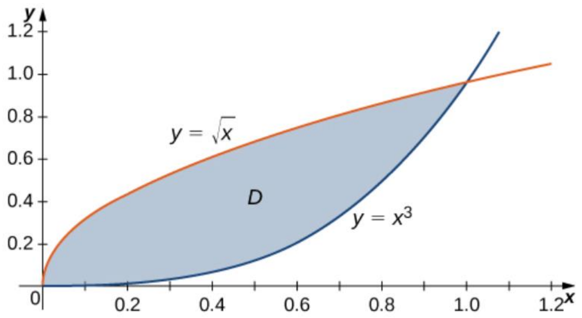
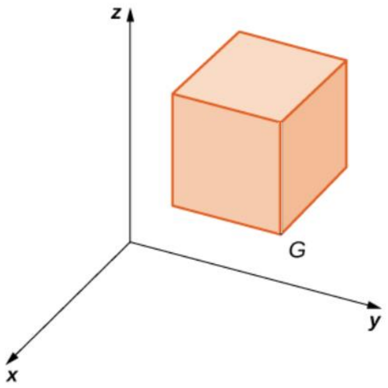
BEKG 2433
ENGINEERING MATHEMATICS 2

DOUBLE INTEGRAL IN CARTESIAN COORDINATES





Ser Lee Loh
slloh@utem.edu.my

Lesson Outcomes

Upon completion of this lesson, students should be able to:

- evaluate double integral over rectangular and non-rectangular regions in Cartesian coordinates.
- evaluate area and volume using double integral over rectangular and non-rectangular regions in Cartesian coordinates.

Double Integral

Double integral is mainly used in calculating the surface area of a 2D region. It has wide application in science and engineering, including the computations of volume, mass of 2D plates, force on a 2D plate, center of mass and moment of inertia.

In mathematics, double integral of a continuous function $f(x,y)$ over a region R is defined as

$$\iint_R f(x,y)\,dA = \iint_R f(x,y)\,dydx \quad \text{or} \iint_R f(x,y)\,dxdy.$$

Evaluating Double Integral (Rectangular Regions)

To solve a double integral:

Solve the inner integral before the outer integral.

$$\int_a^b \left[\int_c^d f(x,y)\,dy\right] dx = \int_a^b g(x)\;dx = G(b) - G(a)$$

Integrate $f(x,y)$ w.r.t. y by treating x constant to obtain $F(x,y)$

Substitute the limit d followed by c into y, i.e. $F(x,d) - F(x,c)$ to obtain $g(x)$

Example 6.1:

Evaluate the integral $\int_0^1 \int_{-2}^2 6y^2 + xy^4\, dy\, dx$.

Solution:

$$\int_0^1 \int_{-2}^2 6y^2 + xy^4\, dy\, dx = \int_0^1 \left[2y^3 + \frac{xy^5}{5}\right]_{-2}^{2} dx$$

$$= \int_0^1 \left(2(2)^3 + \frac{x(2)^5}{5}\right) - \left(2(-2)^3 + \frac{x(-2)^5}{5}\right) dx$$

$$= \int_0^1 \left(16 + \frac{32x}{5}\right) - \left(-16 - \frac{32x}{5}\right) dx$$

$$= \int_0^1 32 + \frac{64x}{5}\, dx$$

$$= \left[32x + \frac{32x^2}{5}\right]_0^1 = 32 + \frac{32}{5} = \frac{192}{5}$$

Example 6.2:

Evaluate $\int_0^{\frac{\pi}{2}} \int_{-1}^{1} e^{2x} + \cos y \, dx \, dy$.

Solution:

$$\int_0^{\frac{\pi}{2}} \int_{-1}^{1} e^{2x} + \cos y \, dx \, dy = \int_0^{\frac{\pi}{2}} \left[\frac{e^{2x}}{2} + x \cos y\right]_{-1}^{1} \, dy$$

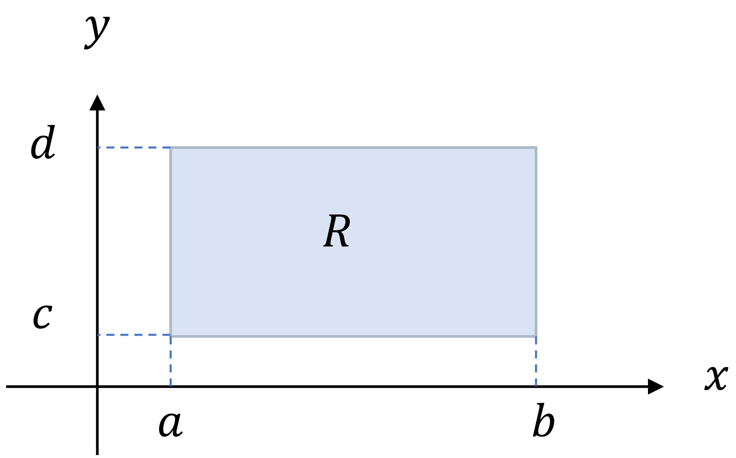
$$= \int_0^{\frac{\pi}{2}} \left(\frac{e^{2(1)}}{2} + (1)\cos y\right) - \left(\frac{e^{2(-1)}}{2} + (-1)\cos y\right) \, dy$$

$$= \int_0^{\frac{\pi}{2}} \frac{e^{2}-e^{-2}}{2} + 2\cos y \, dy$$

$$= \left[\frac{e^{2}-e^{-2}}{2} y + 2\sin y\right]_0^{\frac{\pi}{2}} = 7.6971$$

Let f be continuous on rectangular region

$R = \{(x, y): a \leq x \leq b, c \leq y \leq d\}.$



$$\iint_R f(x,y)\,dA = \int_c^d \int_a^b f(x,y)\,dx\,dy = \int_a^b \int_c^d f(x,y)\,dy\,dx$$

NOTE: For double integral on rectangular region, the interchange of order of integration is done by swapping the limits of x and y directly.

Example 6.3:

Let $f(x,y) = 4xy$ be defined on rectangular region $R = \{(x,y): 0 \le x \le 1, -2 \le y \le 3\}$.

Evaluate

$$\iint_R f(x,y)\,dA$$

by using two different order of integration. Interpret the results.

Hint: Evaluate the following integrals:

i. $\int_0^1 \int_{-2}^3 4xy\,dy\,dx$

ii. $\int_{-2}^3 \int_0^1 4xy\,dx\,dy$

Solution:

$$\int_0^1 \int_{-2}^3 4xy\,dy\,dx = \int_0^1 [2xy^2]_{-2}^3 \; dx$$
$$= \int_0^1 18x - 8x \; dx$$
$$= \int_0^1 10x \; dx$$
$$= [5x^2]_0^1$$
$$= 5$$

$$\int_{-2}^3 \int_0^1 4xy\,dx\,dy = \int_{-2}^3 [2x^2y]_0^1 \; dx$$
$$= \int_{-2}^3 2y \; dx$$
$$= [y^2]_{-2}^3$$
$$= 9 - 4$$
$$= 5$$

Different order of integration gives the same result.

Exercise 6.1:

Evaluate the following integrals.

1) $\int_0^2 \int_1^4 6x^2 + 4xy^3 \, dy \, dx$ [Ans: 558]

2) $\int_{-1}^1 \int_2^5 x + 3x^2y^2 \, dy \, dx$ [Ans: 78]

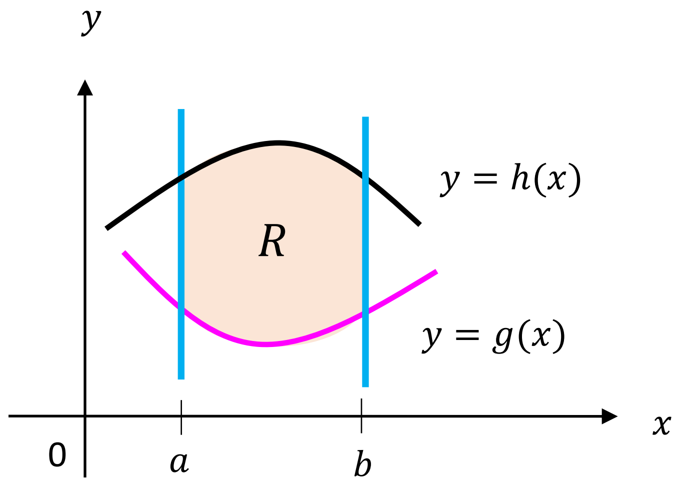
3) $\iint_R (x^2 - 2y) \, dA$, where $R = \{0 \le x \le 2, \ -1 \le y \le 1\}$ [Ans: 16/3]

4) $\iint_R 4xe^{2y} \, dA$, where $R = \{2 \le x \le 4, \ 0 \le y \le 1\}$ [Ans: $12(e^2 - 1)$]

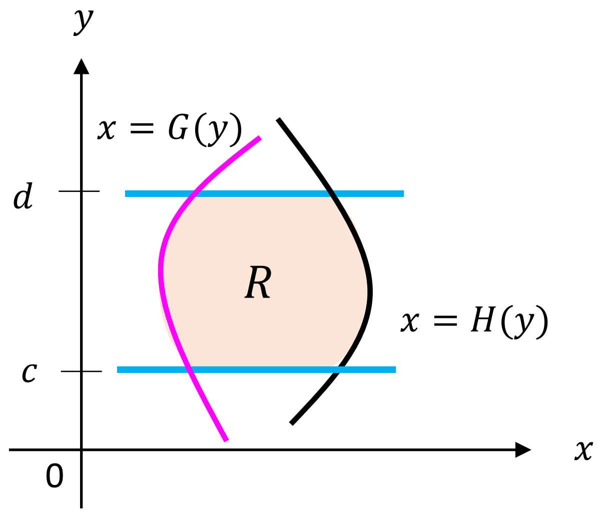
5) $\iint_R ye^{xy} \, dA$, where $R = \{(x, y): 0 \le x \le 1, \ 0 \le y \le \ln 2\}$ [Ans: $1 - \ln 2$]

6) $\iint_R x \sec^2 xy \, dA$, where $R = \{(x, y): 0 \le x \le \pi/3, \ 0 \le y \le 1\}$ [Ans: $\ln 2$]

Double Integral over Nonrectangular Regions



$$\iint_R f(x,y)\,dA = \int_a^b \int_{g(x)}^{h(x)} f(x,y)\,dy\,dx$$

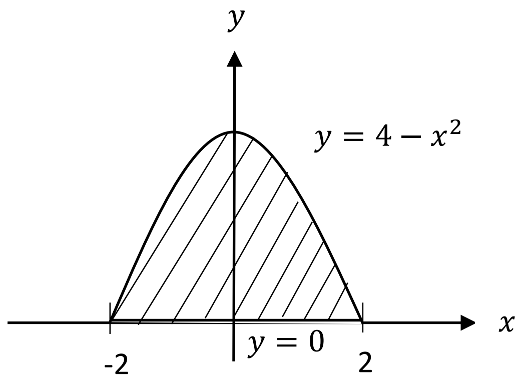


$$\iint_R f(x,y)\,dA = \int_c^d \int_{G(y)}^{H(y)} f(x,y)\,dx\,dy$$

Example 6.4:

Evaluate $\iint_R \; x + y \, dA$ where R is bounded by $y = 4 - x^2$ and $y = 0$.

Solution:



$$\int_{-2}^{2}\int_{0}^{4-x^2} x + y \, dy \, dx$$

$$= \int_{-2}^{2}\left[xy + \frac{y^2}{2}\right]_0^{4-x^2} dx$$

$$= \int_{-2}^{2} x(4 - x^2) + \frac{(4-x^2)^2}{2} \, dx$$

$$= \int_{-2}^{2} 8 + 4x - 4x^2 - x^3 + \frac{1}{2}x^4 \, dx$$

$$= \left[8x + 2x^2 - \frac{4}{3}x^3 - \frac{1}{4}x^4 + \frac{1}{10}x^5\right]_{-2}^{2} = \frac{188}{15} - \left(-\frac{68}{15}\right) = \frac{256}{15}$$

Find the intersection point:

$$4 - x^2 = 0$$

$$x = \pm 2$$

Example 6.5:

Evaluate $\iint_R y\, dA$ where R is bounded by $y = 2x$, $y = 3 - x$ and $y = 0$.

Solution:

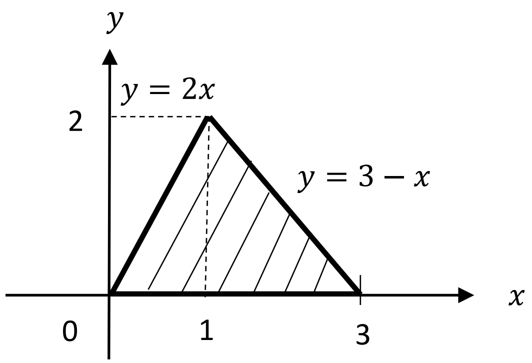
$$\int_0^2 \int_{y/2}^{3-y} y\, dx\, dy$$

$$= \int_0^2 [xy]_{y/2}^{3-y}\, dy$$

$$= \int_0^2 (3-y)y - \left(\frac{y}{2}\right) y\, dy$$

$$= \int_0^2 3y - \frac{3}{2}y^2\, dy$$

$$= \left[\frac{3}{2}y^2 - \frac{1}{2}y^3\right]_0^2 = 6 - 4 = 2$$



Find the intersection points:

$$2x = 3 - x$$

$$3x = 3$$

$$x = 1 \quad \therefore y = 2$$

Exercise 6.2:

1) Evaluate $\iint_R x^2 \, dA$ where R is bounded by $y = x, y = 4$ and $x = 0$.

[Ans: 64/3]

2) Evaluate $\iint_R x^2 + y^2 \, dA$ where R is bounded by $y = x^2$ and $y = 1$.

[Ans: 88/105]

3) Evaluate $\iint_R 6 - x - y \, dA$ where R is bounded by $x = 4 - y^2$ and $x = 0$.

[Ans: 704/15]

Area (Double Integrals)

Double integral

$$\iint_R f(x, y)\, dA$$

can be used to compute for the area of region R by letting $f(x, y) = 1$ as follows:

$$\text{Area} = \iint_R 1\, dA$$

Example 6.6:

Find the area bounded by the graphs of $y = x^2 - 1$ and $y = 3$.

Solution:

$\int_{-2}^{2} \int_{x^2-1}^{3} 1 \, dy \, dx$

$= \int_{-2}^{2} [y]_{x^2-1}^{3} \, dx$

$= \int_{-2}^{2} 3 - (x^2 - 1) \, dx$

$= \int_{-2}^{2} 4 - x^2 \, dx$

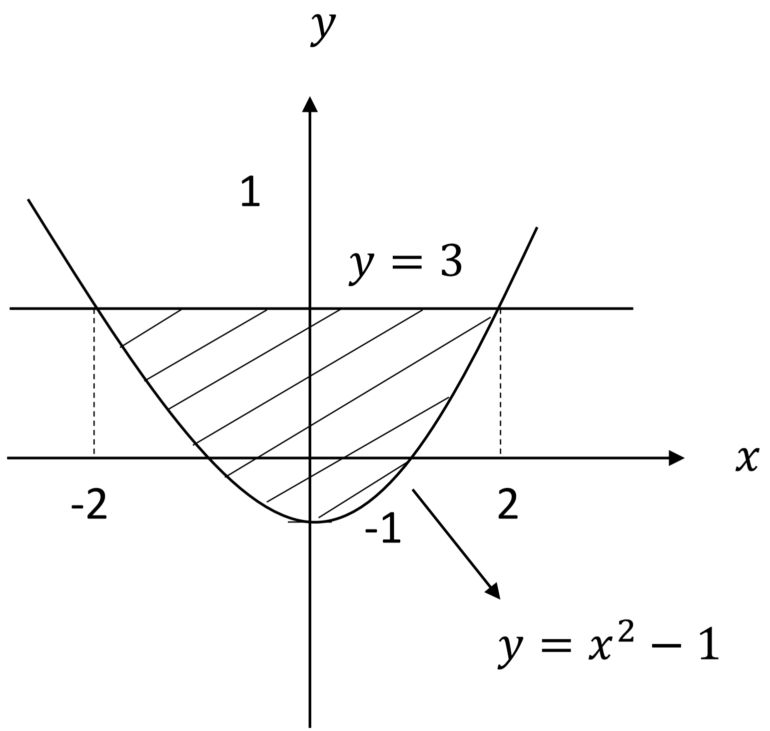
$= \left[4x - \frac{1}{3}x^3\right]_{-2}^{2} = \left(8 - \frac{8}{3}\right) - \left(-8 + \frac{8}{3}\right) = \frac{32}{3}$

Find the intersection points:

$x^2 - 1 = 3$

$x^2 = 4$

$x = \pm 2$



Example 6.7:

Find the area bounded by the graphs of $x = y^2$ and $y = x$.

Solution:

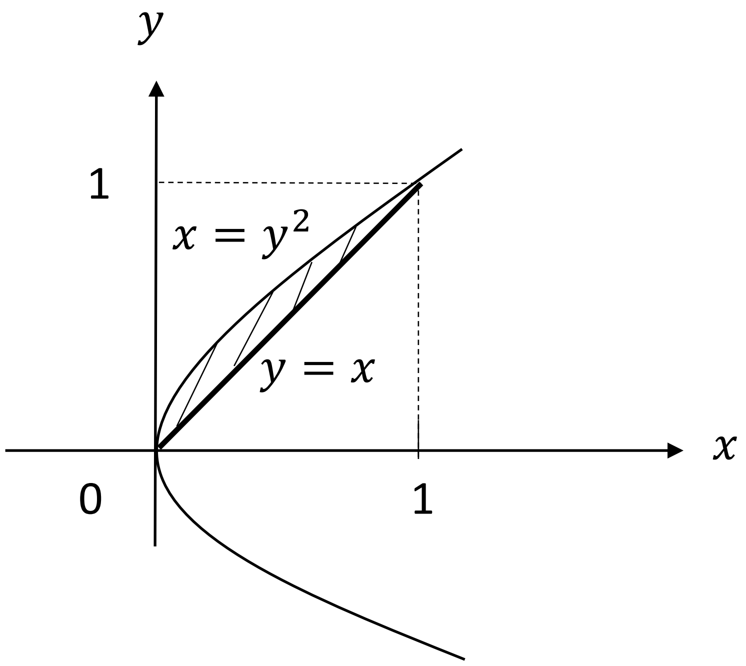
$$\int_0^1 \int_{y^2}^{y} 1 \, dx \, dy$$

$$= \int_0^1 [x]_{y^2}^{y} \, dy$$

$$= \int_0^1 y - y^2 \, dy$$

$$= \left[\frac{1}{2}y^2 - \frac{1}{3}y^3\right]_0^1$$

$$= \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$



Find the intersection points:

$$y^2 = y$$

$$y^2 - y = 0$$

$$y(y-1) = 0$$

$$\therefore y = 0, y = 1$$

Exercise 6.3:

1) Find the area bounded by the graphs of $y = x^2$ and $y = 8 - x^2$.

[Ans: 64/3]

2) Find the area bounded by the graphs of $y = x^2$ and $y = x + 2$.

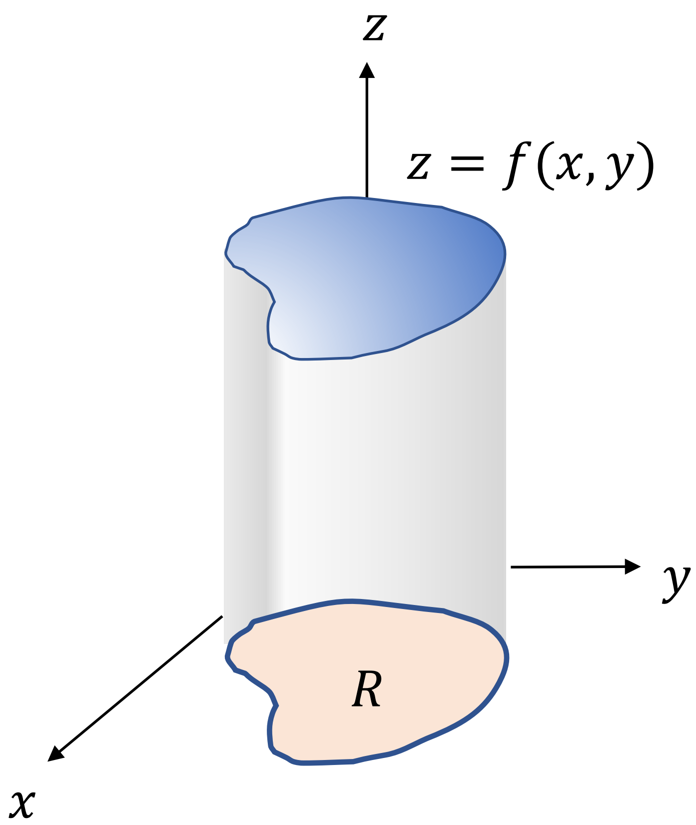
[Ans: 9/2]

3) Find the area bounded by the graphs of $y = 3x,\ \ y = 5 - 2x$ and $y = 0$.

[Ans: 15/4]

Volume (Double Integrals)

Double integral can also be used to compute for volume of a solid as follows:

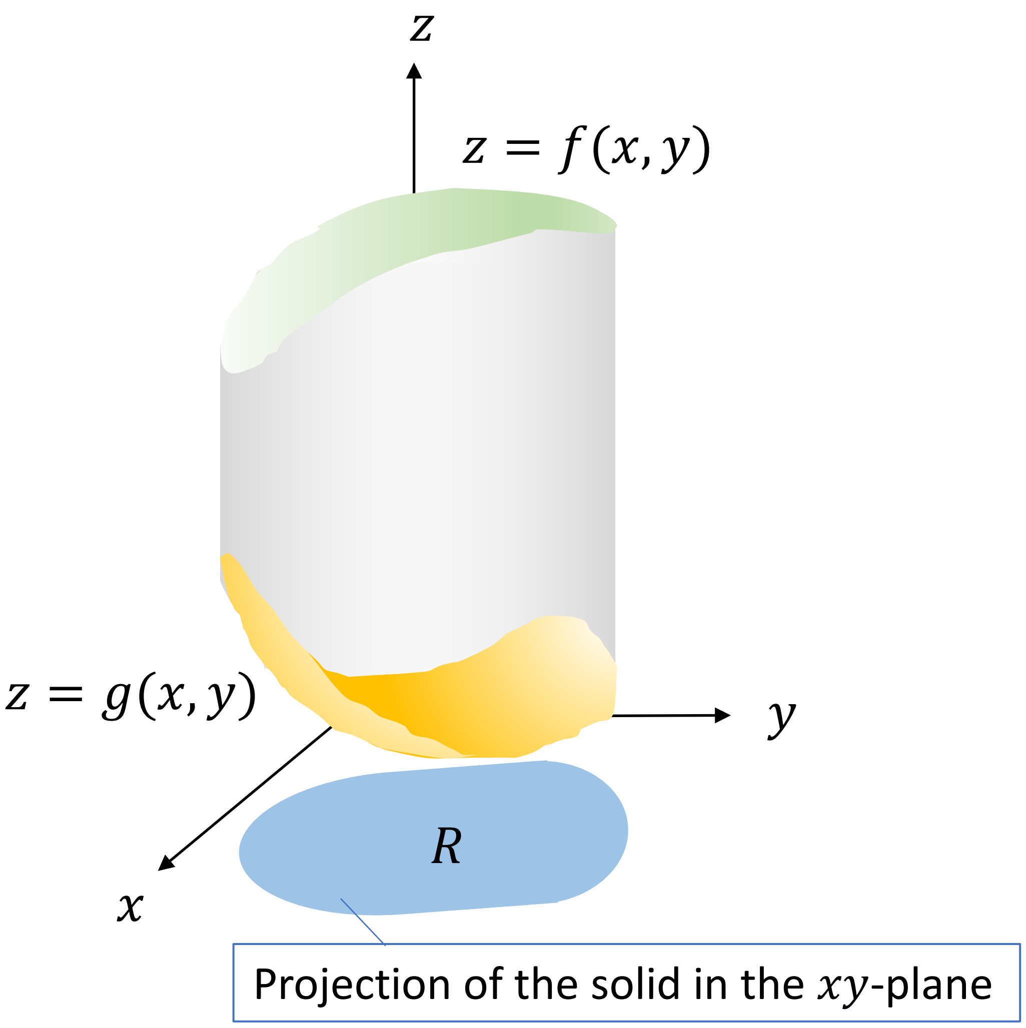


Volume = $\iint_R f(x, y)\, dA.$

Volume (Double Integrals)

Volume = $\iint_R [f(x,y) - g(x,y)]\, dA$

Upper surface – Lower surface

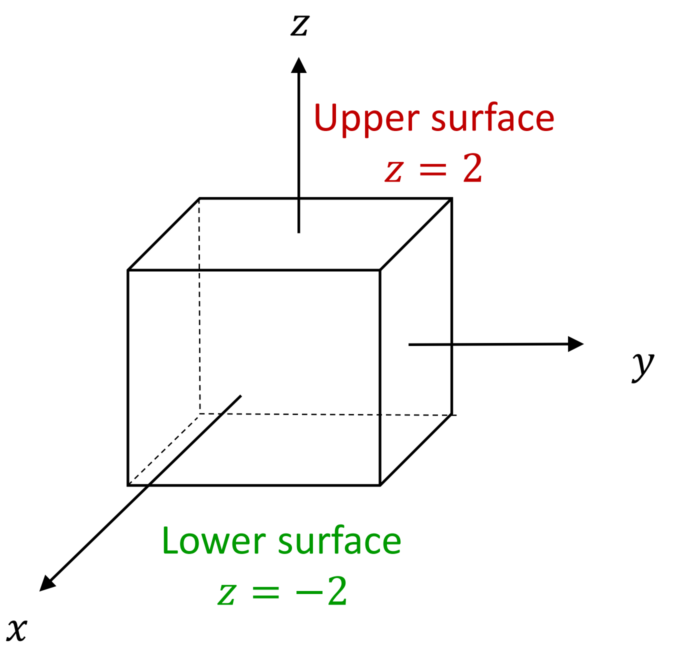


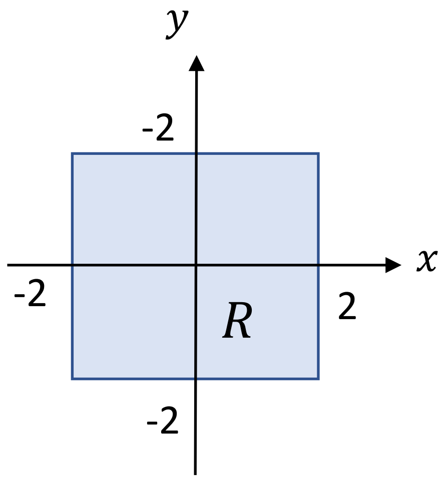
Example 6.8:

Find the volume of the cube bounded by planes $z = \pm 2$, $y = \pm 2$ and $x = \pm 2$ using double integration.

Solution:

$$\int_{-2}^{2}\int_{-2}^{2} 2 - (-2)\, dy\, dx$$
$$= \int_{-2}^{2}\int_{-2}^{2} 4\, dy\, dx$$
$$= \int_{-2}^{2} [4y]_{-2}^{2}\, dx$$
$$= \int_{-2}^{2} 4(2) - 4(-2)\ dx$$
$$= \int_{-2}^{2} 16\ dx$$
$$= [16x]|_{-2}^{2}$$
$$= 16(2) - 16(-2) = 64$$





Example 6.9:

Find the volume of the solid bounded by $x = 2, y = 2, z = y + 1$ and the three coordinate planes using double integration.

Solution:

$$\int_0^2 \int_0^2 (y + 1) - (0)\, dy\, dx$$

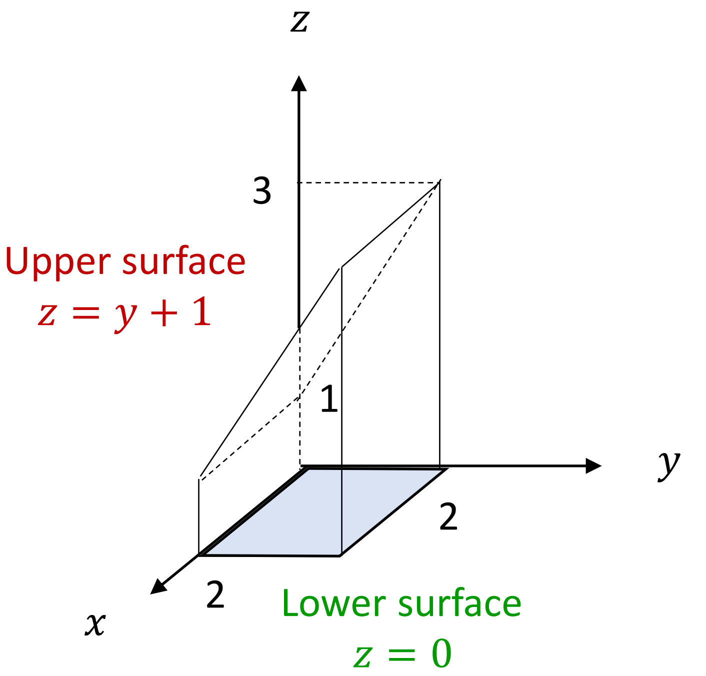
$$= \int_0^2 \int_0^2 y + 1\, dy\, dx$$

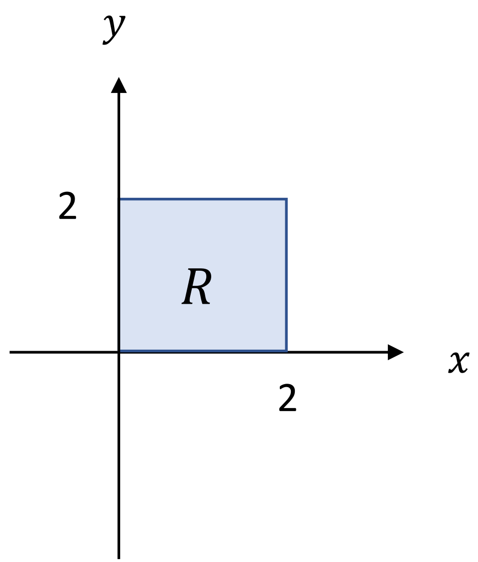
$$= \int_0^2 \left[\frac{y^2}{2} + y\right]_0^2 dx$$

$$= \int_0^2 \frac{2^2}{2} + 2\ dx$$

$$= \int_0^2 4\ dx$$

$$= [4x]|_0^2 = 8$$





Exercise 6.4:

1) Find the volume of the rectangular solid bounded by planes $z = 2$, $y = 1$, $x = 1$ and the three coordinate planes.

[Ans: 2]

2) Find the volume of the solid lying in the first octant and bounded by the graphs of $z = 4 - x^2$, $x = 0$, $y = 4$, $y = 0$ and $z = 0$.

[Ans: 64/3]

3) Find the volume of the tetrahedron bounded by the plane $2x + y + z = 2$ and the three coordinate planes.

[Ans: 2/3]

Changing Order of Integration

When we change the integration order (for nonrectangular region) from

$$\iint \; dydx \text{ to } \iint \; dxdy \quad \text{OR} \quad \iint \; dxdy \text{ to } \iint \; dydx$$

We need to sketch the region and redefine the new limits.

WHY we need to change the order?...

For convenience and possibilities of integration.

Example 6.10:

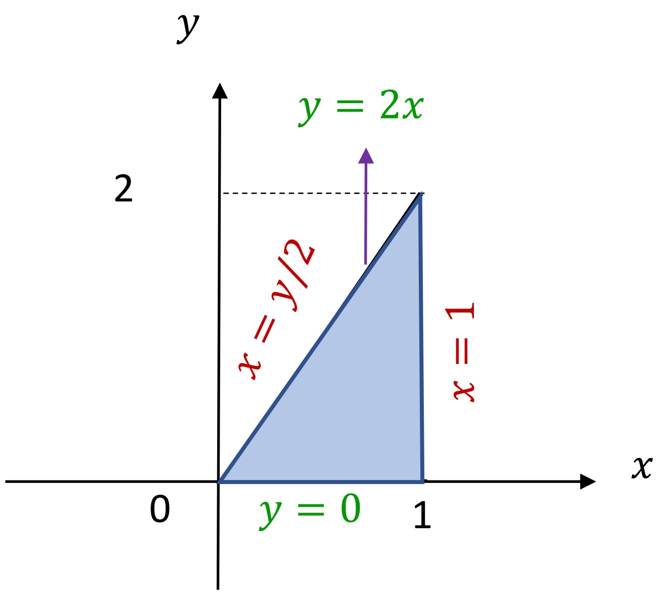
Change the order of integration $\int_0^1 \int_0^{2x} f(x,y)\, dy\, dx$.

Solution:

$$\int_0^1 \int_0^{2x} f(x,y)\, dy\, dx = \int_0^2 \int_{y/2}^{1} f(x,y)\, dx\, dy$$

Lower to Upper curves

LHS to RHS curves



Example 6.11:

Change the order of integration $\int_0^4 \int_{y/2}^2 \cos x^2 \, dx \, dy$, then evaluate it.

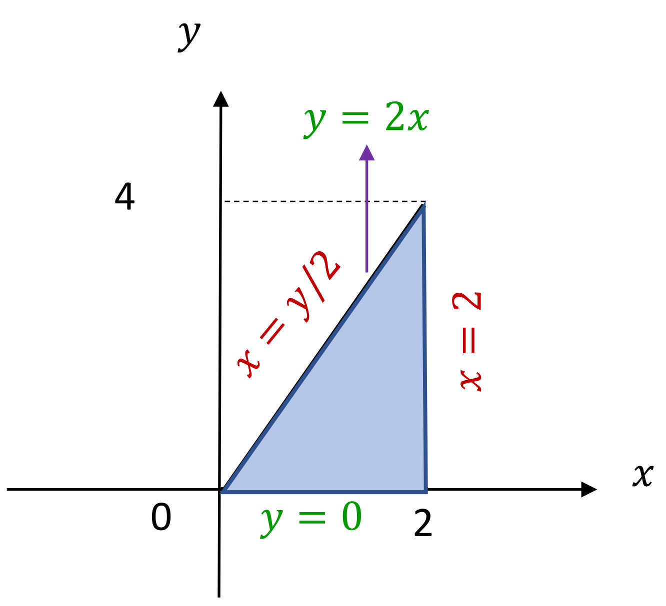
Solution:

$$\int_0^4 \int_{y/2}^2 \cos x^2 \, dx \, dy$$
$$= \int_0^2 \int_0^{2x} \cos x^2 \, dy \, dx$$
$$= \int_0^2 [y \cos x^2]_0^{2x} \, dx$$
$$= \int_0^2 2x \cos x^2 \, dx$$
$$= [\sin x^2]_0^2$$
$$= \sin 4 \approx -0.7568$$

Change the order of integration for possibility of integration

Apply integration by substitution:

$$\int 2x \cos x^2 \, dx \qquad \text{Let } u = x^2$$
$$= \int \cos u \, du \qquad \frac{du}{dx} = 2x$$
$$= \sin u + c \qquad du = 2x \, dx$$
$$= \sin x^2 + c$$



Example 6.12:

Change the order of integration $\int_{-1}^{3}\int_{0}^{\sqrt{y+1}} x\,dx\,dy$, then evaluate it.

Solution:

$$\int_{-1}^{3}\int_{0}^{\sqrt{y+1}} x\,dx\,dy$$

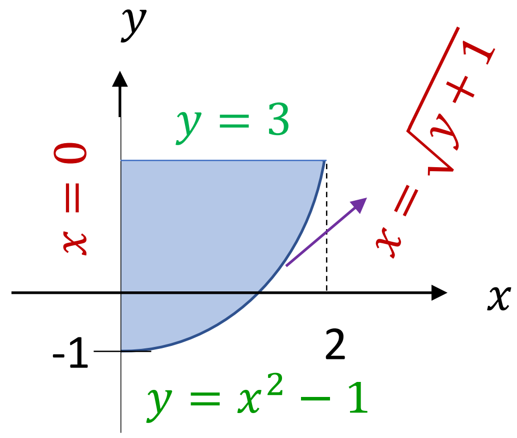
Change the order of integration for convenience of integration

$$= \int_{0}^{2}\int_{x^2-1}^{3} x\,dy\,dx$$

$$= \int_{0}^{2}[xy]_{x^2-1}^{3}\,dx$$

$$= \int_{0}^{2} x(3) - x(x^2-1)\,dx$$

$$= \int_{0}^{2} -x^3 + 4x\,dx = \left[-\frac{x^4}{4} + 2x^2\right]_0^2 = 4$$

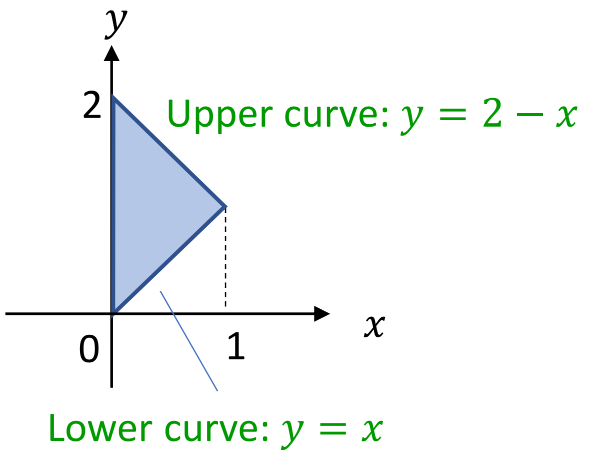


$$x = \sqrt{y+1}$$
$$x^2 = y+1$$
$$y = x^2 - 1$$

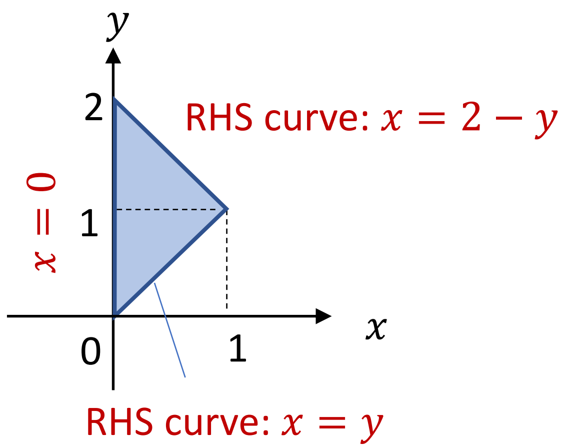
Example 6.13:

Form the definite integral by using different order of integration for a region bounded by $x = 0$, $y = x$ and $x + y = 2$.

Solution:



$$\int_0^1 \int_x^{2-x} f(x,y)\, dy\, dx$$



LHS curve is $x = 0$, but the RHS curve is different for intervals $0 \le y \le 1$ and $1 \le y \le 2$.

$$\int_0^1 \int_0^y f(x,y)\, dx\, dy + \int_1^2 \int_0^{2-y} f(x,y)\, dx\, dy$$

Exercise 6.5:

1) Change the order of integration for each of the following cases.

a) $\int_0^2 \int_{2y}^4 f(x,y)\,dx\,dy$ [Ans: $\int_0^4 \int_0^{x/2} f(x,y)\,dy\,dx$]

b) $\int_1^3 \int_0^{\ln y} f(x,y)\,dx\,dy$ [Ans: $\int_0^{\ln 3} \int_{e^x}^3 f(x,y)\,dy\,dx$]

c) $\int_0^{\ln 4} \int_{e^x}^4 f(x,y)\,dy\,dx$ [Ans: $\int_1^4 \int_0^{\ln y} f(x,y)\,dx\,dy$]

d) $\int_0^1 \int_{2x}^2 f(x,y)\,dy\,dx$ [Ans: $\int_0^2 \int_0^{y/2} f(x,y)\,dx\,dy$]

e) $\int_0^1 \int_0^{1-x} f(x,y)\,dy\,dx$ [Ans: $\int_0^1 \int_0^{1-y} f(x,y)\,dx\,dy$]

f) $\int_{-1}^1 \int_0^{\cos^{-1} y} f(x,y)\,dx\,dy$ [Ans: $\int_0^{\pi} \int_{-1}^{\cos x} f(x,y)\,dy\,dx$]

Exercise 6.5:

2) Sketch the region of integration, reverse the order of integration, and evaluate the integral

a) $\int_0^4 \int_{x/2}^2 e^{y^2}\, dy\, dx.$ [Ans: $e^4 - 1$]

b) $\int_0^1 \int_{\sqrt{x}}^1 \frac{3}{4+y^3}\, dy\, dx.$ [Ans: $\ln \frac{5}{4}$]

Reference

1) W. Briggs, L. Cochran and B. Gillett. (2014). Calculus for Scientists and Engineers Early Transcendentals, Pearson New International Edition.

2) Y. Mohammad Yusof, S. Baharun and R. Abdul Rahman. (2012). Multivariable Calculus for Independent Learners, Pearson Revised Second Edition.

THANK YOU