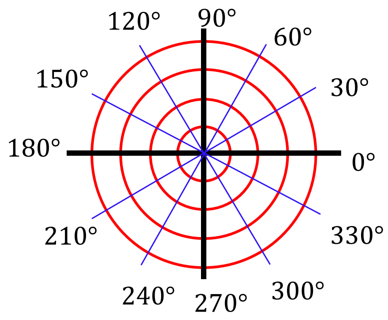
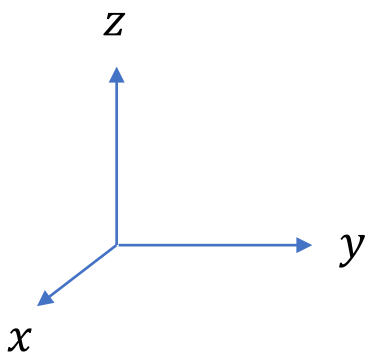
BEKG 2433
ENGINEERING MATHEMATICS 2

DOUBLE INTEGRAL IN POLAR COORDINATES & TRIPLE INTEGRAL IN CARTESIAN COORDINATES





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Lesson Outcomes

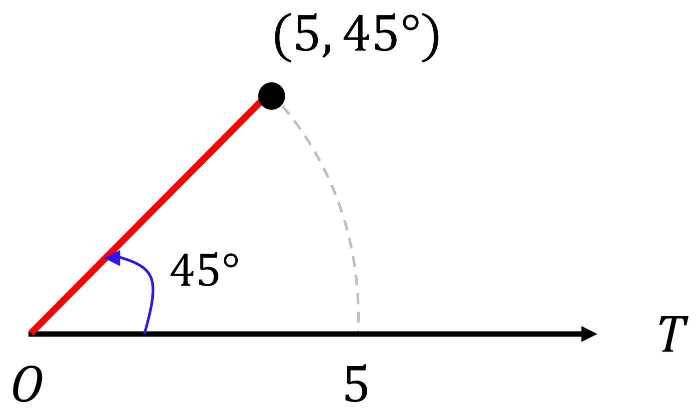
Upon completion of this lesson, students should be able to:

- evaluate double integral in Polar coordinates.
- evaluate triple integral in Cartesian coordinates.

Double Integral in Polar Coordinates

Polar Coordinate System

Polar Coordinate System is a two-dimensional coordinate system where the points (r, θ) are determined by a distance from a reference point, known as pole (analogous to the origin of a Cartesian Coordinate System), and an angle from a reference direction, known as polar axis. The angles are expressed either in degrees or radians.

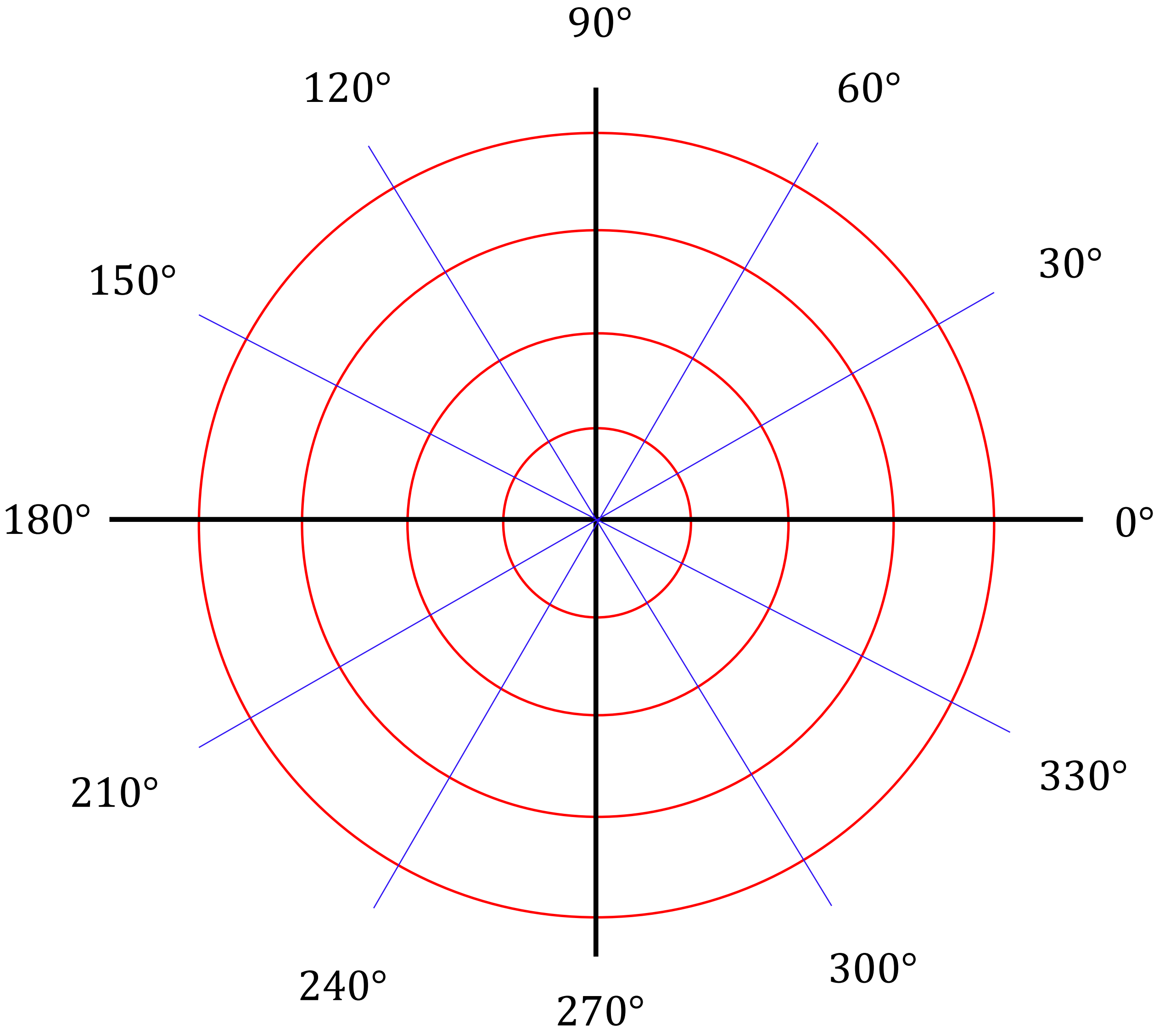


This figure shows a point $(5, 45°)$ in Polar Coordinate System with Pole O and polar axis T.

Polar grid

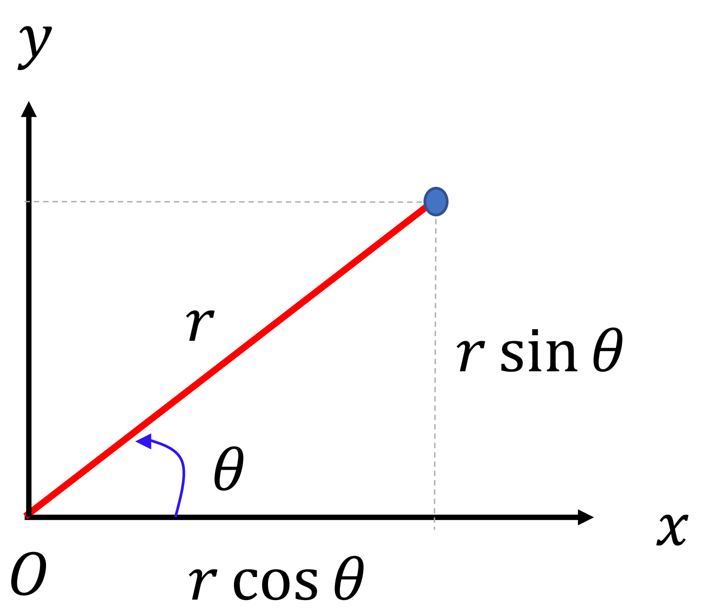
This figure shows a polar grid with several angles, where the angle increases in counterclockwise direction.

Similar to Cartesian Coordinates where a Cartesian curve is plotted on rectilinear axes, a polar plot is plotted on radial axes as shown in Polar Grid.



Cartesian to Polar Coordinates

A function or point in Cartesian Coordinate System can be converted into Polar Coordinate System through a series of formulas.



From the diagram, we have

$$x = r\cos\theta \text{ and } y = r\sin\theta$$

Hence,

$$\begin{aligned} x^2 + y^2 &= (r\cos\theta)^2 + (r\sin\theta)^2 \\ &= r^2\cos^2\theta + r^2\sin^2\theta \\ &= r^2(\cos^2\theta + \sin^2\theta) \\ &= r^2(1) \\ &= r^2 \end{aligned}$$

Example 7.1:

Convert $x^2 + y^2 + 3x + 2y = 1$ into polar coordinate system.

Solution:

Substitute $x = r\cos\theta, y = r\sin\theta$ and $x^2 + y^2 = r^2$ into the equation:

$$x^2 + y^2 + 3x + 2y = 1$$
$$r^2 + 3r\cos\theta + 2r\sin\theta = 1$$

Example 7.2:

Convert $2 + x^3 = 3xy$ into polar coordinate system.

Solution:

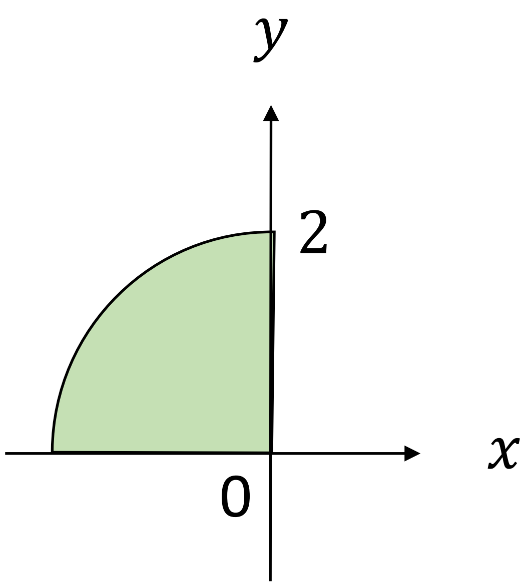
Substitute $x = r\cos\theta$, $y = r\sin\theta$ and $x^2 + y^2 = r^2$ into the equation:

$$2 + x^3 = 3xy$$
$$2 + (r\cos\theta)^3 = 3(r\cos\theta)(r\sin\theta)$$
$$2 + r^3\cos^3\theta = 3r^2\sin\theta\cos\theta$$

Example 7.3:

Sketch the region bounded by $0 \leq r \leq 2$ and $\frac{\pi}{2} \leq \theta \leq \pi$.

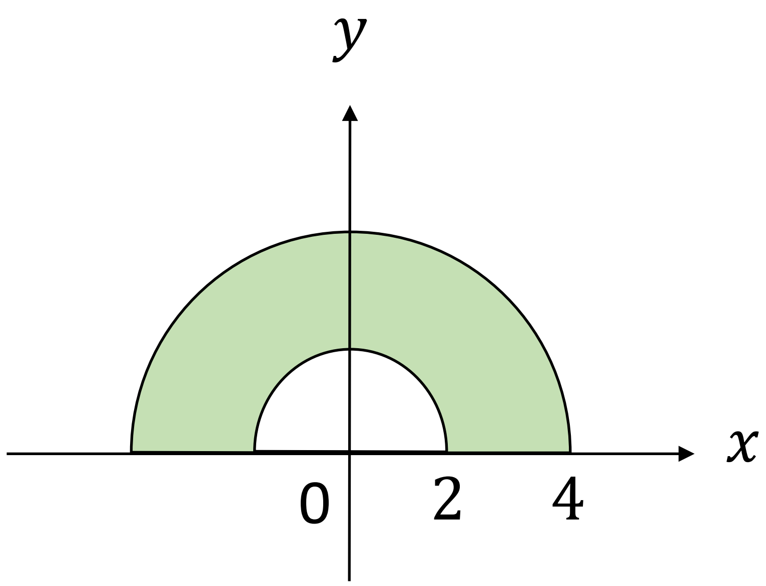
Solution:



Example 7.4:

Sketch the region bounded by $2 \leq r \leq 4$ and $0 \leq \theta \leq \pi$.

Solution:



Example 7.5:

Convert the region of $0 \leq x \leq 3$ and $0 \leq y \leq \sqrt{9-x^2}$ from Cartesian into Polar Coordinate System.

Solution:

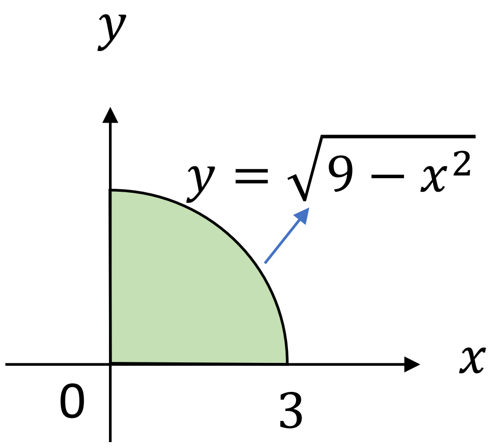
From $y = \sqrt{9-x^2}$,

we know that

$$y^2 = 9 - x^2$$

which is

$$x^2 + y^2 = 9$$



Hence,

$$0 \leq r \leq 3$$

and

$$0 \leq \theta \leq \frac{\pi}{2}$$

Example 7.6:

Convert the region of $0 \leq x \leq 2$ and $-\sqrt{4-x^2} \leq y \leq \sqrt{4-x^2}$ from Cartesian into Polar Coordinate System.

Solution:

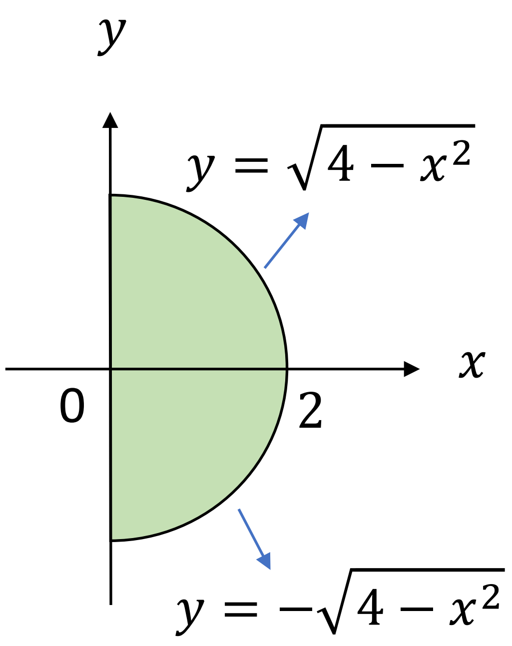
From $y = \sqrt{4-x^2}$,

we know that

$$y^2 = 4 - x^2$$

which is

$$x^2 + y^2 = 4$$



Hence,

$$0 \leq r \leq 2$$

and

$$-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$$

Example 7.7:

Convert the region of $-1 \leq x \leq 1$ and $0 \leq y \leq \sqrt{1-x^2}$ from Cartesian into Polar Coordinate System.

Solution:

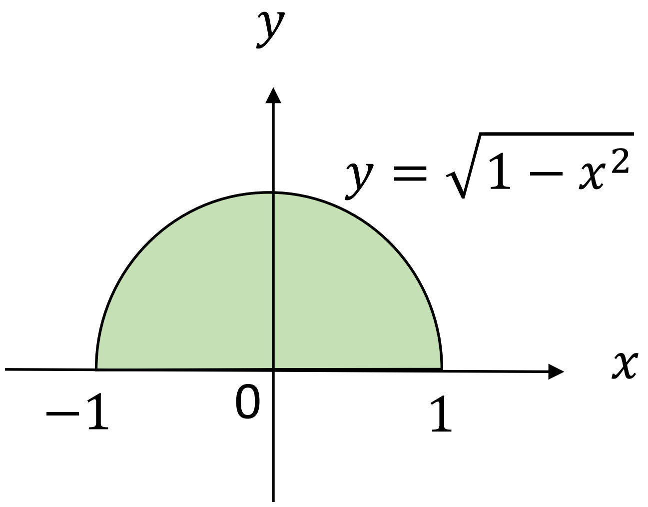
From $y = \sqrt{1-x^2}$,

we know that

$$y^2 = 1 - x^2$$

which is

$$x^2 + y^2 = 1$$



Hence,

$$0 \leq r \leq 1$$

and

$$0 \leq \theta \leq \pi$$

Exercise 7.1:

Convert each of the following regions from Cartesian into Polar Coordinate System.

1) $-3 \leq x \leq 3$ and $-\sqrt{9-x^2} \leq y \leq \sqrt{9-x^2}$
2) $-2 \leq x \leq 2$ and $-\sqrt{4-x^2} \leq y \leq 0$
3) $-1 \leq x \leq 0$ and $-\sqrt{1-x^2} \leq y \leq 0$

[Ans: $0 \leq r \leq 3, 0 \leq \theta \leq 2\pi$;
$0 \leq r \leq 2, \pi \leq \theta \leq 2\pi$;
$0 \leq r \leq 1, \pi \leq \theta \leq \frac{3\pi}{2}$]

Polar Rectangular Regions of Integration

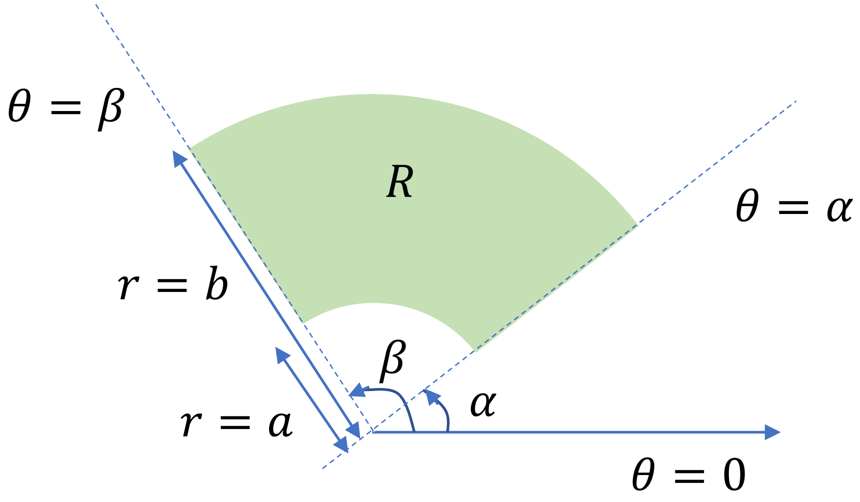
A double integral for a continuous function in Polar Coordinates is defined by dividing the region into sub-polar rectangles with the shape as shown in figure below.

Theorem:

Let f be continuous on the region in the xy-plane.

Given $R = \{(r,\theta): a \le r \le b,\ \alpha \le \theta \le \beta\}$,

$$\iint_R f(x,y)\,dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta.$$



General Polar Regions of Integration

Theorem:

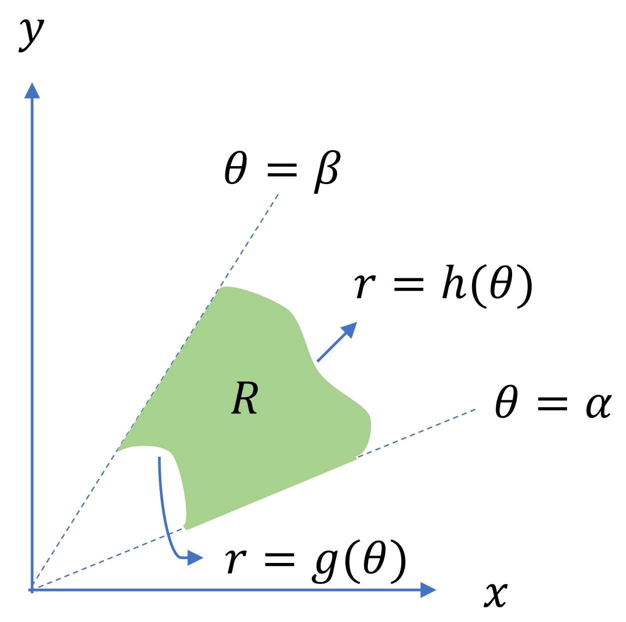
Let f be continuous on the region in the xy-plane

$R = \{(r, \theta): 0 \leq g(\theta) \leq r \leq h(\theta),\ \alpha \leq \theta \leq \beta\}$, where

$\beta - \alpha \leq 2\pi$. Then,

$$\iint_R f(x, y)\, dA = \int_\alpha^\beta \int_{g(\theta)}^{h(\theta)} f(r\cos\theta, r\sin\theta)\, r\, dr\, d\theta.$$

$$x = r\cos\theta, \quad y = r\sin\theta, \quad x^2 + y^2 = r^2$$



Area in Polar Coordinate

Example 7.8:

Evaluate $\iint_R x^2 + y^2 \, dA$, where

$$R = \{(x, y) \colon x^2 + y^2 \leq 4, -2 \leq x \leq 2, y \geq 0\}.$$

Solution:

$$\iint_R x^2 + y^2 \, dA = \int_0^{\pi} \int_0^2 r^2 \, r dr d\theta$$

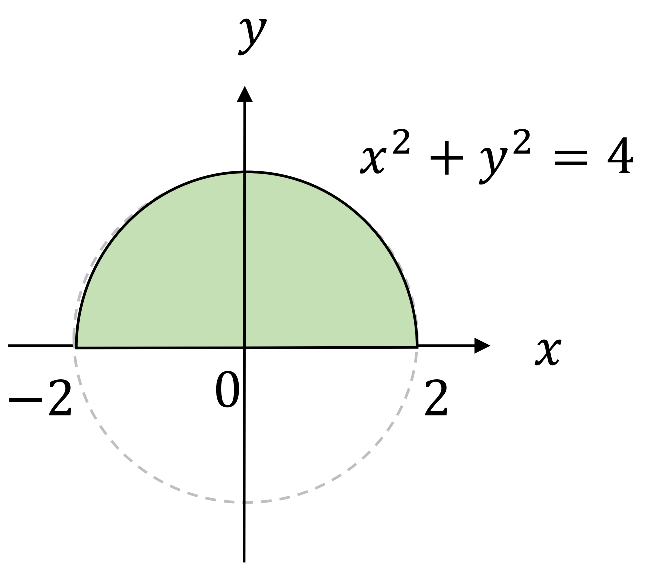
$$= \int_0^{\pi} \int_0^2 r^3 \, dr d\theta$$

$$= \int_0^{\pi} \left[\frac{1}{4} r^4\right]_0^2 d\theta$$

$$= \int_0^{\pi} 4 \, d\theta$$

$$= [4\theta]_0^{\pi}$$

$$= 4\pi$$



Example 7.9:

Evaluate $\iint_R \; 4 + x\, dA$, where

$$R = \{(x, y): 1 \le x^2 + y^2 \le 4, x \ge 0\}.$$

Solution:

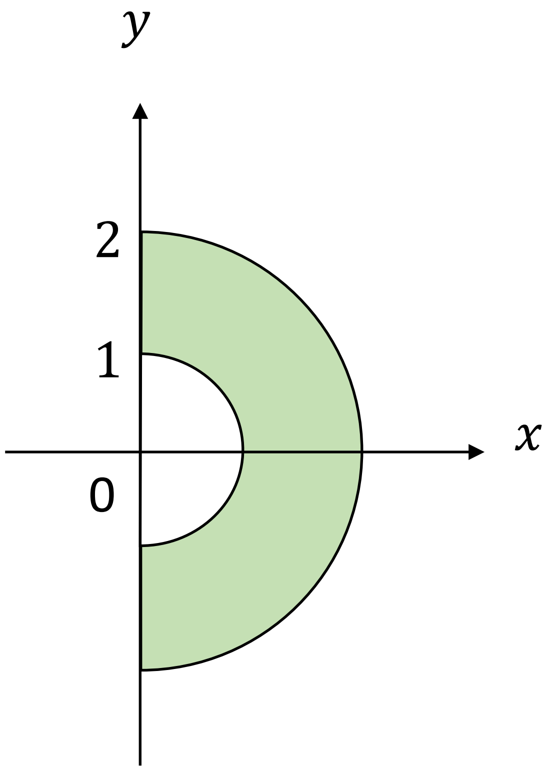
$$\iint_R \; 4 + x\, dA$$

$$= \int_{-\pi/2}^{\pi/2} \int_1^2 \; (4 + r\cos\theta)\, r dr d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \int_1^2 4r + r^2 \cos\theta \; dr d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \left[2r^2 + \frac{r^3}{3}\cos\theta\right]_1^2 d\theta$$

$$= \int_{-\pi/2}^{\pi/2} 6 + \frac{7}{3}\cos\theta \; d\theta = \left[6\theta + \frac{7}{3}\sin\theta\right]_{-\pi/2}^{\pi/2} = 6\pi + \frac{14}{3}$$



Example 7.10:

Evaluate $\int_0^1 \int_0^{\sqrt{1-x^2}} 9 - x^2 - y^2 \, dydx.$

Solution:

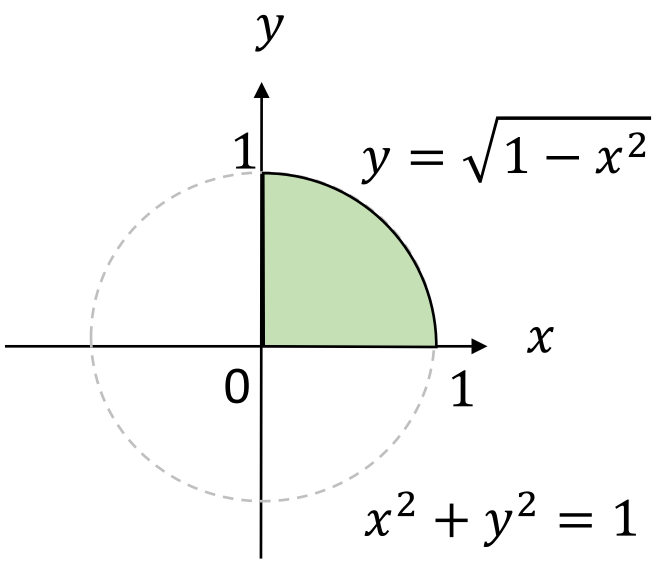
$$\int_0^1 \int_0^{\sqrt{1-x^2}} 9 - x^2 - y^2 \, dydx = \int_0^{\pi/2} \int_0^1 (9 - r^2)\, r dr d\theta$$

$$= \int_0^{\pi/2} \int_0^1 9r - r^3 \, dr d\theta$$

$$= \int_0^{\pi/2} \left[\frac{9}{2} r^2 - \frac{1}{4} r^4\right]_0^1 d\theta$$

$$= \int_0^{\pi/2} \frac{17}{4} d\theta$$

$$= \left[\frac{17}{4}\theta\right]_0^{\pi/2} = \frac{17\pi}{8}$$



Example 7.11:

Evaluate $\int_{-3}^{3}\int_{-\sqrt{9-y^2}}^{0} x + y\,dxdy.$

Solution:

$$\int_{-3}^{3}\int_{-\sqrt{9-y^2}}^{0} x + y\,dxdy$$

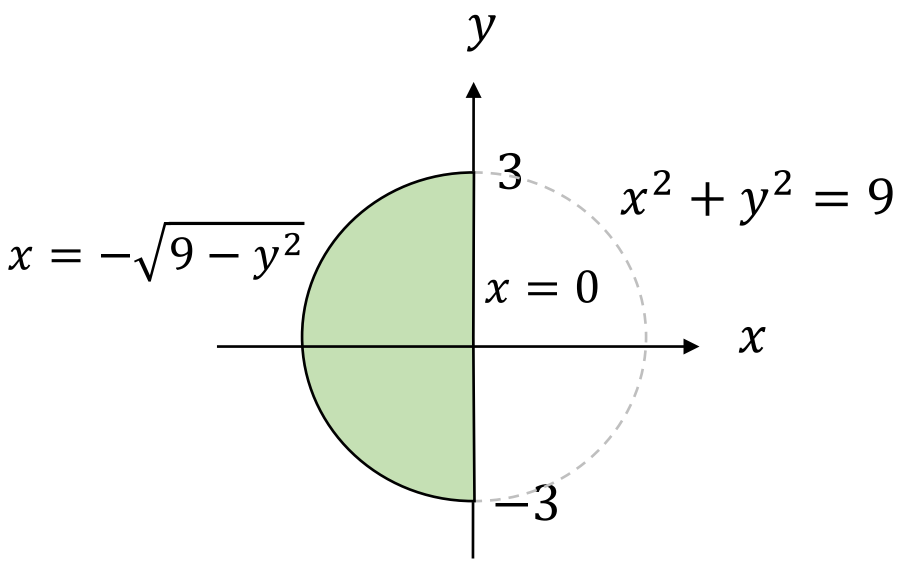
$$= \int_{\pi/2}^{3\pi/2}\int_{0}^{3} r(\cos\theta + \sin\theta)\,rdrd\theta$$

$$= \int_{\pi/2}^{3\pi/2}\int_{0}^{3} r^2(\cos\theta + \sin\theta)\,drd\theta$$

$$= \int_{\pi/2}^{3\pi/2}\left[\frac{r^3}{3}\right]_0^3(\cos\theta + \sin\theta)\,d\theta$$

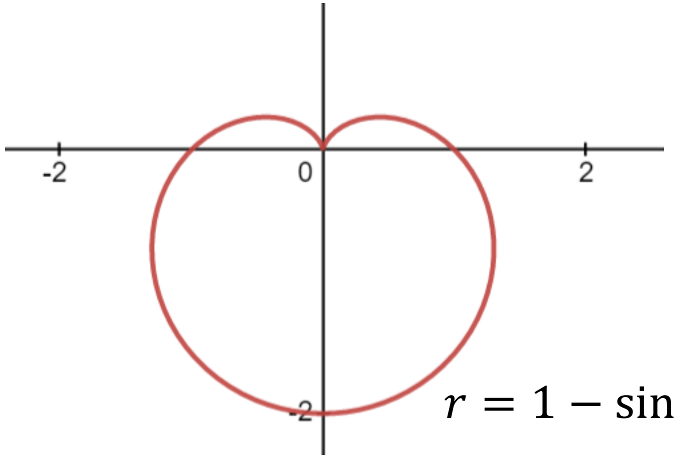
$$= \int_{\pi/2}^{3\pi/2} 9(\cos\theta + \sin\theta)\,d\theta$$

$$= 9[\sin\theta - \cos\theta]_{\pi/2}^{3\pi/2} = -18$$



Example 7.12:

Find the area inside the curve defined by $r = 1 - \sin\theta$.



Solution:

$$\int_0^{2\pi}\int_0^{1-\sin\theta} 1\, r\,dr\,d\theta = \int_0^{2\pi}\left[\frac{r^2}{2}\right]_0^{1-\sin\theta} d\theta$$

$$= \frac{1}{2}\int_0^{2\pi}(1-\sin\theta)^2\, d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} 1 - 2\sin\theta + \sin^2\theta\, d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} \frac{3}{2} - 2\sin\theta - \frac{1}{2}\cos 2\theta\, d\theta$$

Apply trigo identity:
$\cos 2\theta = 1 - 2\sin^2\theta$

$$= \frac{1}{2}\left[\frac{3}{2}\theta + 2\cos\theta - \frac{1}{4}\sin 2\theta\right]_0^{2\pi}$$

$$= \frac{3\pi}{2}$$

Exercise 7.2:

1) Evaluate $\iint_R (x^2 + y^2 + 3)\, dA$, where R is the disk of radius 2 centred at the origin.
[Ans: 20π]

2) Evaluate $\iint_R e^{x^2+y^2}\, dA$,
where $R = \{(x, y) \colon x^2 + y^2 \leq 4, x \geq 0,\ y \geq 0\}$.
[Ans: $\frac{\pi}{4}(e^4 - 1)$]

3) Evaluate $\iint_R \frac{1}{\sqrt{x^2+y^2}} dA$,
where $R = \{(x, y) \colon 1 \leq x^2 + y^2 \leq 4\}$.
[Ans: 2π]

4) Evaluate $\iint_R x + y\, dA$,
where $R = \{(x, y) \colon 4 \leq x^2 + y^2 \leq 9,\ y \geq 0\}$.
[Ans: 38/3]

Exercise 7.2:

5) Sketch the region of integration and evaluate the following integrals, using the method of your choice.

a) $\int_0^1 \int_0^{\sqrt{1-x^2}} dy\, dx$ [Ans: $\frac{\pi}{4}$]

b) $\int_{-2}^2 \int_{-\sqrt{4-y^2}}^{\sqrt{4-y^2}} e^{-(x^2+y^2)}\, dx\, dy$ [Ans: $(1-e^{-4})\pi$]

c) $\int_{-1}^1 \int_0^{\sqrt{1-y^2}} \sqrt{x^2+y^2}\, dx\, dy$ [Ans: $\frac{\pi}{3}$]

Triple Integral

Applications of Triple Integral

Triple integrals are applied in various areas of science and engineering, including computations of

- Volume
- Mass of solids
- Force on a 3D object
- Average of a function over 3D region
- Center of Mass in 3D and Moment of Inertia

Fubini's Theorem for Triple Integrals

If $f(x, y, z)$ is continuous on a rectangular box D, where $D = \{(x, y, z): a \leq x \leq b, c \leq y \leq d, e \leq z \leq f\}$, then

$$\iiint_D f(x, y, z)\, dV = \int_e^f \int_c^d \int_a^b f(x, y, z)\, dx\, dy\, dz$$

This integral is equivalent to another five different orderings as follows:

$$\begin{aligned}
\int_e^f \int_c^d \int_a^b f(x, y, z)\, dx\, dy\, dz &= \int_c^d \int_e^f \int_a^b f(x, y, z)\, dx\, dz\, dy \\
&= \int_e^f \int_a^b \int_c^d f(x, y, z)\, dy\, dx\, dz \\
&= \int_a^b \int_e^f \int_c^d f(x, y, z)\, dy\, dz\, dx \\
&= \int_a^b \int_c^d \int_e^f f(x, y, z)\, dz\, dy\, dx \\
&= \int_c^d \int_a^b \int_e^f f(x, y, z)\, dz\, dx\, dy
\end{aligned}$$

Example 7.13:

Evaluate the integral $\int_{-1}^{1}\int_{0}^{2}\int_{1}^{3} xy - 2z\, dz\, dy\, dx$.

Solution:

$$\int_{-1}^{1}\int_{0}^{2}\int_{1}^{3} xy - 2z\, dz\, dy\, dx = \int_{-1}^{1}\int_{0}^{2}[xyz - z^2]_1^3\ dy\, dx$$

Integrate with respect to z

$$= \int_{-1}^{1}\int_{0}^{2}(3xy - 9) - (xy - 1)\ dy\, dx$$

$$= \int_{-1}^{1}\int_{0}^{2} 2xy - 8\ dy\, dx$$

$$= \int_{-1}^{1}[xy^2 - 8y]_0^2\ dx$$

Integrate with respect to y

$$= \int_{-1}^{1} 4x - 16\ dx$$

$$= [2x^2 - 16x]_{-1}^{1}$$

Integrate with respect to x

$$= (2 - 16) - (1 + 16) = -31$$

Example 7.14:

Evaluate the integral $\iiint_R \; x^2 + 3yz\, dV$ where

$$R = \{(x, y, z): 0 \le x \le 1, 0 \le y \le 2, -1 \le z \le 2\}$$

Solution:

$$\int_{-1}^{2}\int_{0}^{2}\int_{0}^{1} x^2 + 3yz\, dx\, dy\, dz = \int_{-1}^{2}\int_{0}^{2}\left[\frac{1}{3}x^3 + 3xyz\right]_0^1 dy\, dz$$

$$= \int_{-1}^{2}\int_{0}^{2} \frac{1}{3} + 3yz \;\; dy\, dz$$

$$= \int_{-1}^{2}\left[\frac{1}{3}y + \frac{3}{2}y^2 z\right]_0^2 dz$$

$$= \int_{-1}^{2} \frac{2}{3} + 6z \;\; dz$$

$$= \left[\frac{2}{3}z + 3z^2\right]_{-1}^{2} = 11$$

Example 7.15:

Evaluate the triple integral $\iiint_R x\sin y\cos z\,dV$ where

$$R=\left\{(x,y,z):0\le x\le 3,0\le y\le \pi,0\le z\le \frac{\pi}{2}\right\}.$$

Solution:

$$\int_0^{\frac{\pi}{2}}\int_0^{\pi}\int_0^{3} x\sin y\cos z\,dx\,dy\,dz=\int_0^{\frac{\pi}{2}}\int_0^{\pi}\left[\frac{1}{2}x^2\right]_0^3\sin y\cos z\,dy\,dz$$

$$=\int_0^{\frac{\pi}{2}}\int_0^{\pi}\frac{9}{2}\sin y\cos z\;dy\,dz$$

$$=\frac{9}{2}\int_0^{\frac{\pi}{2}}[-\cos y]_0^{\pi}\cos z\;dz$$

$$=\frac{9}{2}\int_0^{\frac{\pi}{2}}2\cos z\;dz$$

$$=9[\sin z]_0^{\frac{\pi}{2}}=9$$

Exercise 7.3:

Evaluate the following integrals.

a) $\int_0^2 \int_{-2}^2 \int_0^2 2x + y - z \; dx\, dy\, dz$

b) $\int_{-1}^1 \int_{-1}^2 \int_0^1 6xyz \; dy\, dx\, dz$

c) $\int_1^2 \int_y^{y^2} \int_0^{\ln x} ye^z \; dz\, dx\, dy$

[Ans: $16; 0; \frac{47}{24}$]

Reference

1) W. Briggs, L. Cochran and B. Gillett. (2014). Calculus for Scientists and Engineers Early Transcendentals, Pearson New International Edition.

2) Y. Mohammad Yusof, S. Baharun and R. Abdul Rahman. (2012). Multivariable Calculus for Independent Learners, Pearson Revised Second Edition.

THANK YOU