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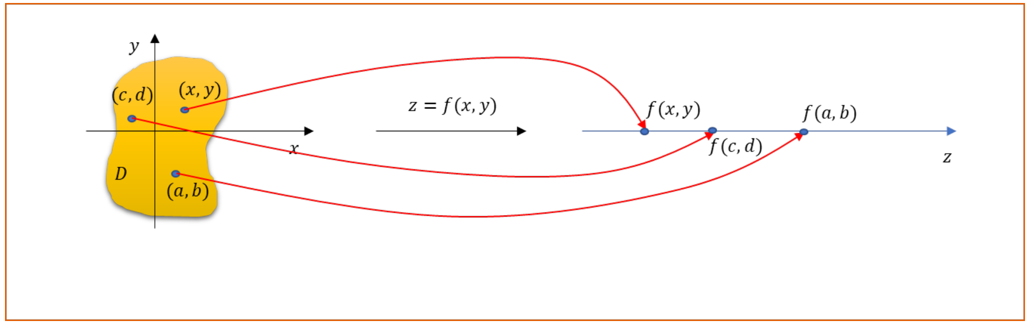
Universiti Teknikal Malaysia Melaka

BEKG 2433
ENGINEERING MATHEMATICS 2

FUNCTIONS OF SEVERAL VARIABLES



Rahifa Ranom
rahifa@utem.edu.my

Lesson Outcomes

Upon completion of this lesson, students should be able to:

- know the concept of a function of more than one variable
- use coordinate system to analyze the graphs of such functions

- In Calculus, YOU have dealt with the calculus of functions of a single variable.
- However, in the real world, physical quantities often depend on two or more variables.
- Examples:
  - The temperature T at a point on the surface of the earth at any given time depends on the longitude x and latitude y of the point, $T(x, y)$.
  - The volume V of a circular cylinder depends on its radius r and its height h, $V = \pi r^2 h$.
  - The potential difference between two points which include a resistance R, $V = IR$.

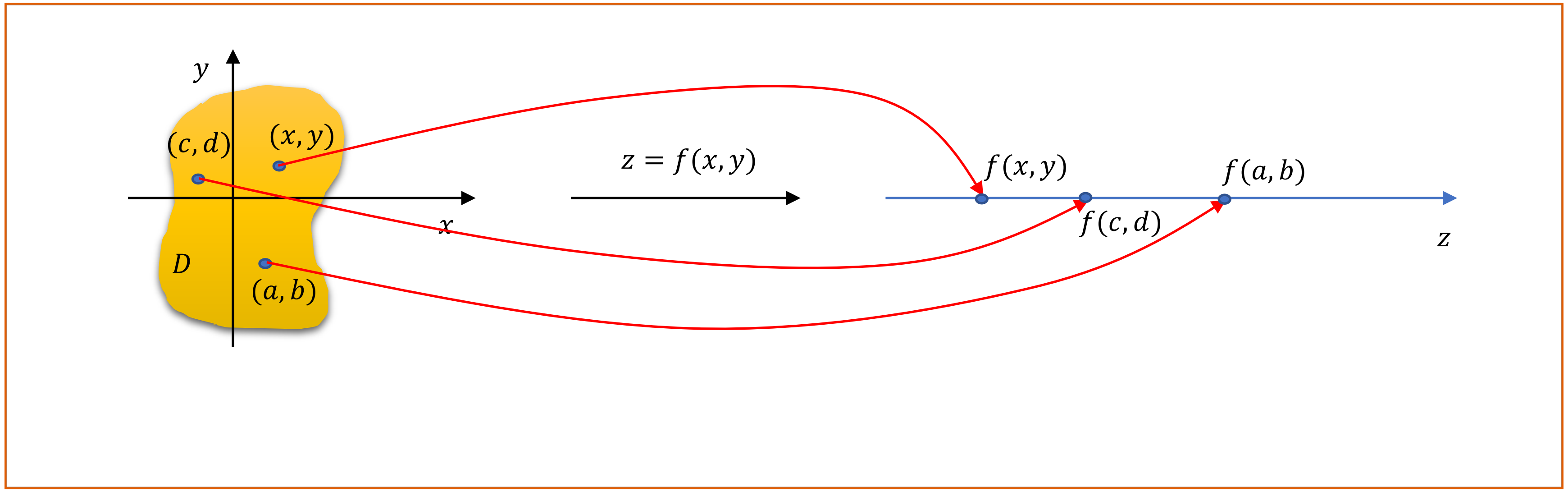
Functions with Two Variables

Definition:

Suppose D is a set of order pairs of real numbers, (x, y) which is the domain of a function with 2 independent variables x and y denotes as $z = f(x, y)$. The set of z −values that associates with the (x, y) in D is the range of the function.

Domain (D): The set of all pairs (x, y) for which the given expression is a well-defined real number.

One way of visualizing such a function is by means of an arrow diagram, where the domain D is represented as a subset of the xy −plane.



Functions with Two Variables

Example 1.1:

For $f(x,y) = 2x^3 + \sqrt{y}$. Find $f(-1,9)$, $f\left(\frac{1}{2},0\right)$, domain and range of f.

Solution:

By substituting the values of x and y into $f(x,y)$, we obtain

$$f(-1,9) = 2(-1)^3 + \sqrt{9} = -2 + 3 = 1$$

$$f\left(\frac{1}{2},0\right) = 2\left(\frac{1}{2}\right)^3 + \sqrt{0} = 2\left(\frac{1}{8}\right) = \frac{1}{4}$$

Domain: x can be any values but y is defined only for $y \geq 0$. Hence the domain of $f(x,y)$ is the set $D_f = \{(x,y): x, y \in \Re, y \geq 0\}$.

Range: The values of $2x^3$ and $\sqrt{y}$ are real numbers and nonnegative values, respectively. Hence the range of $f(x,y)$ is the set $R_f = \{f(x,y) \in \Re\}$.

Functions with Two Variables

Example 1.2:

For $f(x,y) = (x-2)\ln y + e^x$. Find $f(0,e^2)$, domain and range of f.

Solution:

By substituting the values of x and y into $f(x,y)$, we obtain

$$f(0,e^2) = (0-2)\ln e^2 + e^0 = -2(2) + 1 = -3$$

Domain: x can be any values but y is defined only for $y > 0$. Hence the domain of $f(x,y)$ is the set $D_f = \{(x,y): x, y \in \Re, y > 0\}$.

Range: The values of $(x-2)\ln y + e^x$ are real numbers, respectively. Hence the range of $f(x,y)$ is the set $R_f = \{f(x,y) \in \Re\}$.

Functions with Two Variables

Example 1.3:

For $f(x,y) = 2y^2 + \sin 2x$. Find $f(0, e^2)$, domain and range of f.

Solution:

Domain: x and y can be any values. Hence the domain of $f(x,y)$ is the set

$$D_f = \{(x,y): x, y \in \Re\} .$$

Range: The values of $2y^2$ are greater than 0 $(2y^2 \geq 0)$ while the range of $\sin 2x$ is from -1 to 1 $(-1 \leq \sin 2x \leq 1)$. Hence the addition of these two terms determine the range of $f(x,y), R_f = \{f(x,y): f(x,y) \in \Re, f(x,y) \geq -1\}$.

Exercise 1.1:

Find the values at the given point and determine the domain and range of the following functions;

1. $f(x,y) = 1 - \cos xy^2,\ (\pi, -1)$
2. $f(x,y) = \frac{1}{x^3 - y},\ (-2,1)$

[Ans: 1. $D_f = \{(x,y): x, y \in \Re\}, R_f = \{f(x,y): 0 \leq f(x,y) \leq 2\}, 2$
2. $D_f = \{(x,y): x, y \in \Re, y \neq x^3\}, R_f = \{f(x,y): f(x,y) \in \Re, \neq 0\}, -1/9$]

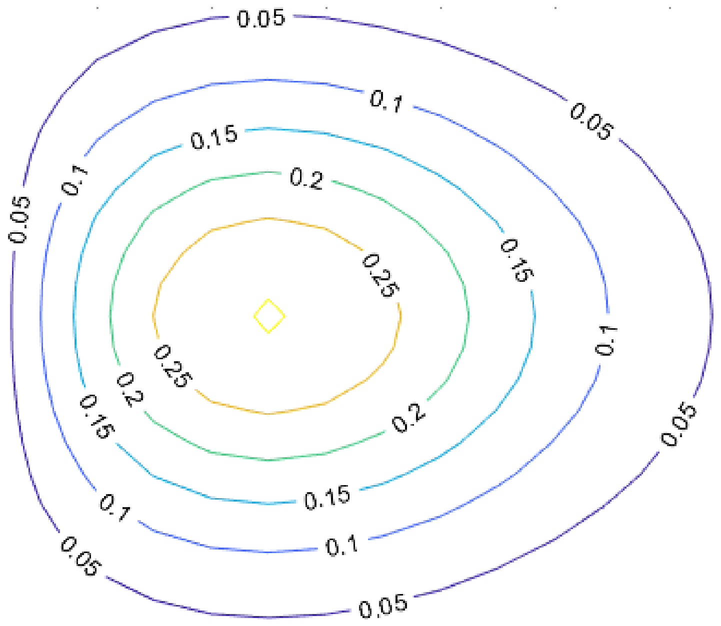
Graphing Functions of Two Variables

Definition:
Level curve is the set of points in the plane $z = f(x, y) = k$ where k are constants. The set of level curves is called **contour curve**. The surface $z = f(x, y)$ is the set of all points $(x, y, f(x, y))$.

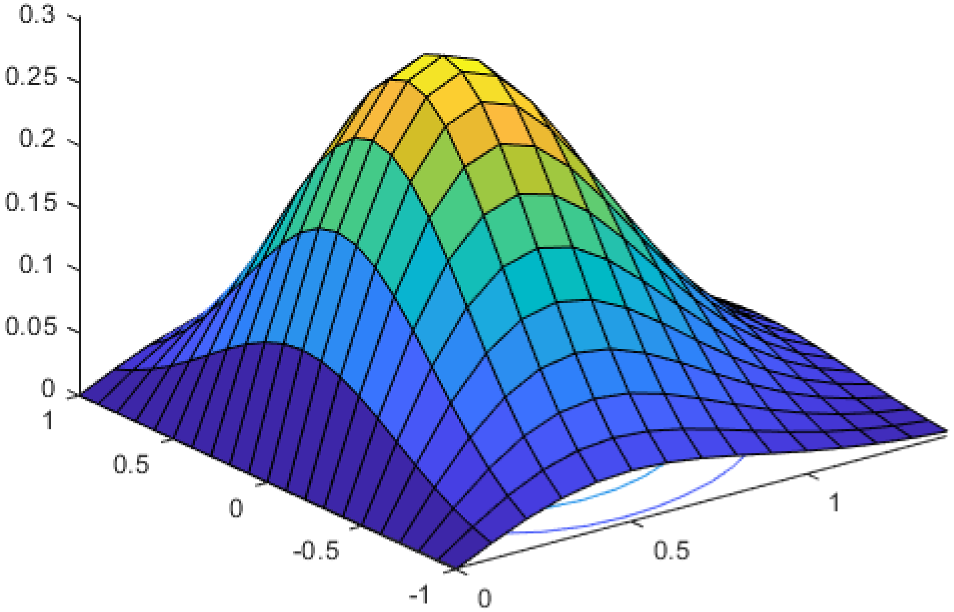
The steps to graph two variables functions:

1. Draw the level curves in the domain (x, y) where f has a constant value where k values must associate with the range of the z.
2. Sketch the surface $z = f(x, y)$ in space (set of level curves).

Graphing Functions of Two Variables



Contour curve (set of level curves)



Surface

Example 1.4:

Given a function $z = f(x, y) = 4 - x^2 - y^2$. Find the domain and range of the function. Show the level curves at $k = 0, k = 1, k = 2, k = 3$. Then, sketch the graph of $z = f(x, y)$.

Solution:

The range of z is $z \leq 4$, thus the values of k must be $k \leq 4$.

From the definition, $z = f(x, y) = k$.

Hence $f(x, y) = 4 - x^2 - y^2 = k$ which gives $x^2 + y^2 = 4 - k$.

When $k = 0$, we obtain $x^2 + y^2 = 4$ (a circle with radius $r = 2$)

$k = 1$, we obtain $x^2 + y^2 = 3$ (a circle with radius $r = \sqrt{3}$)

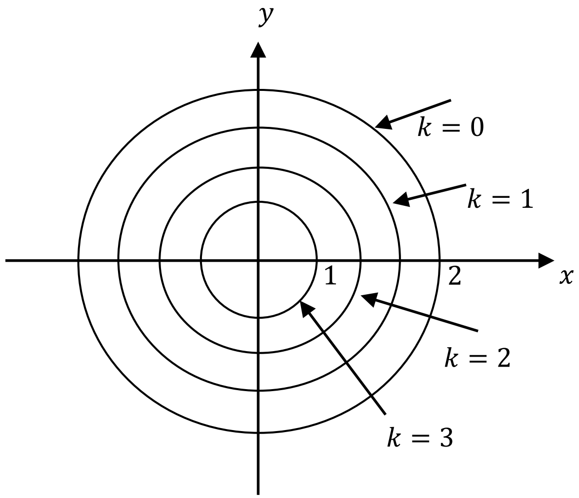
$k = 2$, we obtain $x^2 + y^2 = 2$ (a circle with radius $r = \sqrt{2}$)

$k = 3$, we obtain $x^2 + y^2 = 1$ (a circle with radius $r = 1$)

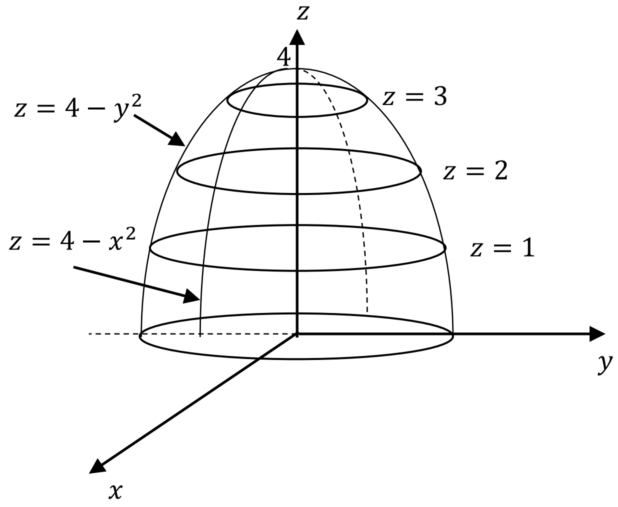
Note: Equation of a circle radius, r is $x^2 + y^2 = r^2$.

Solution:

When $y = 0$, we obtain $z = 4 - x^2$ which is a parabola on the xz −plane.
When $x = 0$, we obtain $z = 4 - y^2$ which is a parabola on the yz −plane.



The contour curve $x^2 + y^2 = 4 - k$



The surface of $f(x, y) = 4 - x^2 - y^2$
is a paraboloid vertex at $(0,0,4)$.

Example 1.5:

Given a function $z = f(x, y) = 1 + \sqrt{x^2 + y^2}$. Find the domain and range of the function. Show the level curves at $k = 2, k = 3$. Then, sketch the graph of $z = f(x, y)$.

Solution:

From the definition, $z = f(x, y) = k$.

Hence $f(x, y) = 1 + \sqrt{x^2 + y^2} = k$ which gives $x^2 + y^2 = (k-1)^2$.

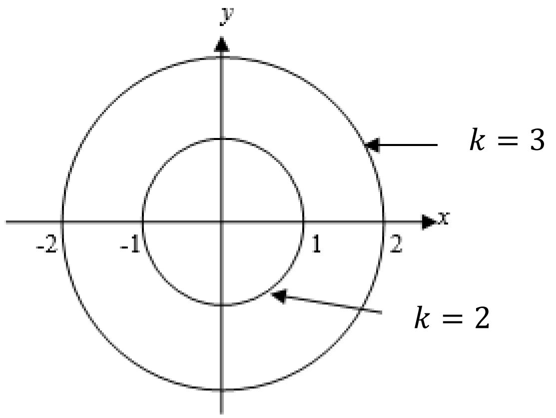
When $k = 2$, we obtain $x^2 + y^2 = 1$ (a circle with radius $r = 1$)

$k = 3$, we obtain $x^2 + y^2 = 2^2$ (a circle with radius $r = 2$)

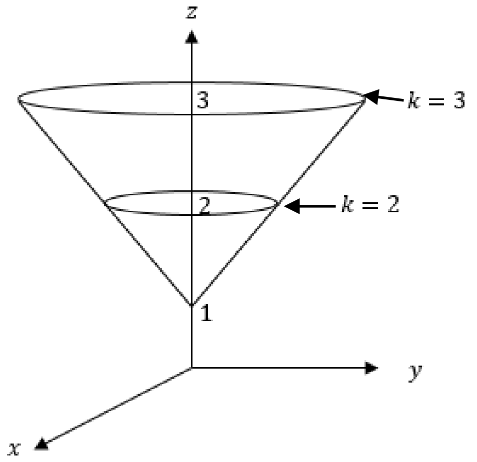
Note: Equation of a circle radius, r is $x^2 + y^2 = r^2$.

Solution:

When $y = 0$, we obtain $z = 1 + x$ which is a linear equation, z −intercept at 1.
When $x = 0$, we obtain $z = 1 + y$ which is a linear equation, z −intercept at 1.
Hence, the surface is a cone vertex at $(0,0,1)$.



The contour curve $x^2 + y^2 = (k-1)^2$



The surface of $f(x,y) = 1 + \sqrt{x^2 + y^2}$
is a cone vertex at $(0,0,1)$

Exercise 1.2:

Sketch the contour curve and the surface for the function

$$f(x, y) = -1 + x^2 + y^2$$

for $k = 0, 3$.

[Ans: A paraboloid with vertex at $(0,0,-1)$]

Exercise 1.3:

Sketch the contour curve and the surface for the function

$$f(x, y) = 2 - x - 2y$$

for $k = 0,1,2$.

[Ans: A tetrahedron plane which x – intercept at $(2,0,0)$, y – intercept at $(0,1,0)$ and z – intercept at $(0,0,2)$]

Functions with Three Variables

Definition:

Suppose G is a set of order pairs of real numbers, (x, y, z), a domain of a function with 3 independent variables x, y and z denotes as $w = f(x, y, z)$. The set of w −values that associates with the numbers in G is the range of the function.

Domain (G): The set of all pairs (x, y, z) for which the given expression is a well-defined real number.

Example 1.6:

For $f(x,y,z)=\frac{xz}{y-1}$. Find $f(1,2,-2\ )$, domain and range of f.

Solution:

Domain: x, y and z can be any values except $y\neq 1$ $(y-1\neq 0)$. Hence the domain of $f(x,y,z)$ is the set

$$D_f=\{(x,y,z): x,y,z\in\Re, y\neq 1\}\ .$$

Range: The function $f(x,y,z)$ can be any values $R_f=\{f(x,y,z): f(x,y,z)\in\Re\}$.

Example 1.7:

For $f(x,y,z) = y^2 - 2\sin xz$. Find $f\left(\pi, 3, \frac{1}{2}\right)$, domain and range of f.

Solution:

Domain: x, y and z can be any values. Hence the domain of $f(x,y,z)$ is the set

$$D_f = \{(x,y,z): x,y,z \in \Re\}\ .$$

Range: The term y^2 has the range of $y \geq 0$, meanwhile the term $-2\sin xz$ has the range $-2 \leq -2\sin xz \leq 2$. Thus, the range of the function $f(x,y,z)$ is

$R_f = \{f(x,y,z): f(x,y,z) \geq -2\}$.

Exercise 1.4:

Find the values at the given point and determine the domain and range of the following functions;

1. $f(x,y,z) = xy - \ln z\,,\ (3,-1,e^2)$
2. $f(x,y,z) = \frac{z}{x-y},\ (2,0,-3)$

[Ans: 1. $D_f = \{(x,y,z): x,y,z \in \Re, z > 0\}, R_f = \{f(x,y,z): f(x,y), z \in \Re\}, -5$
2. $D_f = \{(x,y,z): x,y,z \in \Re, x \neq y\}, R_f = \{f(x,y,z): f(x,y,z) \in \Re\}, -3/2$]

Graphing Functions of Three Variables

Definition:
The set of points in the space $w = f(x, y, z) = k$ where k are constants is called level surface.

If we set values of k are k_1, k_2, and k_3, thus we will get 3 level surfaces presented by S_1, S_2, and S_3. Each surface is the same shape of each other.

Example 1.8:

Describe the level surfaces of $w = x^2 + y^2 + z^2$ for $k = 1, 4, 9$.

Solution:

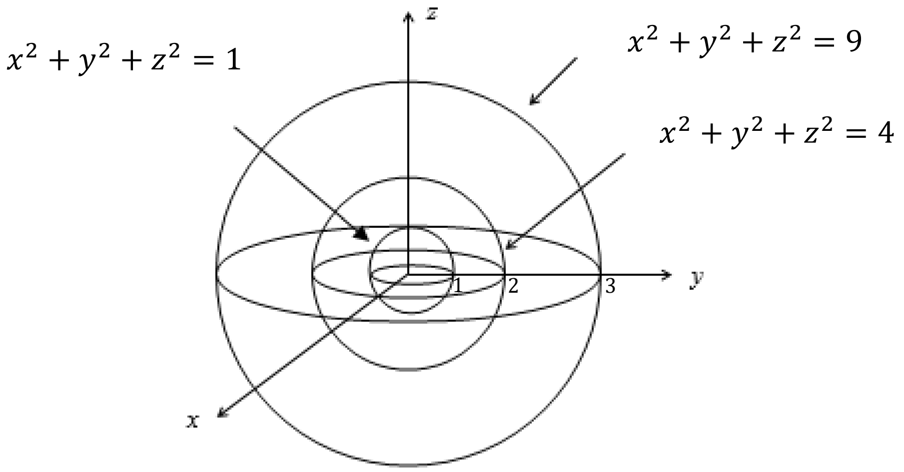
The level surfaces are the graphs of

$$x^2 + y^2 + z^2 = k.$$

When $k = 1$, we obtain $x^2 + y^2 + z^2 = 1$. (A sphere with radius 1)
When $k = 2$, we obtain $x^2 + y^2 + z^2 = 4$. (A sphere with radius 2)
When $k = 9$, we obtain $x^2 + y^2 + z^2 = 9$. (A sphere with radius 3)

Solution:

When $k = 1$, we obtain $x^2 + y^2 + z^2 = 1$.
(A sphere with radius 1)
When $k = 2$, we obtain $x^2 + y^2 + z^2 = 4$.
(A sphere with radius 2)
When $k = 9$, we obtain $x^2 + y^2 + z^2 = 9$.
(A sphere with radius 3)



The surfaces of $f(x, y, z) = x^2 + y^2 + z^2$

Example 1.9:

Describe the level surfaces of $w = z - \sqrt{x^2 + y^2}$ for $k = -1, 0, 1.$

Solution:

The level surfaces are the graphs of $z - \sqrt{x^2 + y^2} = k$, or $z = k + \sqrt{x^2 + y^2}$

When $k = -1$, we obtain $z = -1 + \sqrt{x^2 + y^2}$; (a cone with vertex $(0,0,-1)$)
When $k = 0$, we obtain $z = \sqrt{x^2 + y^2}$; (a cone with vertex $(0,0,0)$)
When $k = 1$, we obtain $z = 1 + \sqrt{x^2 + y^2}$; (a cone with vertex $(0,0,1)$).

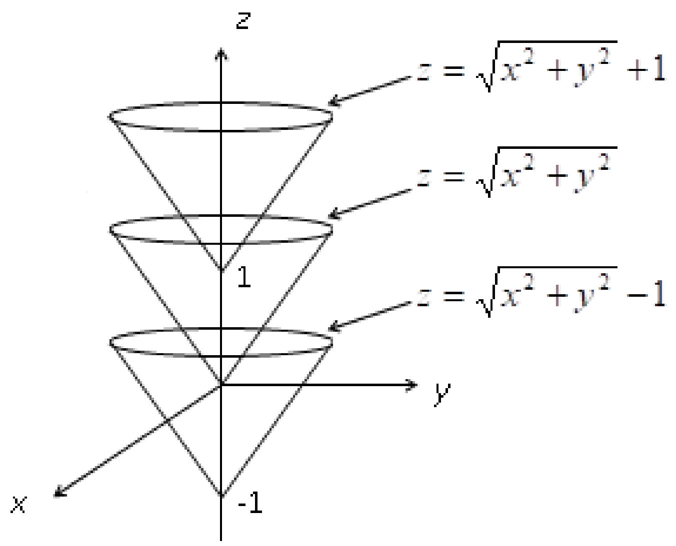
Note: Equation of a circle radius, r is $x^2 + y^2 = r^2$.

Solution:

When $k = -1$, we obtain $z = -1 + \sqrt{x^2 + y^2}$; (a cone with vertex $(0,0,-1)$)

When $k = 0$, we obtain $z = \sqrt{x^2 + y^2}$; (a cone with vertex $(0,0,0)$)

When $k = 1$, we obtain $z = 1 + \sqrt{x^2 + y^2}$; (a cone with vertex $(0,0,1)$).



Exercise 1.5:

Sketch the level surfaces for the function $w = 2 + z - x^2 - y^2$ for $k = 3, 6.$

[Ans: 2 paraboloids vertex at (0,0,1) and (0,0,4)]

Exercise 1.6:

Sketch the level surfaces for the function $w = y^2 + z^2$ for $k = 1, 4$.

[Ans: A cylinder of radius 1 along x-axis when k=1
A cylinder of radius 2 along x-axis when k=4]

Reference

1) R. Ranom, I.W. Jamaludin, N. A. Razak, N.I.A Apandi, Engineering Mathematics, UTeM Press, 2021.

2) Abd Wahid Md. Raji, Ismail Kamis, Mohd Nor Mohamad & Ong Chee Tiong, Advanced Calculus for Science and Engineering Students, UTM Press, 2021.

THANK YOU