BEKG 2433
ENGINEERING MATHEMATICS 2

THE CHAIN RULE AND IMPLICIT DIFFERENTIATION

$$\frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial x^2}\right) = \frac{\partial^3 f}{\partial x^3} = f_{xxx}$$

$$\frac{\partial}{\partial y}\left(\frac{\partial^2 f}{\partial x^2}\right) = \frac{\partial^3 f}{\partial y \partial x^2} = f_{yxx}$$

$$\frac{\partial}{\partial x}\left(\frac{\partial^2 f}{\partial x \partial y}\right) = \frac{\partial^3 f}{\partial x^2 \partial y} = f_{xxy}$$

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Learning outcomes

Upon completion of this lesson, students should be able to:

- Find the derivatives for a given function by using the chain rule.
- Solve the derivatives for a given function by using the implicit differentiation.
- Evaluate the derivatives gained using the chain rule and implicit differentiation in solving engineering problems.

The Chain Rule

Definition:
If y is a differentiable function of x and x is a differentiable function of a parameter t, then the Chain Rule states that

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

The corresponding rule for two variables function is essentially the same except that it involves both variables.

The Chain Rule

Let $z = f(x, y)$ is a function of x and y and suppose that x and y are in turn functions of a single variable t, $x = x(t), y = y(t)$. Then $z = f[x(t), y(t)]$ is a composite function of a parameter t. Thus, we can calculate the derivative $\frac{dz}{dt}$ and its relationship to the derivates $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}, \frac{dx}{dt}$ and $\frac{dy}{dt}$ is given by the following theorem.

The Chain Rule

Theorem: Two Intermediate variables, one parameter

If $z = f(x, y)$ is differentiable and x and y are differentiable functions of t, then z is a differentiable function of t and

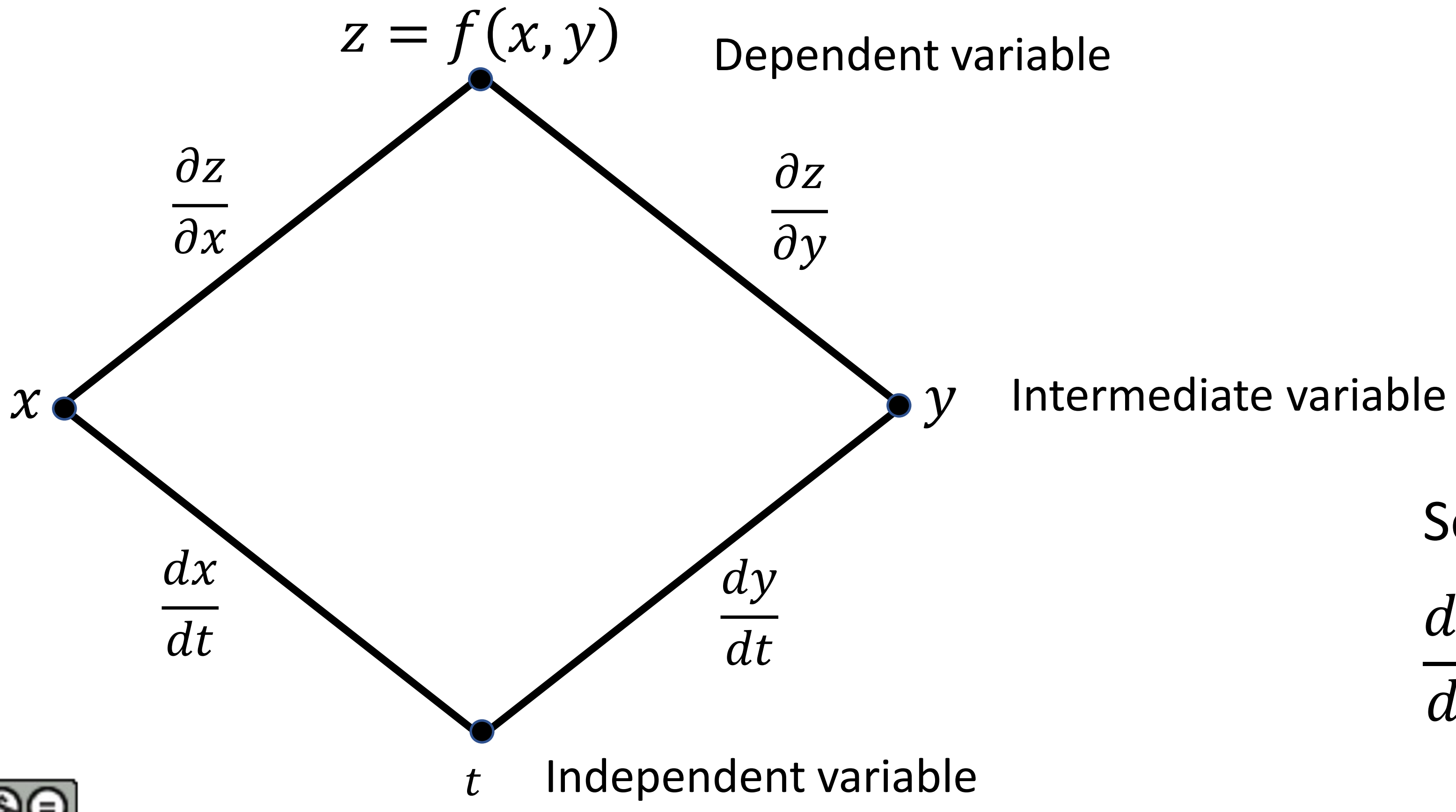
$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

The corresponding rule for two variables function is essentially the same except that it involves both variables.

The Chain Rule

Two Intermediate variables, one independent variable (one parameter)

z= $f(x,y), x = x(t), y = y(t)$. z is a function of t.



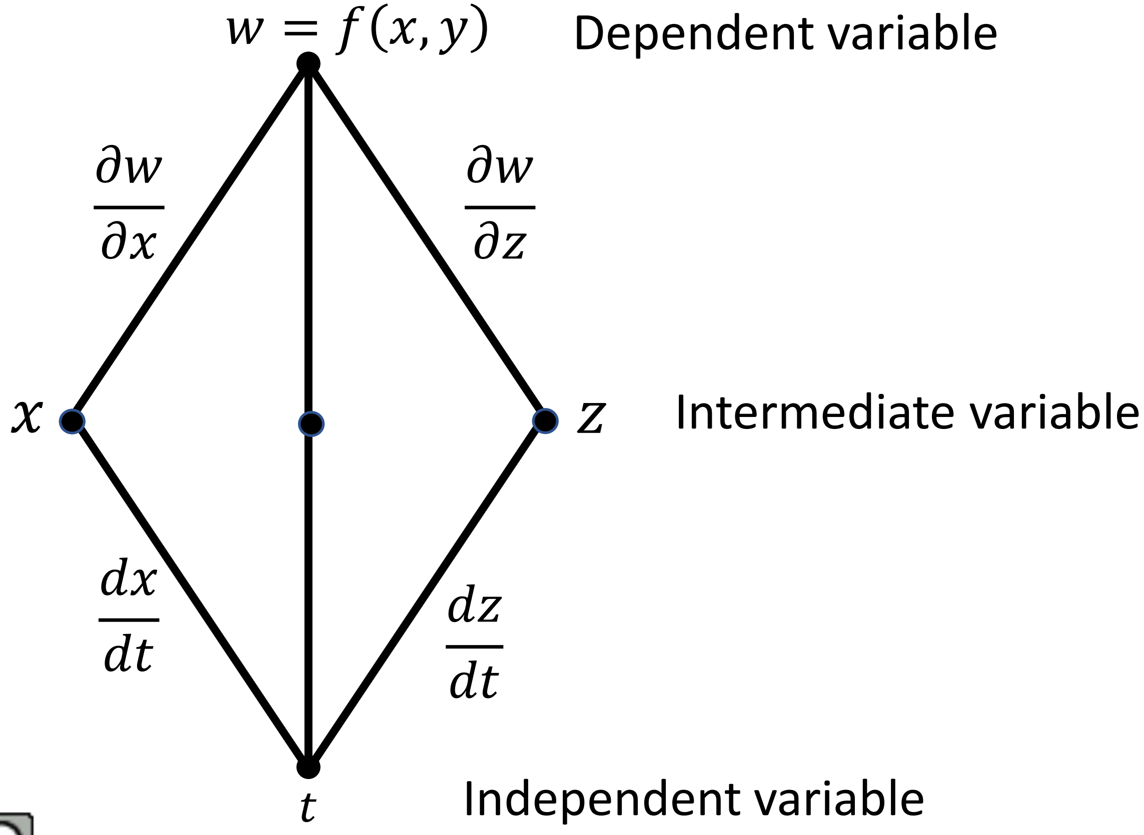
So;

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

The Chain Rule

Three Intermediate variables, one independent variable

$w = f(x, y, z), x = x(t), y = y(t), z = z(t)$. w is a function of t.



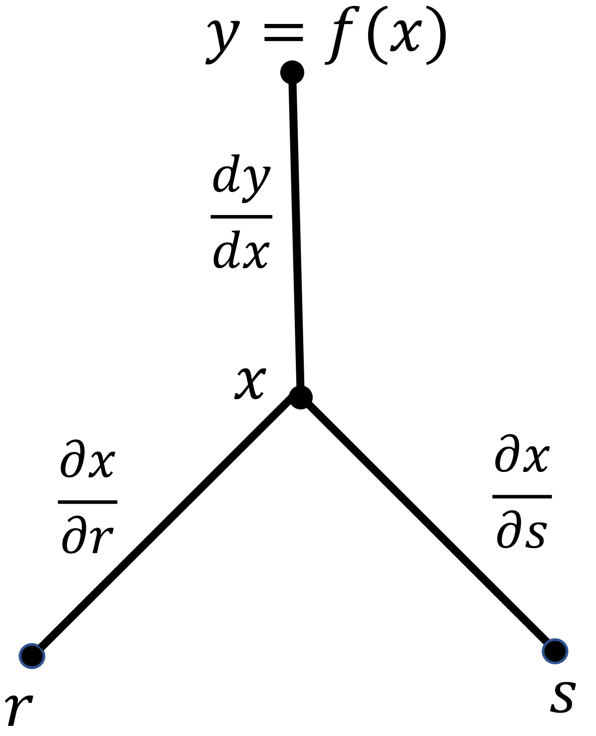
So;

$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial w}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial w}{\partial z} \cdot \frac{dz}{dt}$$

The Chain Rule

One Intermediate variables, two parameters

$y = f(x), x = x(r, s)$. y is a function of two variables r and s.



So; compute $\frac{\partial y}{\partial r}$ and $\frac{\partial y}{\partial s}$

$$\frac{\partial y}{\partial r} = \frac{dy}{dx} \cdot \frac{\partial x}{\partial r}$$

$$\frac{\partial y}{\partial s} = \frac{dy}{dx} \cdot \frac{\partial x}{\partial s}$$

The Chain Rule

The chain rule is a formula for differentiating composition functions.

Let $z = f(u,v), u = g(x,y), v = h(x,y)$ where $f(u,v), g(x,y)$ and $h(x,y)$ are differentiable, then

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x}$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial y}$$

Example 4.1

Use the chain rule to find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if $z = u^2 + v^4$ where $u = x^2 \cos y$; $v = xy^3$

Solution:

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x}$$

$$\begin{aligned}\frac{\partial z}{\partial x} &= 2u(2x \cos y) + 4v^3[y^3] \\ &= 4ux \cos y + 4v^3y^3 \\ &= 4[x^2 \cos y][x \cos y] + 4[xy^3]^3[y^3] \\ &= 4x^3 \cos^2 y + 4x^3y^{12}\end{aligned}$$

Solution continue:

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial y}$$

$$= 2u[-x^2 \sin y] + 4v^3[3xy^2]$$

$$= 2[x^2 \cos y][-x^2 \sin y] + 4[xy^3]^3[3xy^2]$$

$$= -2x^4 \sin y \cos y + 12x^4 y^{11}$$

Example 4.2

Suppose that $w = xy + yz$ where $y = sinx$ and $z = e^x$. Use an appropriate form of the Chain Rule to find $\frac{dw}{dx}$.

Solution:

$$\frac{\partial w}{\partial y} = \frac{\partial w}{\partial x} \cdot \frac{dx}{dx} + \frac{\partial w}{\partial y} \cdot \frac{dy}{dx} + \frac{\partial w}{\partial z} \cdot \frac{dz}{dx}$$
$$= y(1) + (x + z)(\cos x) + y(e^x)$$

Given that $y = \sin x$ and $z = e^x$

Hence: $\frac{\partial w}{\partial x} = \sin x + (x + e^x)\cos x + e^x \sin x$
$$= (1 + e^x)\sin x + (x + e^x)\cos x$$

Example 4.3

Let $w = p^3 + q^2 r + s^2$ with $p = y^3 + z^2,\ q = xe^z, r = xz^2, s = (xy)^2$.
Use an appropriate form of the chain rule to find $\frac{\partial w}{\partial x}$.

Solution:

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial p}\cdot\frac{\partial p}{\partial x} + \frac{\partial w}{\partial q}\cdot\frac{\partial q}{\partial x} + \frac{\partial w}{\partial r}\cdot\frac{\partial r}{\partial x} + \frac{\partial w}{\partial s}\cdot\frac{\partial s}{\partial x}$$

$$= 3p^2(0) + 2qr(e^z) + q^2(z^2) + 2s[2(xy)(y)]$$

$$= 0 + 2(xe^z)(xz^2)(e^z) + (xe^z)^2 z^2 + 2(xy)^2 \cdot 2y(xy)$$

$$\frac{\partial w}{\partial x} = 2x^2z^2e^{2z} + x^2z^2e^{2z} + 4x^3y^4$$

$$= 3x^2z^2e^{2z} + 4x^3y^4$$

Example 4.4

Use the chain rule to find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if $z = u^2 + v^2$ where $u = x^2 \cos y$, $v = xy^3$

Solution:

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x}$$
$$= 2u[2x \cos y] + 2v[y^3]$$
$$= 2[x^2 \cos y][2x \cos y] + 2[xy^3][y^3]$$
$$= 4x^3 \cos^2 y + 2xy^6$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y}$$
$$= 2u[-x^2 \sin y] + 2v[3xy^2]$$
$$= 2[x^2 \cos y][-x^2 \sin y] + 2[xy^3][3xy^2]$$
$$= -2x^4 \sin y \cos y + 6x^2y^5$$

Example 4.5

A rectangular has length 5 ft. and width 2 ft. It is changing in such a way that the length is increasing at a rate of 3 ft/s and its width is increasing at a rate of 4 ft/s. At what rate is the area of the rectangle changing?

Solution:

Given length, $l = 5$; width, $w = 2$

$$\frac{\partial l}{\partial t} = 3 \;\; ; \;\; \frac{\partial w}{\partial t} = 4$$

Area of a rectangle, $A = lw$

$$\begin{aligned} \frac{\partial A}{\partial t} &= \frac{\partial A}{\partial l} \cdot \frac{\partial l}{\partial t} + \frac{\partial A}{\partial w} \cdot \frac{\partial w}{\partial t} \\ &= w(3) + l(4) \\ &= 6 + 20 \\ &= 26\, ft/s^2 \end{aligned}$$

Example 4.6

Suppose that $z = \sqrt{xy + y}$ where $x = \cos\theta$ and $y = \sin\theta$. Find $\frac{dz}{d\theta}$ when $\theta = \frac{\pi}{2}$.

Solution:

$$\frac{dz}{d\theta} = \frac{\partial z}{\partial x} \cdot \frac{dx}{d\theta} + \frac{\partial z}{\partial y} \cdot \frac{dy}{d\theta}$$

$$\frac{dz}{d\theta} = \frac{1}{2}(xy + y)^{-\frac{1}{2}}(y)(-\sin\theta) + \frac{1}{2}(xy + y)^{-\frac{1}{2}}(x + 1)(\cos\theta)$$

When $\theta = \frac{1}{2}$; $x = \cos\frac{\pi}{2} = 0$ and $y = \sin\frac{\pi}{2} = 1$

Substituting $x = 1, y = 1$ and $\theta = \frac{\pi}{2}$ yields;

$$\left.\frac{dz}{d\theta}\right|_{\theta=\frac{\pi}{2}} = \frac{1}{2}(1)(1)(-1) + \frac{1}{2}(1)(1)(0) = -\frac{1}{2}$$

Exercise 4.1:

Let $z = x^2 + y^2$, where $x = \frac{1}{t}$ and $y = \ln t$. Find $\frac{dz}{dt}$ by using the chain rule.

[Ans: $-\frac{2}{t^3} + \frac{2\ln t}{t}$]

Exercise 4.2:

If $w = 4x + y^2 + z^3$ where $x = e^{rs^2}$, $y = \ln\frac{r+s}{t}$ and $z = rst^2$, find $\frac{\partial w}{\partial r}$, $\frac{\partial w}{\partial s}$ and $\frac{\partial w}{\partial t}$.

[Ans:

$$\frac{\partial w}{\partial r} = 4s^2e^{rs^2} + \frac{2}{r+s}\ln\left(\frac{r+s}{t}\right) + 3r^2s^3t^6;$$

$$\frac{\partial w}{\partial s} = 8rse^{rs^2} + \frac{2}{r+s}\ln\left(\frac{r+s}{t}\right) + 3r^3s^2t^6; \quad \frac{\partial w}{\partial t} = -\frac{2}{t}\ln\left(\frac{r+s}{t}\right) + 6r^3s^3t^5$$

Exercise 4.3:

A right circular cylinder is changing in such a way that its radius is increasing at the rate of 3 in./min and its height is decreasing at the rate of 5 in./min. At what rate is the volume changing when the radius is 10 in. and the height is 8 in.?

[Ans: $-62.8\ in^3/min$]

Implicit Differentiation

- In equations containing x and y, let y be a dependent variable while x be an independent variable, then y defines implicitly in term of x.
- We call y an implicit function of x.
- A familiar example of the implicit function is $x^2 + y^2 = 9$ which represent a circle with radius 3 and centered at origin.
- To obtain the derivative of implicit function we apply the implicit differentiation.

Implicit Differentiation

- The chain rule can be applied to implicit relationships of the form $F(x, y) = 0$ between two variables, x and y.
- Let x be the single independent variable, then $F(x, y) = 0$ is a function of x and y in which both x and y are functions of x.
- We may determine the derivative of y with respect to x.
- Differentiating $F(x, y) = 0$ with respect to x gives

$$\frac{\partial F}{\partial x} \cdot \frac{dx}{dx} + \frac{\partial F}{\partial y} \cdot \frac{dy}{dx} = 0$$

  In other words, $\dfrac{\partial F}{\partial x} + \dfrac{\partial F}{\partial y} \cdot \dfrac{dy}{dx} = 0$

- Hence, $\dfrac{dy}{dx} = \dfrac{-\partial F/\partial x}{\partial F/\partial y}$

Implicit Differentiation (for one variable function)

Theorem

Suppose that $F(x, y)$ is differentiable and the function $F(x, y) = 0$ defines y as an implicit function x, $y = f(x)$ then

$$\frac{dy}{dx} = \frac{-F_x(x, y)}{F_y(x, y)}, where\ F_y \neq 0$$

Implicit Differentiation (for two variable function)

Theorem

Let $F(x, y, z) = 0$ defines an implicit function f with two variables, $z = f(x, y)$ then

$$\frac{\partial z}{\partial x} = \frac{-F_x(x, y, z)}{F_z(x, y, z)}$$

$$\frac{\partial z}{\partial y} = \frac{-F_y(x, y, z)}{F_z(x, y, z)}$$

provided that $F_x \neq 0, F_y \neq 0$

Example 4.7

Find $\frac{dy}{dx}$ if $y = f(x)$ defines the equation $y^2 - 3y - 2x^3 + 7x = 2$

Solution :

Let $F(x, y) = y^2 - 3y - 2x^3 + 7x - 2 = 0$

Using the theorem,

$$\frac{dy}{dx} = -\frac{F_x(x,y)}{F_y(x,y)}$$

$$= -\frac{-6x^2+7}{2y-3}$$

$$= \frac{6x^2-7}{2y-3}$$

Example 4.8

If y is a differentiable function of x such that $x^3 + 4x^2y - 3xy + y^2 = 0$, find $\frac{dy}{dx}$.

Solution:

Know that $\dfrac{dy}{dx} = \dfrac{-F_x}{F_y}$

Let $F(x, y) = x^3 + 4x^2y - 3xy + y^2$. So $F(x, y) = 0$. Then

$F_x = 3x^2 + 8xy - 3y$ and $F_y = 4x^2 - 3x + 2y$

Thus

$$\frac{dy}{dx} = \frac{-F_x}{F_y} = \frac{-(3x^2+8xy-3y)}{4x^2-3x+2y}$$

Solution (continue):

Alternatively, differentiating the given function implicitly yields

$$3x^2 + \left(8xy + 4x^2\frac{dy}{dx}\right) - \left(3y + 3x\frac{dy}{dx}\right) + 2y\frac{dy}{dx} = 0$$

Thus;

$$\frac{dy}{dx} = \frac{-(3x^2 + 8xy - 3y)}{4x^2 - 3x + 2y}$$

Example 4.9

If $\sin(x+y)+\cos(x-y)=y$, determine $\frac{dy}{dx}$.

Solution:

Know that $\frac{dy}{dx}=\frac{-F_x}{F_y}$

Let $F(x,y)=\sin(x+y)+\cos(x-y)-y$. So $F(x,y)=0$.

Then;

$$F_x=\cos(x+y)-\sin(x-y) \text{ and } F_y=\cos(x+y)+\sin(x-y)-1$$

Thus

$$\frac{dy}{dx}=\frac{-F_x}{F_y}=\frac{-\cos(x+y)-\sin(x-y)}{\cos(x+y)+\sin(x-y)-1}=\frac{-\cos(x+y)+\sin(x-y)}{\cos(x+y)+\sin(x-y)-1}$$

Example 4.10

Let $z = f(x, y)$. If $z^2xy + zy^2x + x^2 + y^2 = 5$.

Determine the expressions for $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

Solution:

Know that $\frac{\partial z}{\partial x} = \frac{-F_x}{F_z}, \frac{\partial z}{\partial y} = \frac{-F_y}{F_z}$

Let $F(x, y, z) = z^2xy + zy^2x + x^2 + y^2 - 5$. So $F(x, y, z) = 0$.

Then $F_x = z^2y + zy^2 + 2x$; $F_y = z^2x + 2zyx + 2y$ and $F_z = 2zxy + y^2x$

Thus

$$\frac{\partial z}{\partial x} = \frac{-F_x}{F_z} = \frac{-(z^2y+zy^2+2x)}{2zxy+y^2x} \text{ and } \frac{\partial z}{\partial y} = \frac{-F_y}{F_z} = \frac{-(z^2x+2zyx+2y)}{2zxy+y^2x}$$

Exercise 4.4:

Find $\frac{dy}{dx}$ if y an implicit function of x defines the given equation,

$xe^y + ye^{2x} + 2lnx = 3ln2.$

[Ans: $-\frac{\left(e^y+2ye^{2x}+\frac{2}{x}\right)}{xe^y+e^{2x}}$]

Exercise 4.5:

Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if $z = f(x, y)$ defines the equation $3xz - yz^2 + (xy)^2 + x^2z^3 - x^3 = y^2$

[Ans: $\frac{\partial z}{\partial x} = -\frac{3z+2xy^2+2xz^3-3x^2}{3x-2yz+3x^2z^2}, \frac{\partial z}{\partial y} = \frac{z^2-2x^2y+2y}{3x-2yz+3x^2z^2}$]

Exercise 4.6:

Suppose $e^{xyz} + z = 1 + e$ implicitly defines $z = f(x, y)$ as function of x and y.

i. Find f_x and f_y.

ii. Estimate $f(1.01,0.99)$ using linear approximation.

[Ans: i. $f_x = -\frac{yze^{xyz}}{1+xye^{xyz}}$; $f_y = -\frac{xze^{xyz}}{1+xye^{xyz}}$; ii. 1]

Reference

1) A.W. Md. Raji, H. Rahmat, I. Kamis, M.N. Mohamad, O. Chee Tiong (2009). Calculus for Science and Engineering Students, Penerbit UTHM.

2) Y. Mohammad Yusof, S. Baharun and R. Abdul Rahman. (2012). Multivariable Calculus for Independent Learners, Pearson Revised Second Edition.

THANK YOU