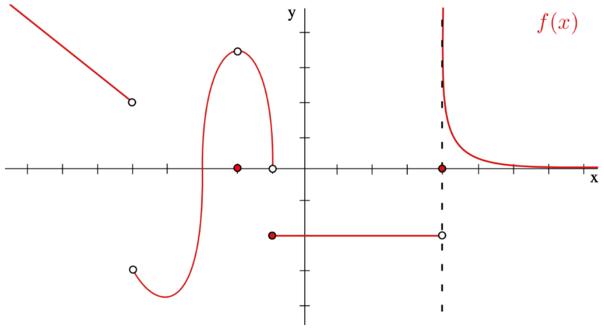
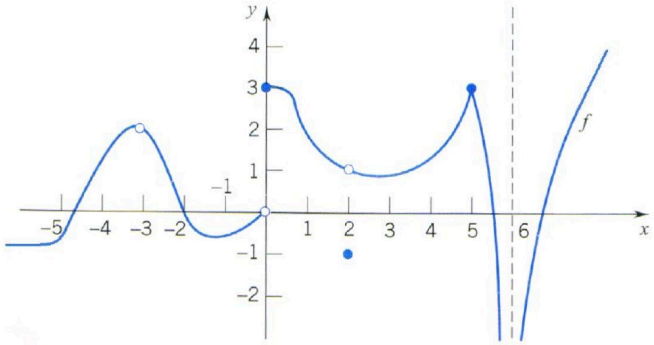
BEKG 2433
ENGINEERING MATHEMATICS 2

LIMITS AND CONTINUITY





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Learning outcomes

Upon completion of this lesson, students should be able to:

- Define the limit for a given function.
- Evaluate the limit for a given function at a given point.
- Evaluate the continuity of a given function.

Limits and Continuity

- The continuity at a point depends on the function's behaviour near the point.
- To study behaviour near a point, we need the idea of a limit of the function.
- The definition of the limit of a function of two or three variables is similar to the definition of the limit of a function of a single variable but with some significant differences.
- We will not go into great detail in discussing the concepts of limits and continuity.

Limits

For a function with one variable:

$$\lim_{x \to a} f(x) = L$$

Provided that:

$$\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = L$$

Move from the right ($x \to a^+$) — Move from the left ($x \to a^-$)

In this case, there are only two paths that we can take as we move in towards $x = a$.

We can either move in from the left or the right.

Limits

- What does it mean for $f(x, y)$ to have a limit L as (x, y) approaches (a, b)?

- In taking a limit of a function of two variables, we want to determine the value of $f(x, y)$ as we move the point (x, y) closer to the point (a, b) without actually letting it be (a, b).

Definition

- The function $f(x, y)$ has a limit L at the point (a, b), written as

$$\lim_{(x,y)\to(a,b)} f(x, y) = L$$

if $f(x, y)$ is close to a fixed real number L for all points (x, y) that are sufficiently close to the point $(a. b)$ but not equal to (a, b).

Theorem

- If $\lim_{(x,y)\to(a,b)} f(x,y) = L$, then $\lim_{(x,y)\to(a,b)} f(x,y) = L$ along any smooth curve.

The theorem says that if the limit exists, then $f(x, y)$ must approach the same limit no matter how (x, y) approaches the point (a, b).

The two-path test

If

$\lim_{(x,y)\to(a,b)} f(x,y) = L_1$, along a path C_1 and

$\lim_{(x,y)\to(a,b)} f(x,y) = L_2$, along a path C_2 with $L_1 \neq L_2$,

then;

$$\lim_{(x,y)\to(a,b)} f(x,y) \text{ does not exist.}$$

Squeeze theorem

Assume $g(x,y) \leq f(x,y) \leq h(x,y)$ for all (x,y) near (a,b).

Assume $\lim_{(x,y)\to(a,b)} g(x,y) = L = \lim_{(x,y)\to(a,b)} h(x,y)$

Then $\lim_{(x,y)\to(a,b)} f(x,y) = L$

Mathematical Operation

Similar to the single variable limit, limits of multivariable functions can be

- Added
- Subtracted
- Multiplied
- Composed
- Divided (provided the limit of the denominator is not zero)

Theorem for properties of two variables function

Let $\lim_{(x,y)\to(a,b)} f(x,y) = A$ and $\lim_{(x,y)\to(a,b)} g(x,y) = B$

1. $\lim_{(x,y)\to(a,b)} cf(x,y) = \mathrm{c} \lim_{(x,y)\to(a,b)} f(x,y) = cA$

2. $\lim_{(x,y)\to(a,b)} [f(x,y) \pm g(x,y)] = \lim_{(x,y)\to(a,b)} f(x,y) \pm \lim_{(x,y)\to(a,b)} g(x,y) = A \pm B$

3. $\lim_{(x,y)\to(a,b)} [f(x,y) \cdot g(x,y)] = \lim_{(x,y)\to(a,b)} f(x,y) \cdot \lim_{(x,y)\to(a,b)} g(x,y) = AB$

Theorem for properties of two variables function

Let $\lim_{(x,y)\to(a,b)} f(x,y) = A$ and $\lim_{(x,y)\to(a,b)} g(x,y) = B$

4. $\lim_{(x,y)\to(a,b)} \left[\frac{f(x,y)}{g(x,y)}\right] = \frac{\lim_{(x,y)\to(a,b)} f(x,y)}{\lim_{(x,y)\to(a,b)} g(x,y)} = \frac{A}{B}$, if $\lim_{(x,y)\to(a,b)} g(x,y) \neq 0$

5. $\lim_{(x,y)\to(a,b)} \sqrt[n]{f(x,y)} = \sqrt[n]{\lim_{(x,y)\to(a,b)} f(x,y)} = \sqrt[n]{A}$ if $\sqrt[n]{A}$ is defined.

6. $\lim_{(x,y)\to(a,b)} f(x,y) = \mathrm{c}$, c a constant, $\lim_{(x,y)\to(a,b)} f(x,y) = c$.

Example 2.1

Find the limit for

$$\lim_{(x,y)\to(5,-2)} (x^5 + 4x^3y - 5xy^2)$$

Solution:

$$\lim_{(x,y)\to(5,-2)} (x^5 + 4x^3y - 5xy^2)$$
$$= (5)^5 + 4(5)^3(-2) - 5(5)(-2)^2$$
$$= 3125 + 4(125)(-2) - 5(5)(4)$$
$$= 2025$$

Example 2.2

Find the limit if it exists.

$$\lim_{(x,y)\to(1,1)} \frac{x^2 - y^2}{x - y}$$

Solution:

Simplify $f(x,y)$ as $f(x,y) = \frac{x^2-y^2}{x-y} = \frac{(x-y)(x+y)}{x-y} = x + y$, for $x \neq y$

Thus, $\lim_{(x,y)\to(1,1)} \frac{x^2-y^2}{x-y} = \lim_{(x,y)\to(1,1)} x + y = 1 + 1 = 2$

Example 2.3

Verify the following limits computations:

a) $$\lim_{(x,y)\to(2,3)} \frac{2x^2+3y^2}{5xy+4y}$$

b) $$\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x+y}$$

Solution (a):

First, consider the denominator of the limit:

$$\lim_{(x,y)\to(2,3)} (5xy + 4y) = (5)(2)(3) + 4(3) = 42 \neq 0$$

thus, we have

$$\lim_{(x,y)\to(2,3)} \frac{2x^2 + 3y^2}{5xy + 4y} = \frac{\lim_{(x,y)\to(2,3)} 2x^2 + 3y^2}{\lim_{(x,y)\to(2,3)} 5xy + 4y} = \frac{35}{42}$$

Solution (b):

The limit in the denominator is:

$$\lim_{(x,y)\to(0,0)} (x+y) = 0 + 0 = 0$$

Simplify the function

$$\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x+y} = \lim_{(x,y)\to(0,0)} \frac{(x+y)(x-y)}{x+y} = \lim_{(x,y)\to(0,0)} (x-y) = 0$$

The existence of a limit

- We can define the $\lim_{(x,y)\to(a,b)} f(x,y)$ exists if and only of the limits along every direction as (x,y) approaches to (a,b) are equivalent.
- Then, to show that $\lim_{(x,y)\to(a,b)} f(x,y)$ does not exist in (a,b), we only need to show that the function has different limits along two different directions when (x,y) approaches to (a,b).

Example 2.4

Show that $\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{x^2-y^2}$ does not exist.

Solution:

Let $x = 0$, thus determine the limit as $y \to 0$.

$$\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{x^2-y^2} = \lim_{(x,y)\to(0,0)} \frac{y^2}{-y^2} = \lim_{(x,y)\to(0,0)} -1 = -1$$

Next, $y = 0$, thus determine the limit as $x \to 0$.

$$\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{x^2-y^2} = \lim_{(x,y)\to(0,0)} \frac{x^2}{x^2} = \lim_{(x,y)\to(0,0)} 1 = 1$$

Since the limits from two different directions are not the same, therefore $\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{x^2-y^2}$ does not exist.

Example 2.5

Show that $\lim_{(x,y)\to(0,0)} \frac{xy}{x^2-y^2}$ does not exist.

Solution:

Consider $x = 0$, compute the limits $y \to 0$.Thus,

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2 - y^2} = \lim_{(x,y)\to(0,0)} \frac{0}{-y^2} = \lim_{(x,y)\to(0,0)} 0 = 0$$

$y = 0$, compute the limits $x \to 0$.Thus,

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2 - y^2} = \lim_{(x,y)\to(0,0)} \frac{0}{-x^2} = \lim_{(x,y)\to(0,0)} 0 = 0$$

Solution continue:

However, if the $x = 0$ and $y = 0$ get the same value does not means the limits exists.

Look at other direction, $y = 2x$

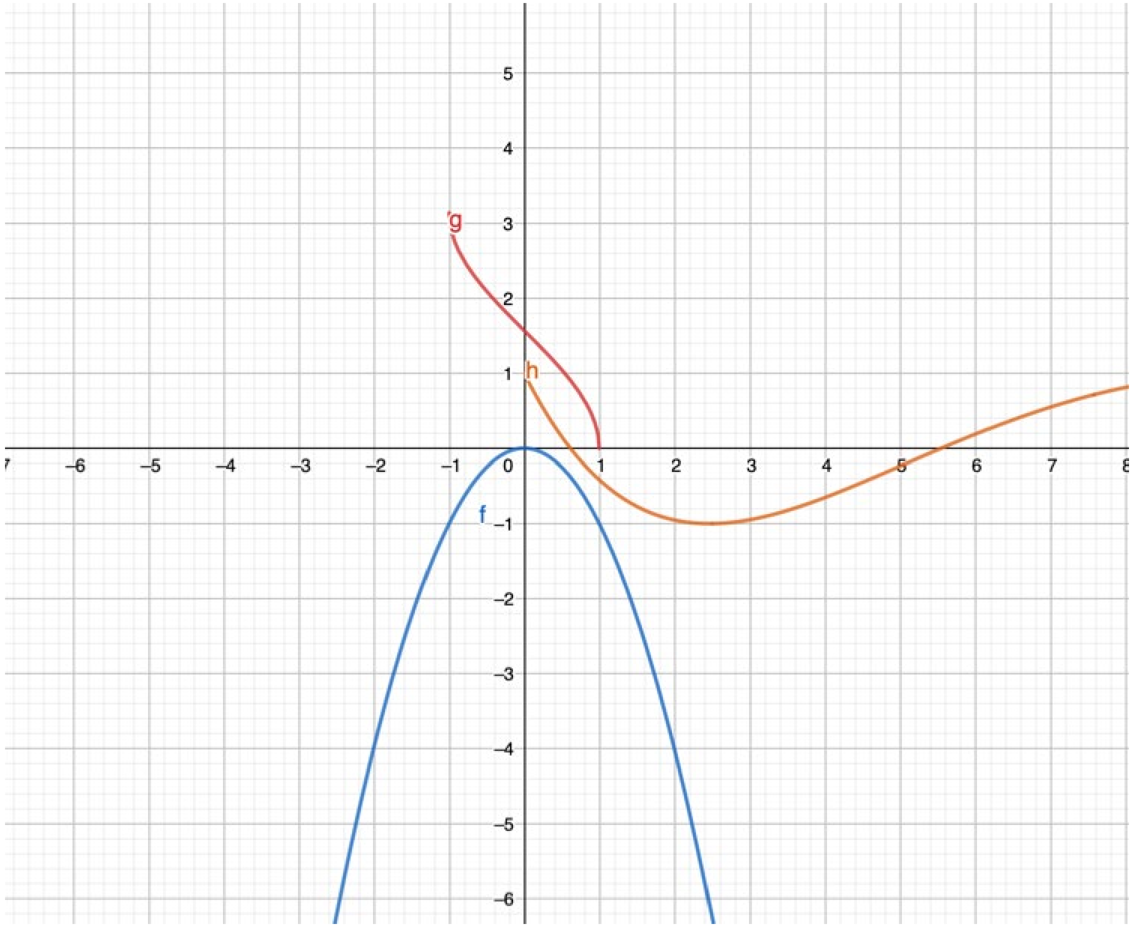
$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2 - y^2} = \lim_{(x,y)\to(0,0)} \frac{2x^2}{-3x^2} = \lim_{(x,y)\to(0,0)} -\frac{2}{3} = -\frac{2}{3}$$

Since the limits from two different directions are not the same,

$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2-y^2}$ does not exist.

Continuity

- A function of two variables is continuous if it represents a surface **without any holes, tears or gaps**.
- Small changes in the independent variable will result in small changes in the dependent variable.



Continuity

A function $f(x, y)$ is defined to be continuous at (a, b) if the following three conditions hold.

i. $f(a, b)$ is define

ii. $\lim_{(x,y)\to(a,b)} f(x, y)$ exists

iii. $\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b)$

Continuity

Some common examples of continuous functions are:

- Polynomial functions are continuous in $\Re^2$

  for example $f(x, y) = x + 2xy + 5y^2$
- Rational functions are continuous on their domain

  for example $f(x, y) = \frac{x-y}{x^2+y^2}$, $(x, y) \neq (0,0)$
- Composition of continuous functions are continuous

  for example $f(x, y) = \sin(x^2 + y^2)$

Example 2.6

Find the area R in which the following function is continuous:

$$f(x,y) = \sqrt{x-5}\,\ln(y+2)$$

Solution:

x: the value of x is real for $x - 5 \geq 0 \Rightarrow x \geq 5$
y: the value of y is real for $y + 2 > 0 \Rightarrow y > -2$
Thus the area $R = \{(x,y): (x,y) \in \mathcal{R}, x \geq 5, y > -2\}$

Example 2.7

Show that $f(x,y) = \begin{cases} \frac{xy}{x^2-y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$ is not continuous at the origin.

Solution:

Condition 1: $f(0,0) = 0$. Thus, the value of $f(x,y)$ is defined at $(0,0)$

Condition 2: Refer to Example 2.5, limit of $f(x,y)$ does not exist.

Condition 3: $\lim_{(x,y) \to (0,0)} f(x,y) \neq f(0,0)$ not exist at $(0,0)$.

Since the limit is fail to exist when (x,y) approaches to $(0,0)$, therefore the function is not continuous at origin.

Example 2.8

Show that $f(x, y) = 3(x + y)^2$ is continuous at the origin.

Solution:

Condition 1: $f(0,0) = 0$. Thus, the value of $f(x, y)$ is defined at $(0,0)$

Condition 2: Is $\lim_{(x,y)\to(a,b)} f(x, y)$ exists?.

Let $z = x + y$. Therefore $f(z) = 3z^2$.

Thus it can be shown that $\lim_{z\to0^+} 3z^2 = \lim_{z\to0^-} 3z^2 = 0$,

that is $\lim_{z\to0} 3z^2 = 0$

Solution continue:

Condition 3:

From Condition 1 and 2, $\lim_{(x,y)\to(0,0)} f(x,y) = f(0,0)$.

Since all the conditions are fulfill, therefore the function is continuous at origin.

Exercise 2.1:

Find the limits of the functions:

(a) $\lim_{(x,y)\to(1,1)} (xy^2 + x^2y + 5)$

(b) $\lim_{(x,y)\to(1,3)} \frac{x+y}{x-y}$

(c) $\lim_{(x,y)\to(1,0)} e^{xy}$

(d) $\lim_{(x,y)\to(1,1)} ln|1 + x^2y^2|$

[Ans: (a) 7 (b) -2 (c) 1 (d) ln 2]

Exercise 2.2:

Evaluate the limits, if it is exists.

(a) $\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$

(b) $\lim_{(x,y)\to(0,2)} \frac{\sin xy}{x}$

[Ans: (a) limit does not exists (b) 2]

Exercise 2.3:

Find the points of discontinuity of the functions

(a) $f(x,y) = ln\sqrt{x^2 + y^2}$

(b) $f(x,y) = \frac{1}{(y-x)^2}$

[Ans: (a) $(x,y) = (0,0)$ (b) All points where $y = x$]

Reference

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2) Y. Mohammad Yusof, S. Baharun and R. Abdul Rahman. (2012). Multivariable Calculus for Independent Learners, Pearson Revised Second Edition.

THANK YOU