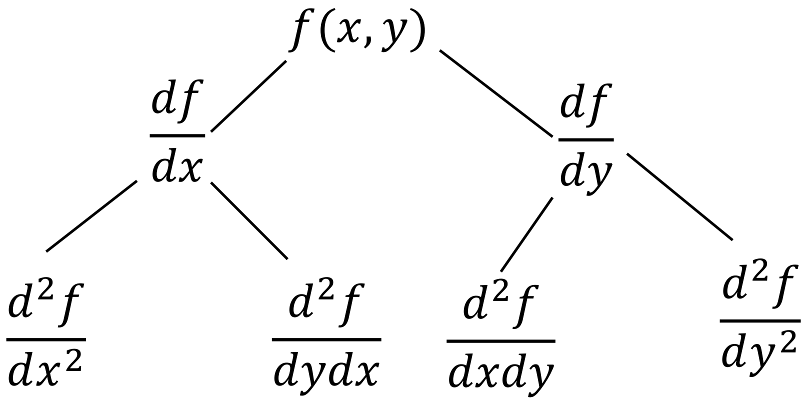
BEKG 2433
ENGINEERING MATHEMATICS 2

PARTIAL DERIVATIVES AND TOTAL DIFFERENTIAL



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Learning outcomes

Upon completion of this lesson, students should be able to:

- Find the partial derivatives for a given function.
- Solve the total differential for a given function at a given point.
- Evaluate the partial derivatives and total differential in solving engineering problems.

Partial Derivatives

- Partial differentiation is the process of differentiating a function of several variables with respect to one of its variables while keeping the other variables fixed.
- The partial derivatives is the resulting derivative of the function.
- Partial Derivatives of f with **respect to x** can be written as f_x or $\frac{\partial f}{\partial x}$.
- Partial derivative of f with **respect to y** can be written as f_y or $\frac{\partial f}{\partial y}$.

Partial Derivatives

Definition:

Let $z = f(x, y)$ function of two variables.

Then, 1st order **partial derivative** $f(x, y)$ with respect **to** $\boldsymbol{x}$:

$$\frac{\partial f}{\partial x} = \lim_{\delta x \to 0} \frac{f(x + \delta x, y) - f(x, y)}{\delta x}$$

1st order **partial derivative** $f(x, y)$ with respect **to** $\boldsymbol{y}$:

$$\frac{\partial f}{\partial y} = \lim_{\delta y \to 0} \frac{f(x, y + \delta y) - f(x, y)}{\delta y}$$

with the limits exist.

Notation of Partial Derivatives

- $\frac{\partial}{\partial x}$ to denote the partial derivative with respect to x.

- Other notation for the partial derivative of $f(x, y)$ with respect to x are:

$$\frac{\partial z}{\partial x}, \frac{\partial}{\partial x} f(x, y), f_x(x, y), z_x$$

- The values of the partial derivatives at the point (a, b):

$$\left.\frac{\partial f}{\partial x}\right|_{(a,b)} \text{ or } f_x(a, b)$$

Notation of Partial Derivatives

- ∂ means partial derivatives rather than d for ordinary derivatives.

Conclusion:

f_x: **differentiate f with respect to x** by assume y is constant.

f_y: **differentiate f with respect to y** by assume x is constant.

Example 3.1

Find $f_x(1,0)$ and $f_y(1,0)$ of $f(x,y) = x^2 + 3x^2y + 4y^2 - xy + 2$.

Solution:

$$\frac{\partial f}{\partial x} = 2x + 6xy - y$$

$$f_x(1,0) = 2(1) + 6(1)(0) - 0$$
$$= 2$$

$$\frac{\partial f}{\partial y} = 3x^2 + 8y - x$$

$$f_y(1,0) = 3(1) + 8(0) - 1$$
$$= 2$$

Example 3.2

Find $f_x(1,0)$ and $f_y(1,0)$ of $f(x,y) = \frac{\cos(\pi x+y)}{1+x+y}$

Solution:

$u = cos(\pi x + 2y)$ $\qquad$ $v = 1 + x + y$

$\frac{\partial u}{\partial x} = -\pi \, sin(\pi x + 2y)$ $\qquad$ $\frac{\partial v}{\partial x} = 1$

$$\frac{\partial f}{\partial y} = \frac{(1 + x + y)[-\pi sin(\pi x + 2y)] - cos(\pi x + 2y)}{(1 + x + y)^2}$$

. $f_x\,(1,0) = \frac{1}{4}$

Solution:

$u = cos(\pi x + 2y)$ $\qquad$ $v = 1 + x + y$

$\frac{\partial u}{\partial y} = -2\, sin(\pi x + 2y)$ $\qquad$ $\frac{\partial v}{\partial x} = 1$

$$\frac{\partial f}{\partial y} = \frac{(1 + x + y)[-2sin(\pi x + 2y)] - cos(\pi x + 2y)}{(1 + x + y)^2}$$

$f_y(1,0) = \frac{1}{4}$

Example 3.3

Find $f_x(1,0)$ and $f_y(1,0)$ of $f(x, y) = x\ln(y+1) + e^{2y}\tan x$

Solution:

$$\frac{\partial f}{\partial x} = \ln(y+1) + e^{2y} sec^2 x$$

$$f_x(1,0) = 1$$

$$\frac{\partial f}{\partial y} = \frac{x}{y+1} + 2e^{2y}\tan x$$

$$f_y(1,0) = 4.1148$$

Example 3.4

Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ if the equation $z = f(x, y)$ defines $yz + xz^3 - 2z^2 = 3$

Solution:

$$\frac{\partial}{\partial x}[yz] + \frac{\partial}{\partial x}[xz^3] - \frac{\partial}{\partial x}[2z^2] = \frac{\partial}{\partial x}[3]$$

$$y\frac{\partial z}{\partial x} + z^3 + 3xz^2\frac{\partial z}{\partial x} - 4z\frac{\partial z}{\partial x} = 0$$

$$\frac{\partial z}{\partial x} = \frac{-z^3}{y+3xz^2-4z}$$

Solution continue:

$$\frac{\partial}{\partial y}[yz] + \frac{\partial}{\partial y}[xz^3] - \frac{\partial}{\partial y}[2z^2] = \frac{\partial}{\partial y}[3]$$

$$z + y\frac{\partial z}{\partial y} + 3xz^2\frac{\partial z}{\partial y} - 4z\frac{\partial z}{\partial y} = 0$$

$$\frac{\partial z}{\partial y} = \frac{-z}{y + 3xz^2 - 4z}$$

Higher Order Partial Derivatives

Let $f(x, y)$ is the function with two variables.

If the derivative more then one, then $f(x, y)$ become

$$f_{xx} = \frac{\partial}{\partial x} f_x = \frac{\partial}{dx}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}$$

$$f_{xy} = \frac{\partial}{\partial y} f_x = \frac{\partial}{dy}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}$$

Higher Order Partial Derivatives

Let $f(x, y)$ is the function with two variables.

If the derivative more then one, then $f(x, y)$ become

$$f_{yx} = \frac{\partial}{\partial x} f_y = \frac{\partial}{dx}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y}$$

$$f_{yy} = \frac{\partial}{\partial y} f_y = \frac{\partial}{dy}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2}$$

Higher Order Partial Derivatives

Notes: $f_{xy} = (f_x)_y \Rightarrow$ Differentiate from left to right

$\frac{\partial^2 f}{\partial x \partial y} \Rightarrow$ Differentiate from right $\left(\frac{\partial}{dy}\right)$ to left $\left(\frac{\partial}{\partial x}\right)$.

Example 3.5

Given the following functions. Find the $f_{xx}, f_{xy}, f_{yy}, f_{yx}$

a) $f(x,y) = x^5y^3 + x^2y^4.$

b) $f(x,y) = x\ln y + e^x \sin y$

Solution :

(a)

$$f_x = 5x^4y^3 + 2xy^4 \quad ; \quad f_{xx} = 20x^3y^3 + 2y^4 \quad ; f_{xy} = 15y^2x^4 + 8xy^3$$

$$f_y = 3y^2x^5 + 4y^3x^2 \quad ; \quad f_{yy} = 6yx^5 + 12y^2x^2 \quad ; \quad f_{yx} = 15y^2x^4 + 8xy^3$$

(b)

$$f_x = \ln y + e^x \sin y \quad ; \quad f_{xx} = e^x \sin y \quad ; \quad f_{xy} = \frac{1}{y} + e^x \cos y$$

$$f_y = \frac{x}{y} + e^x \cos y \, ; \quad f_{yy} = -\frac{x}{y^2} - e^x \sin y \quad ; \quad f_{yx} = \frac{1}{y} + e^x \cos y$$

Example 3.6

Find the $f_{xx}, f_{xy}, f_{yy}, f_{yx}$ for the function $f(x,y) = \ln(2x+3y)$. Determine the value of f_{yy} and f_{xx} when $x = 3$ and $y = 5$.

Solution:

$$f_x = \frac{2}{2x+3y} \quad ; \quad f_{xx} = \frac{-4}{(2x+3y)^2} \quad ; \quad f_{xy} = \frac{-6}{(2x+3y)^2}$$

$$f_y = \frac{3}{2x+3y} \quad ; \quad f_{yy} = \frac{-9}{(2x+3y)^2} \quad ; \quad f_{yx} = \frac{-6}{(2x+3y)^2}$$

When $x = 3$ and $y = 5$;

$$f_{xx} = \frac{-4}{[2(3)+3(5)]^2} = \frac{-4}{441}$$

$$f_{yy} = \frac{-9}{[2(3)+3(5)]^2} = \frac{-9}{441}$$

Example 3.7

Find the f_{xx} and f_{yy} for the function $f(x,y) = \frac{y^2+x^2}{x-y}$

Solution:

$$f_x = \frac{x^2 - 2xy - y^2}{(x-y)^2} \quad ; \quad f_{xx} = \frac{(2x-2)(x-y) - 2(x^2 - 2xy - y^2)}{(x-y)^3}$$

$$f_y = \frac{2xy - y^2 + x^2}{(x-y)^2} \quad ; \quad f_{yy} = \frac{2(x-y)^2 + 2(2xy - y^2 + x^2)}{(x-y)^3}$$

Example 3.8

Given the following functions. Find the f_{xyz} for both functions.

a) $f(x,y,z) = z^2 e^{5x} \sin 2y$

b) $f(x,y,z) = x^3 cosyz$

Solution:

(a) $f_x = 5e^{5x}z^2 \sin 2y$; $f_{xy} = 10e^{5x}z^2 \cos 2y$; $f_{xyz} = 20e^{5x}z \cos 2y$

(b) $f_x = 3x^2 \cos yz$; $f_{xy} = -3x^2 z \sin yz$; $f_{xyz} = -3x^2 \sin yz$

Total Differential

- Let $P(x, y)$ at surface $z = f(x, y)$. If there have change of x and y,then there have change in z.
- This change in z called total differential of $f(x, y)$.

Definition:

Let $z = f(x, y)$, $f_x(x, y)$ and $f_y(x, y)$ exist. If δx and δy denotes the change in the values of x and y respectively, then the corresponding change of z, δz called the total differential of f as given by;

$$\delta z = f_x \delta x + f_y \delta y$$

Example 3.9

Let $z = f(x, y) = 3x^2y - y^2$, use partial derivatives to estimate the change, δz if the values of (x, y) change from $(1,2)$ to $(1.02,1.97)$

Solution:

The partial derivatives is $f_x = 6xy, f_y = 3x^2 - 2y$

The change of values of x and y

$$\delta x = 1.02 - 1 = 0.02; \delta y = 1.97 - 2 = -0.03$$

Hence, the change in z; $\delta z = f_x\delta x + f_y\delta y$

$$= 6xy(\delta x) + (3x^2 - 2y)\delta y$$

$$= 6(1)(2)(0.02) + [3(1)^2 - 2(1)](-0.03)$$

$$= 0.27$$

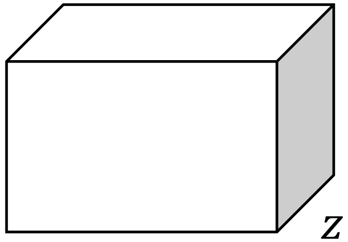
Total Differential for function with three variables

Let $w = f(x, y, z)$ with f_x, f_y and f_z exist. If $\delta x, \delta y$ and δz denotes the changes in values of x, y and z value respectively, thus the change value in w, δw also called the total differential of f is given as

$$\delta w = f_x \delta x + f_y \delta y + f_z \delta z$$

Example 3.10

The dimension (in cm) of a box changes from 9, 6, 2 to 8.98,6.03,1,99. Use partial derivatives to estimate the increment of its volume.



Solution:

Volume of a box, $V(x, y, x) = xyz$

$$V_x = yz \,;\, V_y = xz \;\;;\, V_z = xy$$

$$\delta_x = 8.98 - 9.0 = -0.02 \,;\; \delta_y = 6.03 - 6.0 = 0.03 \;\;;\delta_z = 1.99 - 2.0 = -0.01$$

$$\delta V = V_x\delta_x + V_y\delta_y + V_z\,\delta_z$$

$$= yz\delta x + xz\delta y + xy\delta z$$

$$= 6(2)(-0.02) + 9(2)(0.03) + 9(6)(-0.01)$$

$$= -0.24 \text{ (the volume of the box is reduced)}$$

Example 3.11

The measurement of radius and height of a cone are known to have a maximum error $0.1cm$, in each measurement. Use partial derivatives to approximate the maximum possible error in the calculated value of the volume of the cone if the radius and height to be 5cm and 20cm.

Solution:

Volume of a cone, $V = \frac{1}{3}\pi r^2 h$

Given $\delta r = 0.1$; $\delta h = 0.1$

$\frac{\partial V}{\partial r} = \frac{2}{3}\pi rh$; $\frac{\partial V}{\partial h} = \frac{1}{3}\pi r^2$

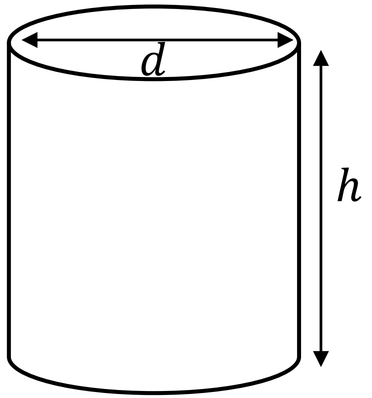
$$\begin{aligned}\delta V &= V_r \delta r + V_h \delta h \\ &= \frac{2}{3}\pi(5)(20)(0.1) + \frac{1}{3}\pi(5^2)(0.1) \\ &= 7.5\pi\ cm^3\end{aligned}$$

(the volume of the cone is increasing)

Example 3.12

The diameter of the base and the height of a cylindrical are measured, the measurements are known to have errors of at most 0.5cm. If the diameter and height are taken to be $4cm$ and $6cm$ respectively. Estimate the maximum possible error in

a. The volume V of the cylinder
b. The surface area S of the cylinder



Solution:

Volume of a cylinder, $V = \pi r^2 h$

Area of a cylinder, $A = 2\pi r^2 + 2\pi rh$

Diameter, $d = 2r$, hence $r = \frac{d}{2}$

Thus; $V = \frac{1}{4}\pi d^2 h$ and $A = \frac{1}{2}\pi d^2 + \pi dh$

Solution:

(a) Maximum error in the volume of the cylinder:

$$V_d = \frac{1}{2}\pi dh \ ; V_h = \frac{1}{4}\pi d^2$$

$$\delta V = V_d \delta d + V_h \delta h$$

$$= \frac{1}{2}\pi(4)(6)(0.5) + \frac{1}{4}\pi(4^2)(0.5)$$

$$= 6\pi + 2\pi$$

$$= 8\pi \ cm^3$$

(b) Maximum error in the area of the cylinder:

$$A_d = \pi d + \pi h \ ; A_h = \pi d$$

$$\delta A = A_d \delta d + A_h \delta h$$

$$= (4\pi + 6\pi)(0.5) + (4\pi)(0.5)$$

$$= 7\pi \ cm^2$$

Example 3.13

Suppose that a cylinder can is designed to have a radius of 1 inch. And a height of 5 inch. But that the radius and height are off by the amounts $\delta r = 0.03$ and $\delta h = -0.1$. Estimate the resulting, relative and percentage changes in the volume of the can.

Solution:

Volume of a cylinder, $V = \pi r^2 h$

$$\frac{\partial V}{\partial r} = 2\pi r h \quad ; \frac{\partial V}{\partial h} = \pi r^2$$

$$\begin{aligned}\partial V &= \frac{\partial V}{\partial r}(\delta r) + \frac{\partial V}{\partial h}(\delta h) \\ &= 2\pi(1)(5)(0.03) + \pi(1)(-0.1) \\ &= 0.2\pi\end{aligned}$$

The resulting changes in volume, $V = 0.2\pi$

Relative change; $\frac{\delta V}{V} = \frac{0.2\pi}{\pi(1)(5)} = 0.04$

Percentage of change; $\frac{\delta V}{V} \times 100 = 4\%$

Exercise 3.1:

1. If $f(x,y) = x^3y + x^2y^2 + 4x$, find

   a. $\frac{\partial f}{\partial x}$ b. $\frac{\partial f}{\partial y}$ c. $f_y(1,-2)$

2. Suppose $f(x,y,z) = x^2 + 2xy^2 + yz^3$ find

   a. f_x b. f_y c. f_z

3. Find $\frac{\partial f}{\partial y}$ if $f(x,y) = ln(x+y)$

[Ans: 1.(a) $3x^2y + 2xy^2 + 4$ (b)$x^3 + 2x^2y$ (c) -3 2. (a) $2x + 2y^2$ (b) $4xy + z^3$ (c) $3z^2y$ 3. $\frac{1}{x+y}$]

Exercise 3.2:

1. Find $\frac{\partial f}{\partial x}$ if $f(x, y) = x \sin xy$
2. Find f_x and f_y if $f(x, y) = \frac{2y}{y+cosx}$
3. Find $\frac{\partial z}{\partial x}$ if the equation $yz - lnz = x + y$ defines z as a function of two independent variables x and y.

[Ans: (1) $xy \cos xy + \sin xy$ (2) $f_x = \frac{2y \sin x}{(y+\cos x)^2}$; $f_y = \frac{2 \cos x}{(y+\cos x)^2}$ (3) $\frac{z}{yz-1}$]

Exercise 3.3:

Use partial derivatives to find the approximation to $\sqrt{3.94}\left(\sqrt[3]{8.02}\right)$

[Ans: -0.027]

Exercise 3.4:

The resistance R produced by wiring resistors of R_1 and R_2 ohms in parallel (see Figure 1) can be calculated from the formula $\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}$.

a. Show that $dR = \left(\frac{R}{R_1}\right)^2 dR_1 + \left(\frac{R}{R_2}\right)^2 dR_2$

b. In another circuit shown like Figure 1, you plan to change R_1 from 20 to 20.1 ohms and R_2 from 25 to 24.9 ohms. By about what percentage will this change R?

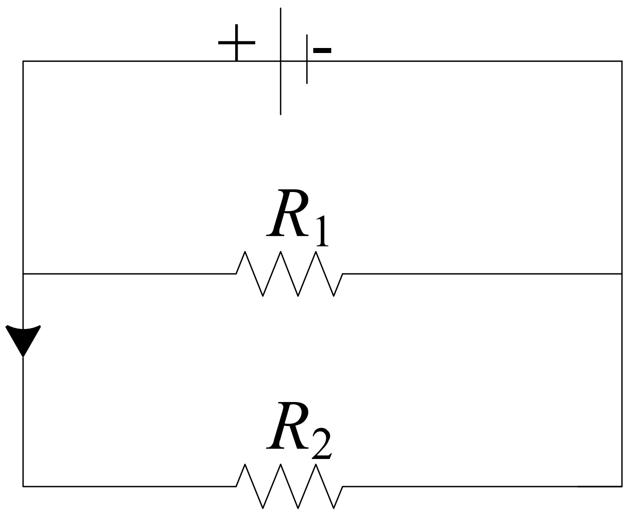


Figure 1

[Ans: (b) 0.1%]

Reference

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THANK YOU