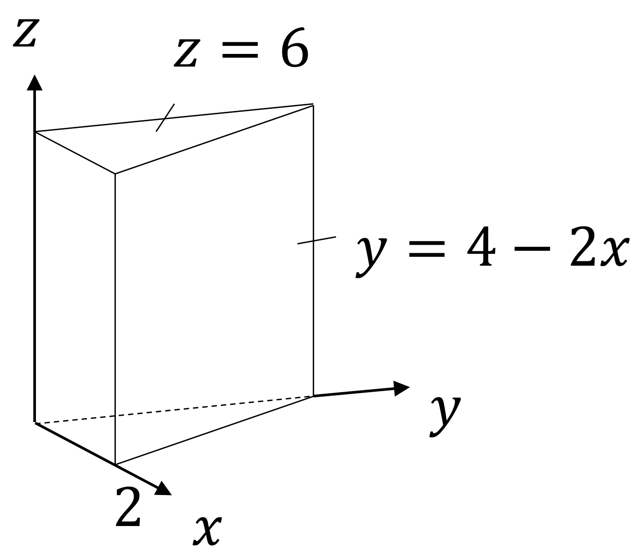




$$\text{Volume} = \int_0^3 \int_0^1 \int_0^2 1\, dy\,dz\,dx$$

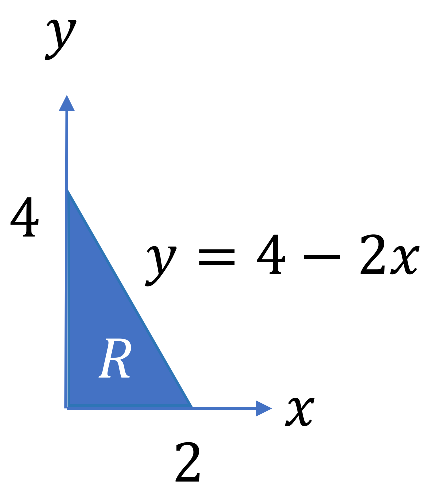
Example 8.2:

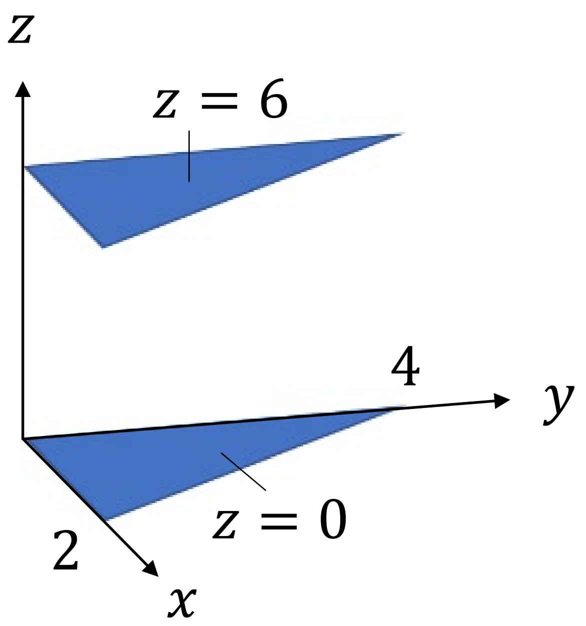
Find all the three possible integrals for the volume of the solid D.



Solution:

The first possible integral has the region lies in xy-plane and the solid is bounded by planes $z = 0$ and $z = 6$:



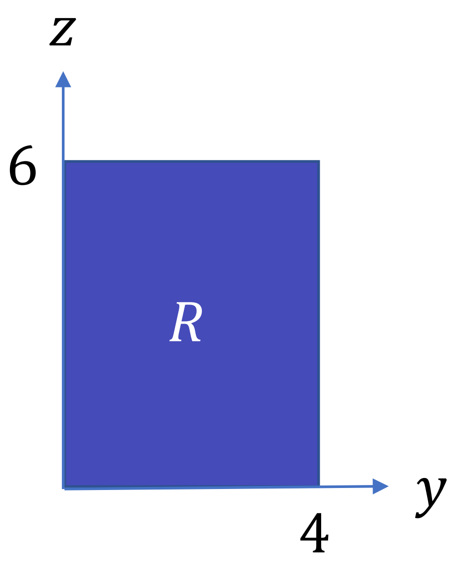


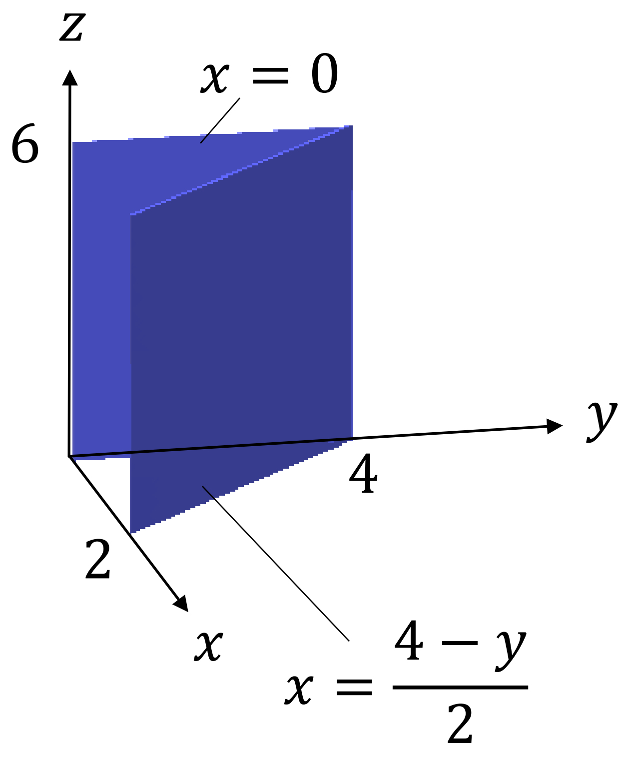
$$\text{Volume} = \int_0^2 \int_0^{4-2x} \int_0^6 1\, dz\, dy\, dx$$

Solution:

The second possible integral has the region

lies in yz-plane and the solid is bounded by planes $x = 0$ and $x = \frac{4-y}{2}$:





$$\text{Volume} = \int_0^4 \int_0^6 \int_0^{\frac{4-y}{2}} 1\, dx\, dz\, dy$$

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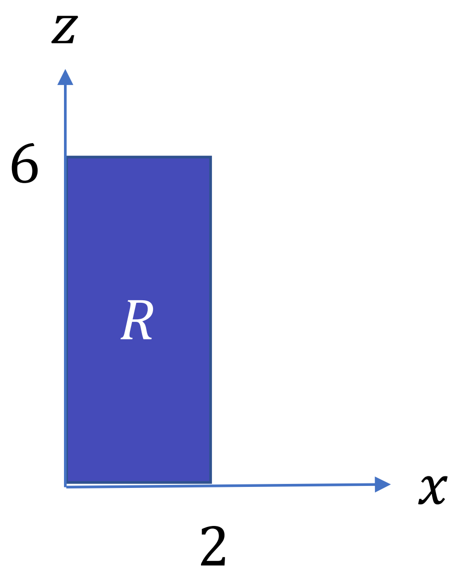
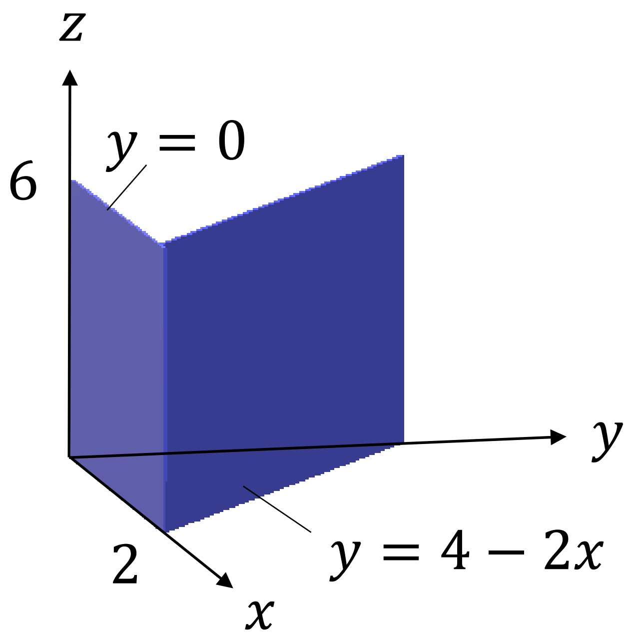
Solution:

The third possible integral has the region

lies in xz-plane and the solid is bounded by planes $y = 0$ and $y = 4 - 2x$:





$$\text{Volume} = \int_0^2 \int_0^6 \int_0^{4-2x} 1 \, dy\, dz\, dx$$

Exercise 8.1:

Write out the three possible triple integrals for the volume of the solid, and then find the volume.

1) Find the volume of the rectangular solid bounded by planes $z = 2$, $y = 1$, $x = 1$ and the three coordinate planes.

[Ans: $\int_0^1 \int_0^1 \int_0^2 1\, dzdydx$ or

$\int_0^1 \int_0^2 \int_0^1 1\, dxdzdy$ or

$\int_0^1 \int_0^2 \int_0^1 1\, dydzdx$;

$V = 2$]

Exercise 8.1:

2) Find the volume of the solid lying in the first octant and bounded by the graphs of $z = 4 - x^2$, $x = 0$, $y = 4$, $y = 0$ and $z = 0$.

[Ans: $\int_0^2 \int_0^4 \int_0^{4-x^2} 1\, dzdydx$ or $\int_0^4 \int_0^4 \int_0^{\sqrt{4-z}} 1\, dxdzdy$

or $\int_0^2 \int_0^{4-x^2} \int_0^4 1\, dydzdx\,; V = \frac{64}{3}$]

3) Find the volume of the tetrahedron bounded by the plane $2x + y + z = 2$ and the three coordinate planes.

[Ans: $\int_0^1 \int_0^{2-2x} \int_0^{2-2x-y} 1\, dzdydx$ or $\int_0^2 \int_0^{2-y} \int_0^{1-\frac{1}{2}y-\frac{1}{2}z} 1\, dxdzdy$

or $\int_0^1 \int_0^{2-2x} \int_0^{2-2x-z} 1\, dydzdx\,; V = \frac{2}{3}$]

Example 8.3:

Find the volume of the solid lying in the first octant bounded by $x + 2y + z = 4$.

Solution:

$$\int_0^4 \int_0^{2-\frac{1}{2}x} \int_0^{4-x-2y} 1 \; dzdydx$$

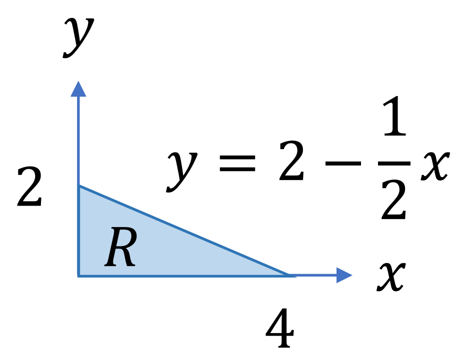
$$= \int_0^4 \int_0^{2-\frac{1}{2}x} 4 - x - 2y \; dydx$$

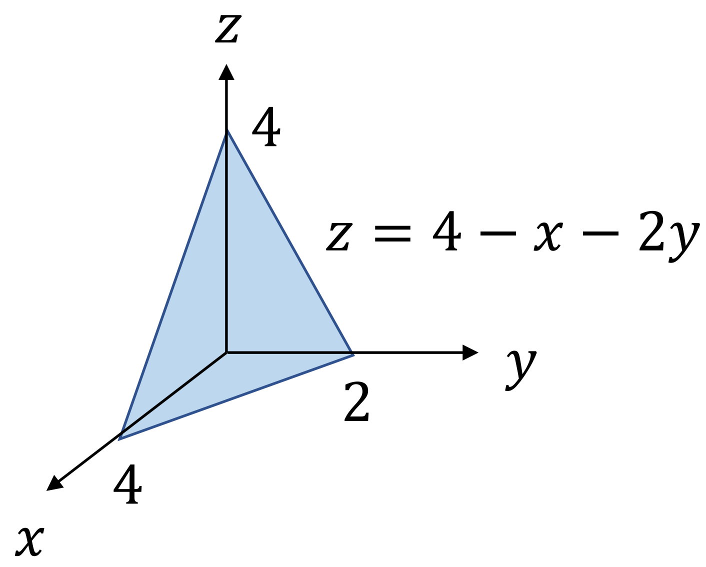
$$= \int_0^4 \left[(4-x)y - y^2\right]_0^{2-\frac{1}{2}x} \; dx$$

$$= \int_0^4 (4-x)\left(2 - \frac{1}{2}x\right) - \left(2 - \frac{1}{2}x\right)^2 \; dx$$

$$= \int_0^4 \frac{1}{4}x^2 - 2x + 4 \; dx$$

$$= \left[\frac{1}{12}x^3 - x^2 + 4x\right]_0^4 = \frac{16}{3}$$





Example 8.4:

Find the volume of the solid lying in the first octant bounded by $x + y + z = 2$ and $z = 1$.

Solution:

$$\int_0^1 \int_0^{1-x} \int_1^{2-x-y} 1 \; dzdydx$$

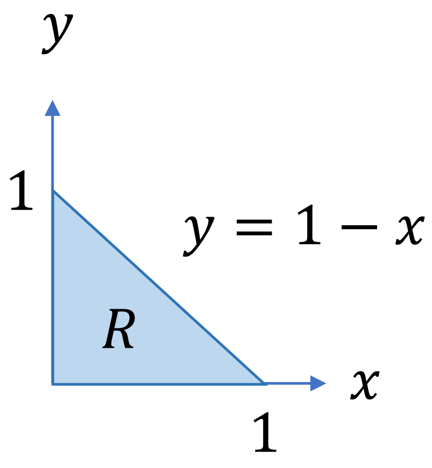
$$= \int_0^1 \int_0^{1-x} [z]_1^{2-x-y} \; dydx$$

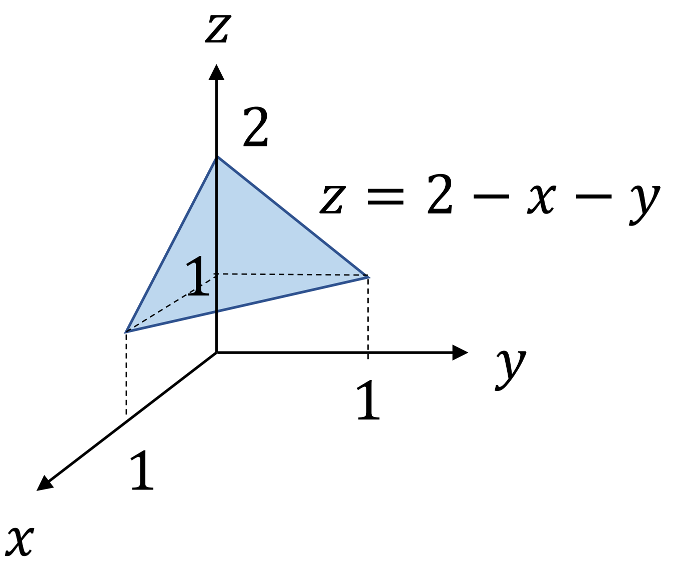
$$= \int_0^1 \int_0^{1-x} 1 - x - y \; dydx$$

$$= \int_0^1 \left[(1-x)y - \frac{y^2}{2}\right]_0^{1-x} \; dx$$

$$= \frac{1}{2}\int_0^1 1 - 2x + x^2 \; dx$$

$$= \frac{1}{2}\left[x - x^2 + \frac{x^3}{3}\right]_0^1 = \frac{1}{6}$$





Example 8.5:

Find the volume of the solid lying in the first octant bounded by the surface $z = 2 - y^2$, and plane $x = 3$.

Solution:

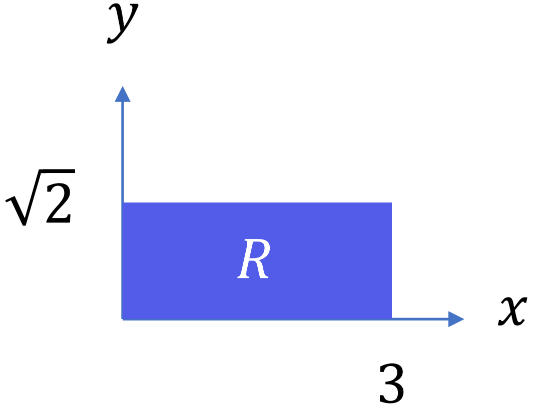
$$\int_0^3 \int_0^{\sqrt{2}} \int_0^{2-y^2} 1 \; dzdydx$$

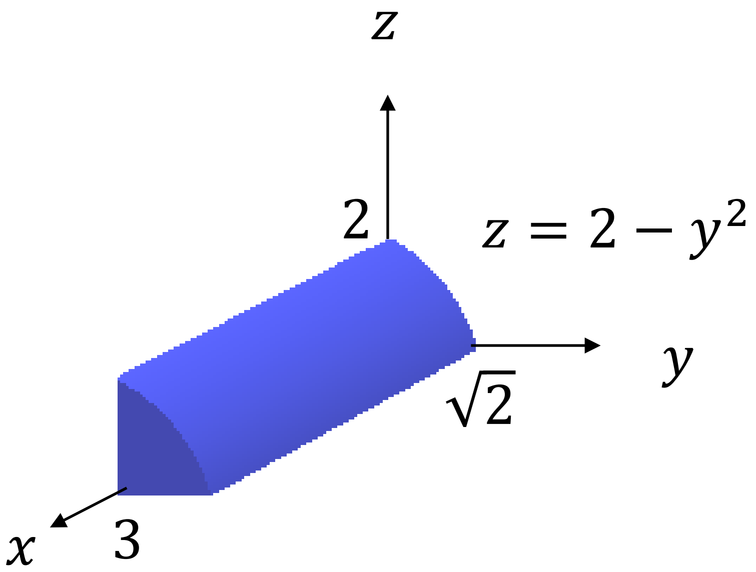
$$= \int_0^3 \int_0^{\sqrt{2}} 2 - y^2 \, dydx$$

$$= \int_0^3 \left[2y - \frac{y^3}{3}\right]_0^{\sqrt{2}} dx$$

$$= \int_0^3 2\sqrt{2} - \frac{2\sqrt{2}}{3} \, dx$$

$$= \left[\frac{4\sqrt{2}}{3}x\right]_0^3 = 4\sqrt{2}$$



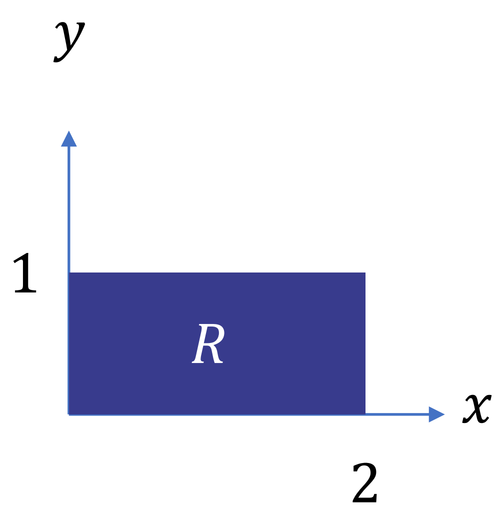


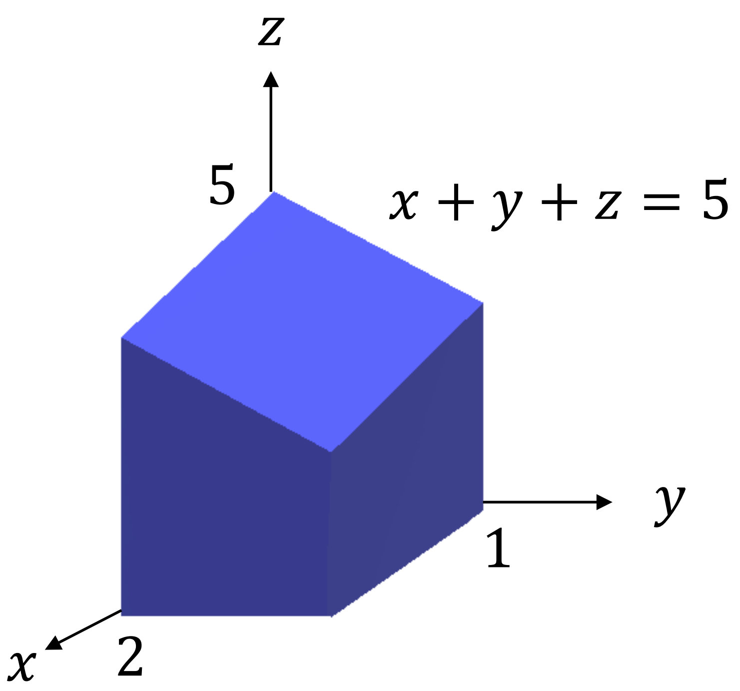
Example 8.6:

Find the volume of the solid lying in the first octant bounded by the planes $x + y + z = 5$, $x = 2$, and $y = 1$.

Solution:

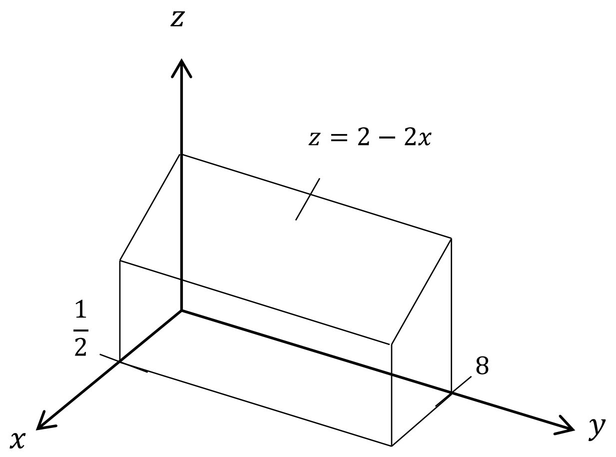
$$\int_0^2 \int_0^1 \int_0^{5-x-y} 1 \; dzdydx$$
$$= \int_0^2 \int_0^1 5 - x - y \, dydx$$
$$= \int_0^2 \left[(5-x)y - \frac{y^2}{2} \right]_0^1 dx$$
$$= \int_0^2 \frac{9}{2} - x \; dx$$
$$= \left[\frac{9}{2}x - \frac{x^2}{2} \right]_0^2 = 7$$





Exercise 8.2:

1) Find the volume of the following solid.



[Ans: 6]

2) Find the volume of the region bounded by $y = x^2$, $z = 3 - y$ and $z = 0$.

[Ans: $24\sqrt{3}/5$]

3) Find the volume of prism in the first octant bounded by $z = 2 - 4x$ and $y = 8$.

[Ans: 4]

Cylindrical Coordinates

The triple integral of f over solid D in cylindrical coordinates is

$$\iiint_D f(r,\theta,z)\,dV = \int_{\alpha}^{\beta}\int_{g(\theta)}^{h(\theta)}\int_{G(r\cos\theta, r\sin\theta)}^{H(r\cos\theta, r\sin\theta)} f(r,\theta,z)\,dz\,r\,dr\,d\theta$$

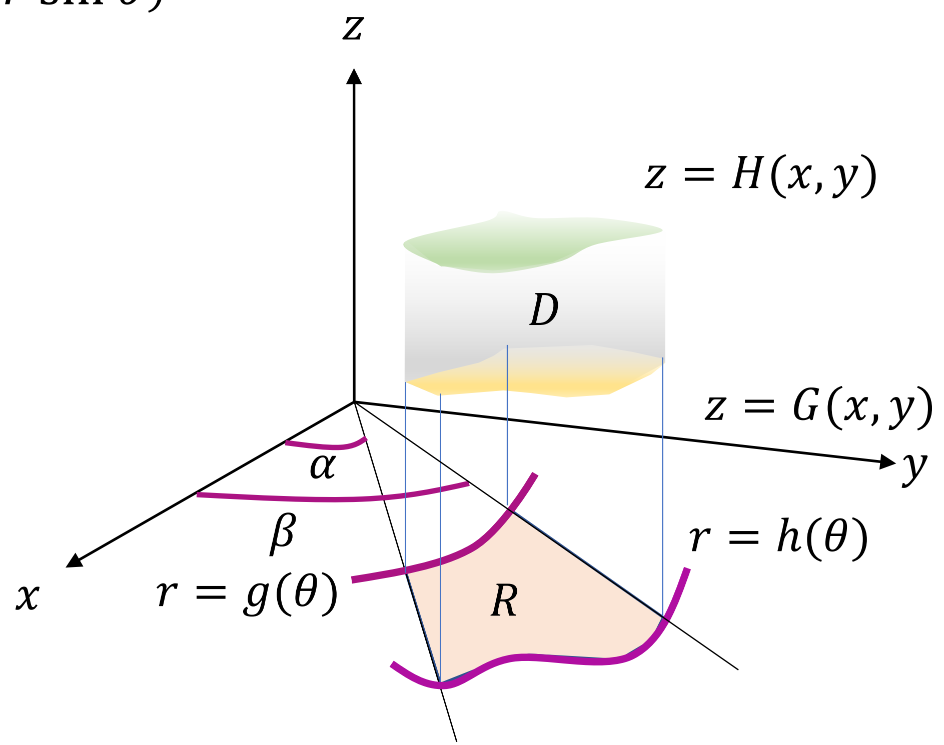
where from the Cartesian Coordinates,

$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$z = z$$

$$x^2 + y^2 = r^2$$



Example 8.7:

Evaluate $\int_0^3 \int_{-2}^{2} \int_0^{\sqrt{4-x^2}} \; x^2 + y^2 + 1 \; dy\, dx\, dz$ in cylindrical coordinates.

Solution:

$$\int_0^3 \int_0^{\pi} \int_0^{2} (r^2 + 1)\; r dr d\theta\; dz$$

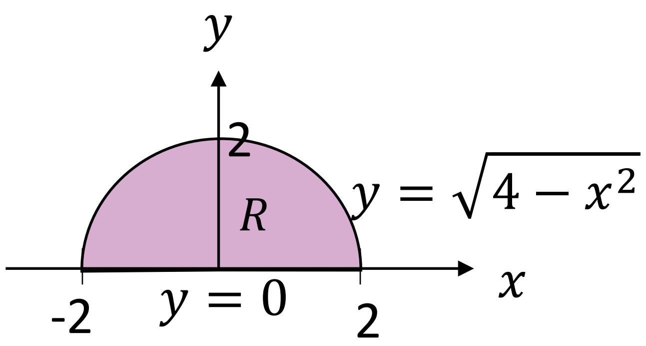
$$= \int_0^3 \int_0^{\pi} \int_0^{2} \; r^3 + r \; dr d\theta\; dz$$

$$= \int_0^3 \int_0^{\pi} \left[\frac{r^4}{4} + \frac{r^2}{2}\right]_0^2 \; d\theta\; dz$$

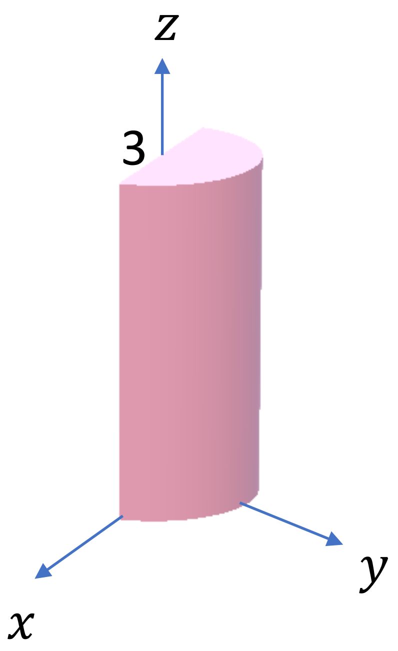
$$= \int_0^3 \int_0^{\pi} 6 \; d\theta\; dz$$

$$= \int_0^3 [6\theta]_0^{\pi} \; dz$$

$$= \int_0^3 6\pi \; dz = [6\pi z]_0^3 = 18\pi$$



$$y = \sqrt{4 - x^2}$$
$$y^2 = 4 - x^2$$
$$x^2 + y^2 = 4$$



Example 8.8:

Evaluate $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^2 \; 4xz \; dzdydx$ in cylindrical coordinates.

Solution:

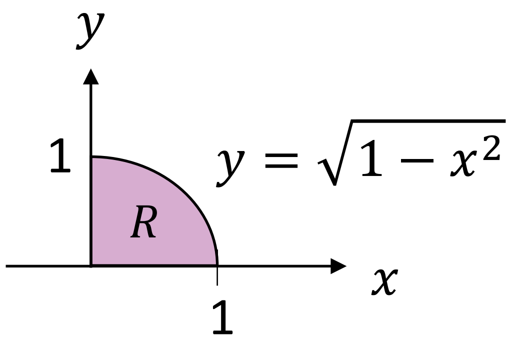
$$\int_0^{\frac{\pi}{2}} \int_0^1 \int_0^2 4(r\cos\theta)z \, dz \, r dr d\theta$$

$$= \int_0^{\frac{\pi}{2}} \int_0^1 4r^2 \cos\theta \left[\frac{z^2}{2}\right]_0^2 dr d\theta$$

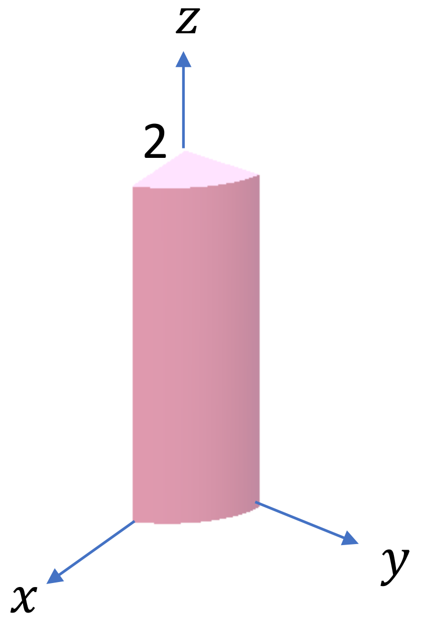
$$= 8 \int_0^{\frac{\pi}{2}} \int_0^1 r^2 \cos\theta \, dr d\theta$$

$$= 8 \int_0^{\frac{\pi}{2}} \cos\theta \left[\frac{r^3}{3}\right]_0^1 d\theta$$

$$= \frac{8}{3} \int_0^{\frac{\pi}{2}} \cos\theta \, d\theta = \frac{8}{3} [\sin\theta]_0^{\frac{\pi}{2}} = \frac{8}{3}$$



$$y = \sqrt{1-x^2}$$
$$y^2 = 1 - x^2$$
$$x^2 + y^2 = 1$$



Example 8.9:

Evaluate $\int_0^1 \int_{-2}^{2} \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} 1 \; dzdydx$ in cylindrical coordinates.

Solution:

$$\int_0^1 \int_0^{2\pi} \int_0^2 1\, r dr d\theta \, dx$$

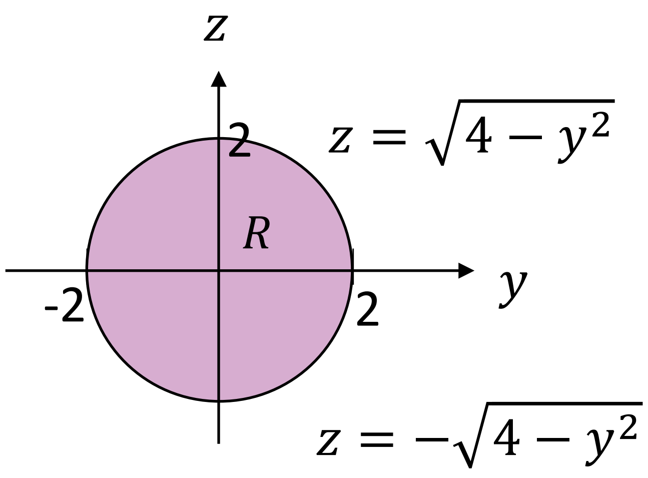
$$= \int_0^1 \int_0^{2\pi} \left[\frac{r^2}{2}\right]_0^2 d\theta \, dx$$

$$= \int_0^1 \int_0^{2\pi} 2 \, d\theta \, dx$$

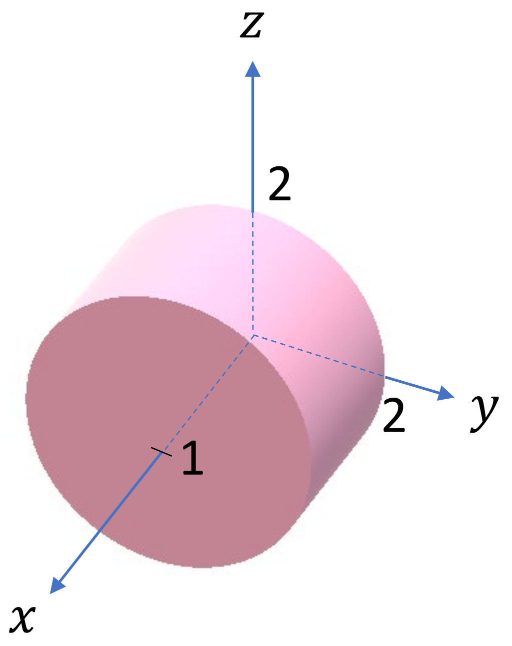
$$= \int_0^1 [2\theta]_0^{2\pi} \; dx$$

$$= \int_0^1 4\pi \; dx$$

$$= [4\pi x]_0^1 = 4\pi$$



$$z = \pm\sqrt{4 - y^2}$$
$$z^2 = 4 - y^2$$
$$y^2 + z^2 = 4$$



Exercise 8.3:

1) Evaluate $\int_0^1 \int_{-3}^3 \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} x^2 + y^2 \; dy\, dx\, dz.$

[Ans: $\frac{81\pi}{2}$]

2) Evaluate $\int_0^3 \int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} \int_0^{9-\sqrt{x^2+y^2}} dz\, dx\, dy.$

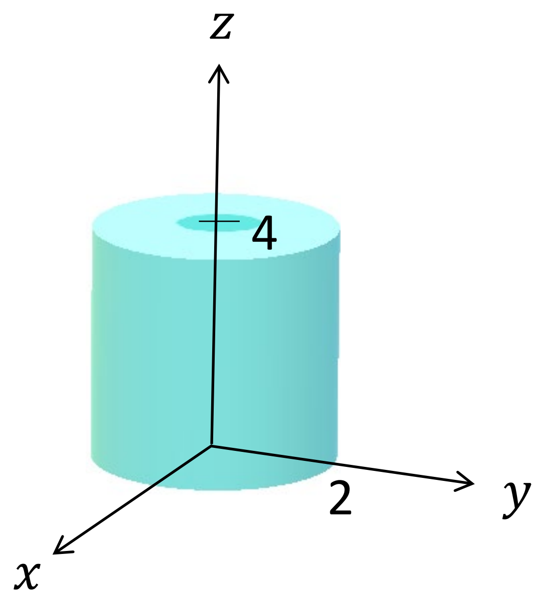
[Ans: $\frac{63\pi}{2}$]

3) Evaluate $\int_{-1}^1 \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{x^2+y^2}^{2-x^2-y^2} (x^2 + y^2)^{\frac{3}{2}} \; dz\, dy\, dx.$

[Ans: $\frac{8\pi}{35}$]

Example 8.10:

Find the volume of the given cylinder where the inner ring has a radius of 1.



Solution:

$$\int_0^4 \int_0^{2\pi} \int_1^2 1\ r dr d\theta\ dz$$

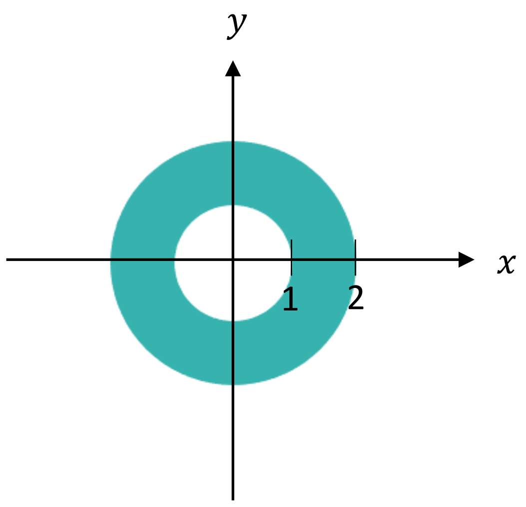
$$= \int_0^4 \int_0^{2\pi} \left[\frac{r^2}{2}\right]_1^2\ d\theta\ dz$$

$$= \int_0^4 \int_0^{2\pi} \frac{3}{2}\ d\theta\ dz$$

$$= \int_0^4 \left[\frac{3}{2}\theta\right]_0^{2\pi}\ dz$$

$$= \int_0^4 3\pi\ dz$$

$$= [3\pi z]_0^4 = 12\pi$$



Example 8.11:

Find the volume of the solid where the cylinder $x^2 + y^2 = 4$ is bounded by the planes $z = 3 - x$ and $z = x - 3$ as shown in diagram.

Solution:

$$\int_0^{2\pi}\int_0^2\int_{r\cos\theta-3}^{3-r\cos\theta} 1\ dz\, r dr d\theta$$

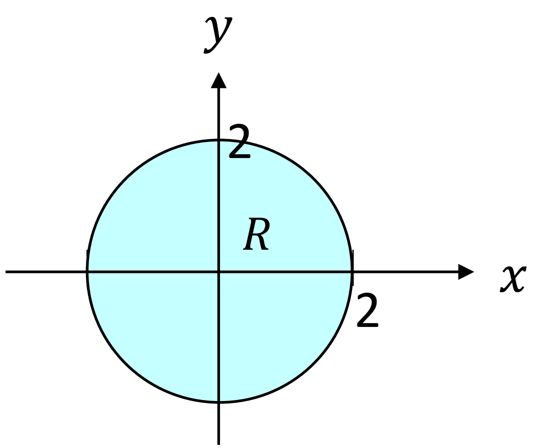
$$= \int_0^{2\pi}\int_0^2 [z]_{r\cos\theta-3}^{3-r\cos\theta}\ r dr d\theta$$

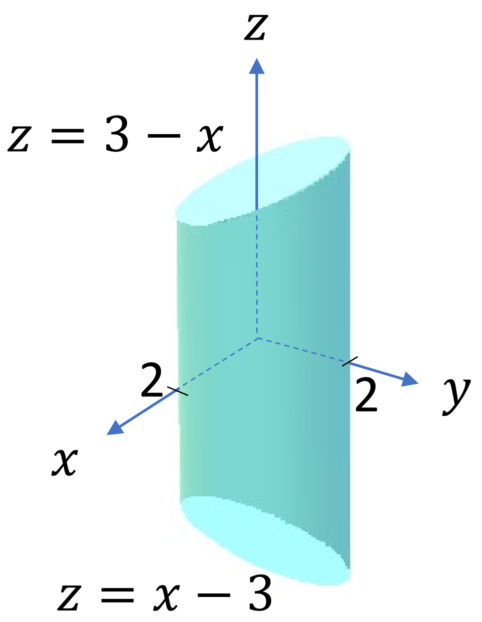
$$= \int_0^{2\pi}\int_0^2 6r - 2r^2\cos\theta\ dr d\theta$$

$$= \int_0^{2\pi}\left[3r^2 - \frac{2r^3}{3}\cos\theta\right]_0^2\ d\theta$$

$$= \int_0^{2\pi} 12 - \frac{16}{3}\cos\theta\ d\theta$$

$$= \left[12\theta - \frac{16}{3}\sin\theta\right]_0^{2\pi} = 24\pi$$





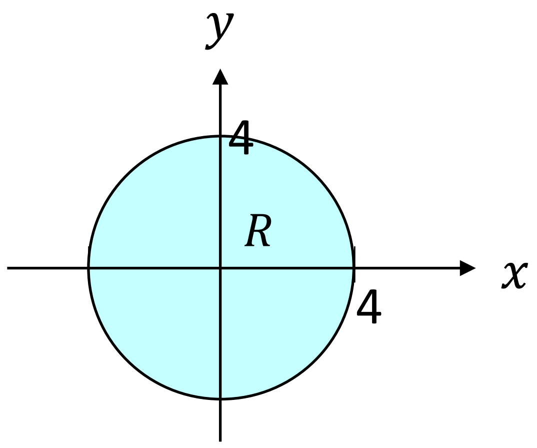
Example 8.12:

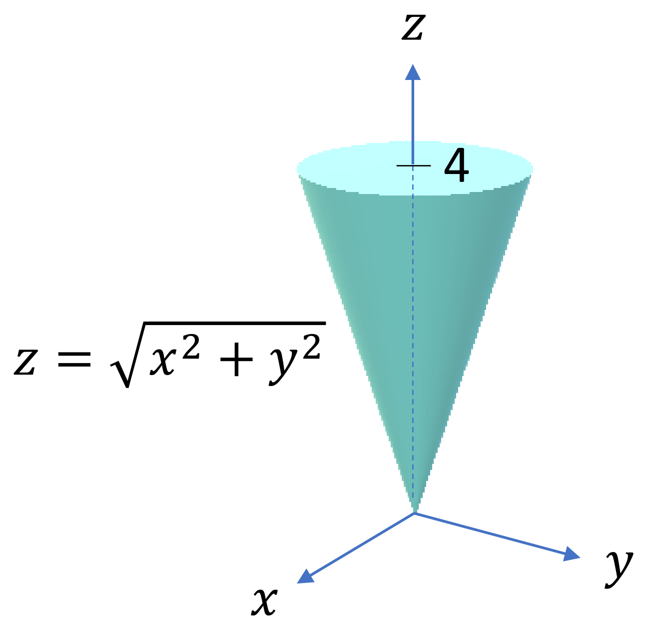
Find the volume of the solid where the region is bounded below by the cone $z=\sqrt{x^2+y^2}$ and bounded above by the plane $z=4$.

Solution:

$$\int_0^{2\pi}\int_0^4\int_r^4 1\ dz\,rdrd\theta$$
$$=\int_0^{2\pi}\int_0^4[z]_r^4\ rdrd\theta$$
$$=\int_0^{2\pi}\int_0^4 4r-r^2\ drd\theta$$
$$=\int_0^{2\pi}\left[2r^2-\frac{r^3}{3}\right]_0^4\ d\theta$$
$$=\int_0^{2\pi}\frac{32}{3}\ d\theta$$
$$=\left[\frac{32}{3}\theta\right]_0^{2\pi}=\frac{64}{3}\pi$$

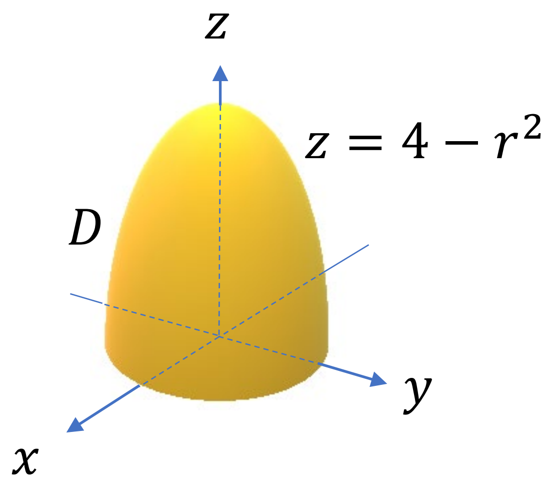
$$z=\sqrt{x^2+y^2}=\sqrt{r^2}=r$$





Exercise 8.4:

1) Find the volume of the following solid D. [Ans:8π]



2) Find the volume of the solid D between the cone $z = \sqrt{x^2 + y^2}$ and the inverted paraboloid $z = 12 - x^2 - y^2$.

[Ans: $\frac{99\pi}{2}$]

3) Find the volume of the solid bounded above by the sphere $x^2 + y^2 + z^2 = 9$, the cylinder $x^2 + y^2 = 4$ and bounded below by $z = 1$.

[Ans: $2\pi\left(7 - \frac{5}{3}\sqrt{5}\right)$]

Reference

1) W. Briggs, L. Cochran, B. Gillett and E. Schulz. (2018). Calculus: Early Transcendentals, Pearson.

2) Y. Mohammad Yusof, S. Baharun and R. Abdul Rahman. (2012). Multivariable Calculus for Independent Learners, Pearson Revised Second Edition.

THANK YOU