ENGINEERING MATHEMATICS 1

BMFG 1313

VECTOR-VALUED FUNCTIONS (UNIT BINORMAL VECTOR, CURVATURE & TORSION)

Ser Lee Loh[1], Nur Ilyana Anwar Apandi[2],

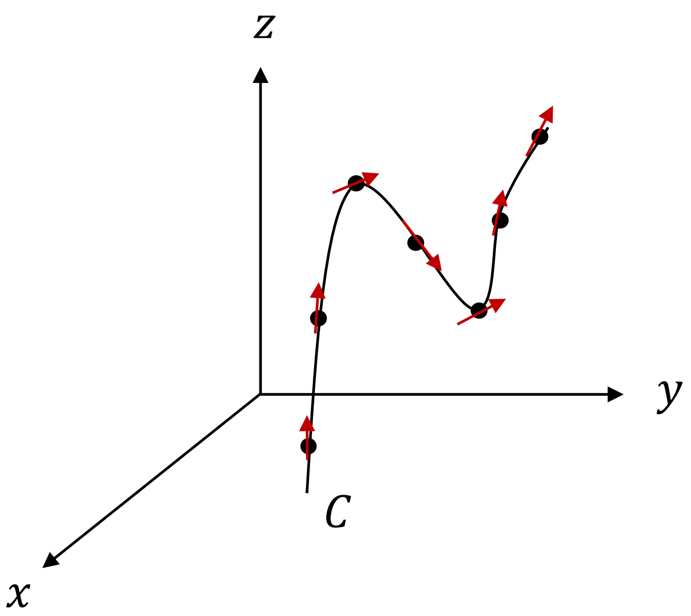
[1]slloh@utem.edu.my, [2]ilyana@utem.edu.my

Lesson Outcome

Upon completion of this lesson, the student should be able to:

- Compute Curvature of a curve
- Compute Binormal vector of a curve
- Compute Torsion of a curve

8.7 Curvature



Unit tangent vectors at equally spaced points on C

The **curvature** of C at a given point measures how quickly the curve changes at that point.

Tangent vector, $\mathbf{T}(t)$ changes direction slowly when the curve C bends a bit as shown in the first two points in the diagram. When C twists more sharply, for example the third point in the diagram, $\mathbf{T}(t)$ changes direction more quickly.

8.7 Curvature

Curvature is the magnitude of the rate of change of the unit tangent vector with respect to arc length,

$$\kappa = \left\| \frac{d\mathbf{T}}{ds} \right\|.$$

*Curvature is determined only in magnitude except for plane curves.

8.7.1 Proof of $\kappa = \frac{\|\mathbf{T}'(\mathrm{t})\|}{\|\mathbf{r}'(\mathrm{t})\|}$:

$$\kappa = \left\| \frac{d\mathbf{T}}{ds} \right\| = \left\| \frac{\frac{d\mathbf{T}}{dt}}{\frac{ds}{dt}} \right\| = \frac{\|\mathbf{T}'(t)\|}{\left|\frac{ds}{dt}\right|} = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} \qquad (8.1)$$

Chain Rule

$\|f(t)\mathbf{F}(t)\| = |f(t)|\|\mathbf{F}(t)\|$

8.7 Curvature

8.7.2 Proof of $\kappa = \frac{\|\mathbf{r}' \times \mathbf{r}''\|}{\|\mathbf{r}'\|^3}$:

Rearrange tangent vector,

$$\mathbf{T} = \frac{\mathbf{r}'}{\|\mathbf{r}'\|}$$

$$\mathbf{r}' = \|\mathbf{r}'\|\mathbf{T}$$

$$= \frac{ds}{dt}\mathbf{T} \qquad (8.2)$$

Differentiate $\mathbf{r}$ with respect to t,

$$\mathbf{r}'' = \frac{d^2s}{dt^2}\mathbf{T} + \frac{ds}{dt}\mathbf{T}'$$

$$= \frac{d^2s}{dt^2}\mathbf{T} + \frac{ds}{dt}\frac{d\mathbf{T}}{ds}\frac{ds}{dt}$$

(From Frenet-Serret Formula)

$$= \frac{d^2s}{dt^2}\mathbf{T} + \left(\frac{ds}{dt}\right)^2 \kappa\mathbf{N} \qquad (8.3)$$

8.7 Curvature

Find the cross product between $\mathbf{r}'$ and $\mathbf{r}''$,

$$\mathbf{r}' \times \mathbf{r}'' = \frac{ds}{dt}\mathbf{T} \times \left[\frac{d^2s}{dt^2}\mathbf{T} + \left(\frac{ds}{dt}\right)^2 \kappa\mathbf{N}\right]$$

$$= \boxed{\frac{ds}{dt}\mathbf{T} \times \frac{d^2s}{dt^2}\mathbf{T}} + \frac{ds}{dt}\mathbf{T} \times \left(\frac{ds}{dt}\right)^2 \kappa\mathbf{N}$$

$$\boxed{\mathbf{T} \times \mathbf{T} = \mathbf{0}}$$

$$= \kappa\left(\frac{ds}{dt}\right)^3 (\mathbf{T} \times \mathbf{N})$$

$$= \kappa\left(\frac{ds}{dt}\right)^3 \mathbf{B} \qquad (8.4)$$

Compute the magnitude of $\mathbf{r}' \times \mathbf{r}''$,

$$\|\mathbf{r}' \times \mathbf{r}''\| = \kappa\left(\frac{ds}{dt}\right)^3 \|\mathbf{B}\| = \kappa\left(\frac{ds}{dt}\right)^3 \quad \text{since } \|\mathbf{B}\| = 1 \qquad (8.5)$$

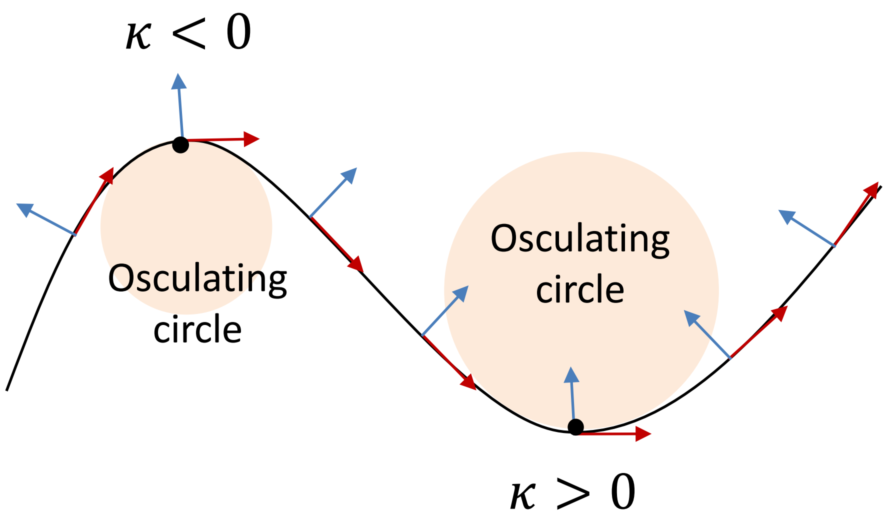
Rearrange the equation,

$$\kappa = \frac{\|\mathbf{r}' \times \mathbf{r}''\|}{\left(\frac{ds}{dt}\right)^3} = \frac{\|\mathbf{r}' \times \mathbf{r}''\|}{\|\mathbf{r}'\|^3} \qquad (8.6)$$

8.7 Curvature

8.7.3 Curvature of Plane Curves

In a plane curve, the sign of curvature indicates the direction of rotation of the tangent vector, $\mathbf{T}$ as a function of the parameter along the curve.
If $\mathbf{T}$ rotates in counterclockwise direction (or the unit Normal vector, $\mathbf{N}$ points to the osculating circle), it gives positive curvature. If it rotates in clockwise, it gives negative curvature.

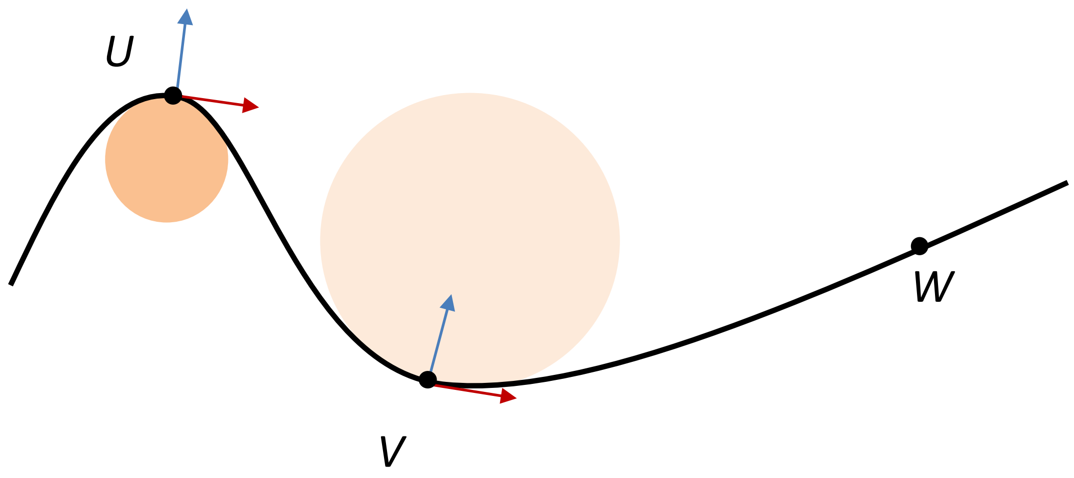


Red colour represents **unit tangent vectors, T** and blue colour represents **unit normal vectors, N.**

8.7 Curvature

8.7.3 Curvature of Plane Curves

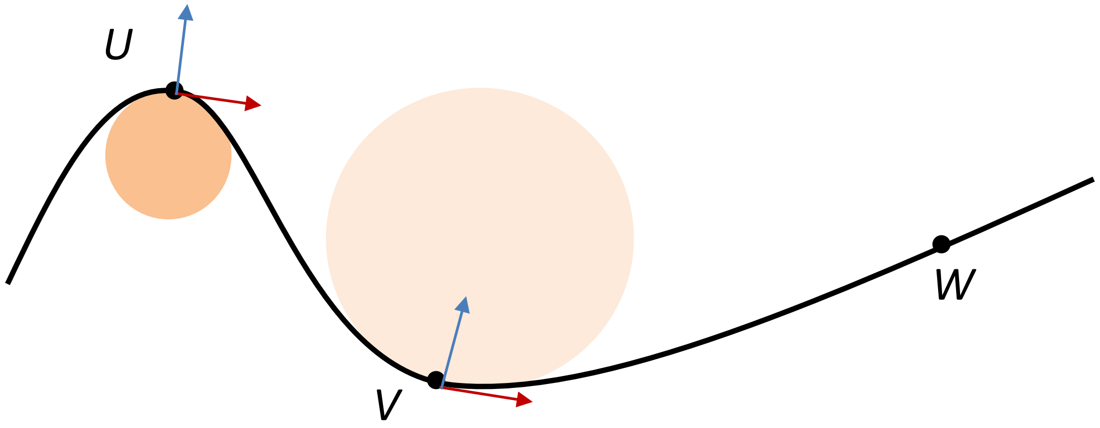
Curvature measures on the non-linearity at a point *P*.



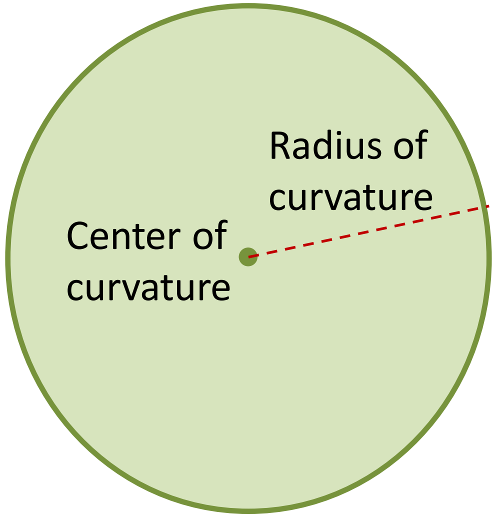
- A curve with zero curvature is a line. Hence, curvature at point *W* is zero.
- Based on the rotation direction of the Tangent vector (or direction of Normal vector), curvature at point *U* is negative while curvature at point *V* is positive.

8.7 Curvature

8.7.3 Curvature of Plane Curves



Radius of curvature, $R = \frac{1}{\kappa}$

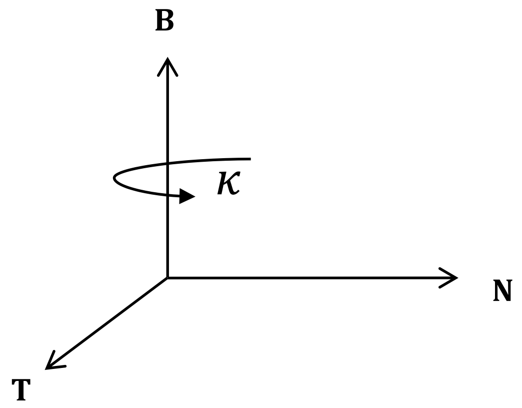


- Curvature at point *U* is higher than the one at point *V* since tangent vector bends more at point *U* compared to point *V*.
- Radius of curvature at point *U* is smaller compared to point *V* as radius of curvature is inversely proportional to the curvature.

8.7 Curvature

8.7.4 Curvature of Space Curves

For a moving particle with unit speed along a space curve, curvature is the magnitude of its acceleration. Different from curvature in a plane curve, curvature of a space curve is always non-negative $[0, \infty)$. Hence, in a space curve, tangent vector points to the moving direction, and normal vector points to the inner side of the curve.



8.7 Curvature

Example:

Find the curvature of $\mathbf{r}(t) = \langle 3\cos t\,, 3\sin t\,, t\rangle$ by using the formula $\kappa = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|}$.

Solution:

$\mathbf{r}(t) = \langle 3\cos t\,, 3\sin t\,, t\rangle$

$\mathbf{r}'(t) = \langle -3\sin t\,, 3\cos t\,, 1\rangle$

$\|\mathbf{r}'(t)\| = \sqrt{(-3\sin t)^2 + (3\cos t)^2 + 1^2} = \sqrt{9\sin^2 t + 9\cos^2 t + 1} = \sqrt{10}$

$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{1}{\sqrt{10}}\langle -3\sin t\,, 3\cos t\,, 1\rangle$

$\mathbf{T}'(t) = \frac{1}{\sqrt{10}}\langle -3\cos t\,, -3\sin t\,, 0\rangle$

$\|\mathbf{T}'(t)\| = \frac{1}{\sqrt{10}}\sqrt{(-3\cos t)^2 + (-3\sin t)^2 + 0^2} = \frac{3}{\sqrt{10}}$

$$\therefore\ \kappa = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = \frac{3/\sqrt{10}}{\sqrt{10}} = \frac{3}{10}$$

8.7 Curvature

Example:

Find the curvature of $\mathbf{r}(t) = \langle 4\cos 2t\,, 4\sin 2t\rangle$ by using the formula $\kappa = \frac{\|\mathbf{r}'\times\mathbf{r}''\|}{\|\mathbf{r}'\|^3}$.

Solution:

$\mathbf{r}(t) = \langle 4\cos 2t\,, 4\sin 2t\rangle$

$\mathbf{r}'(t) = \langle -8\sin 2t\,, 8\cos 2t\rangle$

$\|\mathbf{r}'(t)\| = \sqrt{(-8\sin 2t)^2 + (8\cos 2t)^2} = 8$

$\mathbf{r}''(t) = \langle -16\cos 2t\,, -16\sin 2t\rangle$

$$\mathbf{r}' \times \mathbf{r}'' = \begin{bmatrix} -8\sin 2t \\ 8\cos 2t \\ 0 \end{bmatrix} \times \begin{bmatrix} -16\cos 2t \\ -16\sin 2t \\ 0 \end{bmatrix} = (128\sin^2 2t + 128\cos^2 2t)\mathbf{k} = 128\mathbf{k}$$

$\|\mathbf{r}' \times \mathbf{r}''\| = 128$

$$\therefore \kappa = \frac{\|\mathbf{r}' \times \mathbf{r}''\|}{\|\mathbf{r}'\|^3} = \frac{128}{8^3} = \frac{1}{4}$$

8.7 Curvature

Exercise 8.10:

1) Find the curvature of $\mathbf{r}(t) = \langle a\cos t, a\sin t\rangle$ by using the formula $\kappa = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|}$.

[Ans: $\kappa = \frac{1}{a}$]

2) Find the curvature of $\mathbf{r}(t) = \langle 3t, 4\sin t, 4\cos t\rangle$ by using the formula $\kappa = \frac{\|\mathbf{r}'\times\mathbf{r}''\|}{\|\mathbf{r}'\|^3}$.

[Ans: $\kappa = \frac{4}{25}$]

3) Find the curvature of the curve $\mathbf{r}(t) = \langle t, 2t^2\rangle$.

[Ans: $\kappa = \frac{4}{(1+16t^2)^{3/2}}$]

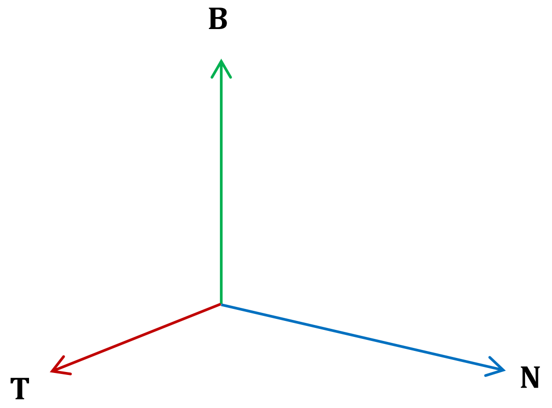
8.8 Unit Binormal Vector

A tangent vector $\mathbf{T}$ is orthogonal to a normal vector $\mathbf{N}$. Binormal vector $\mathbf{B}$ is perpendicular to both tangent and normal vectors as shown in the diagram.

Given a smooth curve $\mathbf{r}(t)$,

Unit Binormal Vector:

$$\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$$



Vectors $\mathbf{T}(t), \mathbf{N}(t)$ and $\mathbf{B}(t)$

Note: Since both tangent and normal vectors are unit vectors, hence the binormal vector from the cross product is also a unit vector.

8.8 Unit Binormal Vector

$$\mathbf{r}' \times \mathbf{r}'' = \kappa \left(\frac{ds}{dt}\right)^3 \mathbf{B} \qquad (8.4)$$

$$\kappa = \frac{\|\mathbf{r}' \times \mathbf{r}''\|}{\left(\frac{ds}{dt}\right)^3} = \frac{\|\mathbf{r}' \times \mathbf{r}''\|}{\|\mathbf{r}'\|^3} \qquad (8.6)$$

Alternate formula of Binormal vector:

Rearrange Eqn. (8.4),

$$\mathbf{B} = \frac{\mathbf{r}' \times \mathbf{r}''}{\kappa \left(\frac{ds}{dt}\right)^3}$$

By substituting Eqn. (8.6),

$$\mathbf{B} = \frac{\mathbf{r}' \times \mathbf{r}''}{\left[\frac{\|\mathbf{r}' \times \mathbf{r}''\|}{\left(\frac{ds}{dt}\right)^3}\right]\left(\frac{ds}{dt}\right)^3}$$

$$= \frac{\mathbf{r}' \times \mathbf{r}''}{\|\mathbf{r}' \times \mathbf{r}''\|} \qquad (8.7)$$

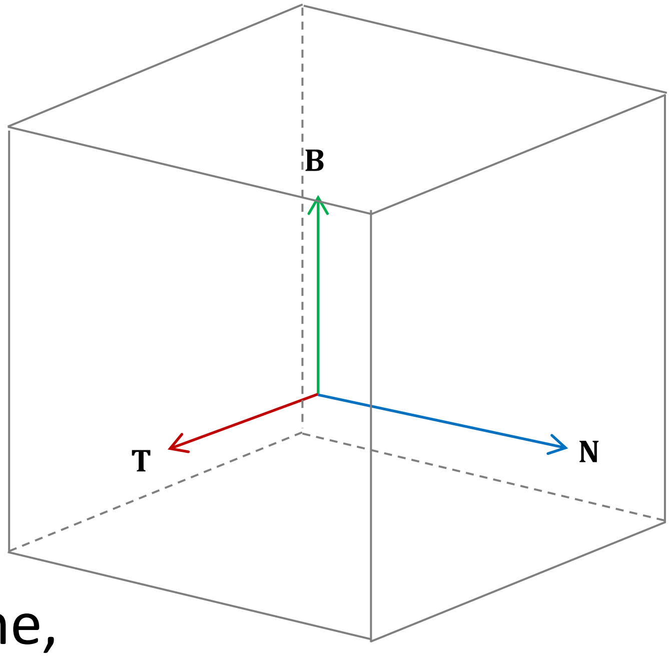
8.8 Unit Binormal Vector

A Binormal vector, $\mathbf{B}$ forms an orthogonal coordinate system with Tangent vector, $\mathbf{T}$ and Normal vector, $\mathbf{N}$ along a space curve where $\mathbf{T}$, $\mathbf{N}$ and $\mathbf{B}$ satisfy the Right-Hand Thumb Rule as follows:

$$\mathbf{T}(t) = \mathbf{N}(t) \times \mathbf{B}(t)$$

$$\mathbf{N}(t) = \mathbf{B}(t) \times \mathbf{T}(t)$$

$$\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$$



Frenet-Serret frame

Vectors $\mathbf{T}$, $\mathbf{N}$ and $\mathbf{B}$ forms a frame, called Frenet-Serret frame.

<u>Frenet-Serret formulas:</u>

$$\frac{d\mathbf{T}}{ds} = \kappa\mathbf{N} \qquad (8.8)$$

$$\frac{d\mathbf{N}}{ds} = -\kappa\mathbf{T} + \tau\mathbf{B} \qquad (8.9)$$

$$\frac{d\mathbf{B}}{ds} = -\tau\mathbf{N} \qquad (8.10)$$

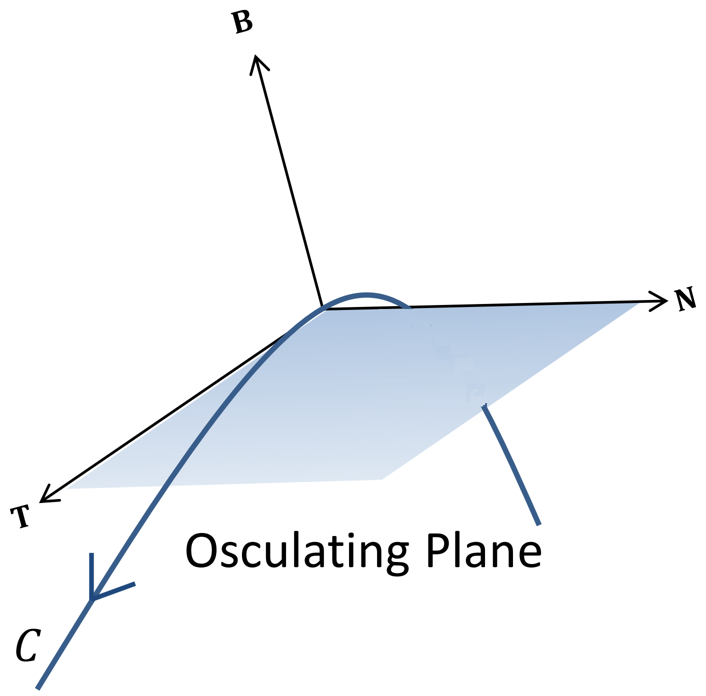
8.8 Unit Binormal Vector

Let $\mathbf{r}$ be a vector-valued function that represents a smooth curve, C.

<u>The Osculating Plane</u>:

Spanned by Tangent vector, $\mathbf{T}$ and Normal vector, $\mathbf{N}$.

Binormal Vector $\mathbf{B}$ is normal to the osculating plane.



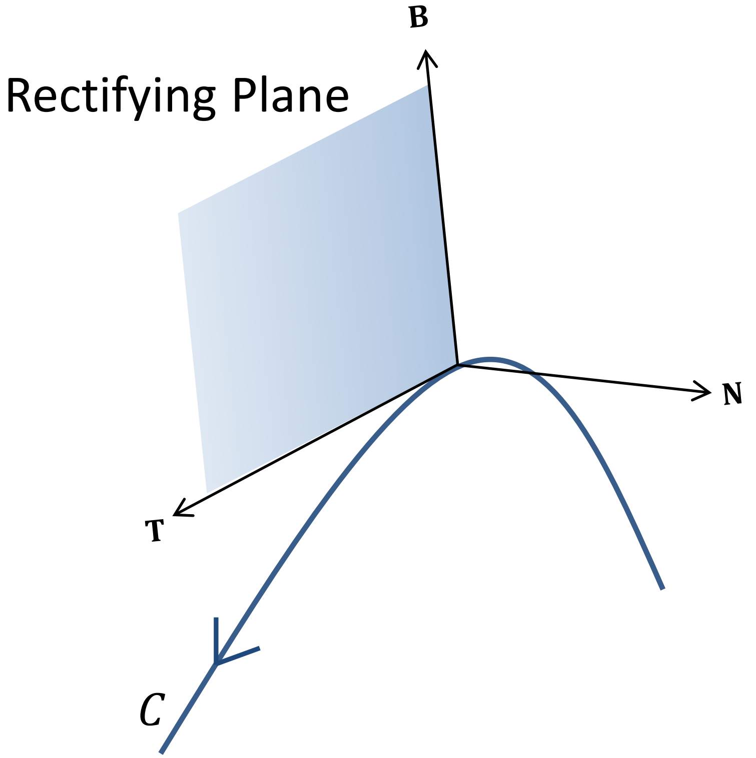
8.8 Unit Binormal Vector

Let $\mathbf{r}$ be a vector-valued function that represents a smooth curve, C.

<u>The Rectifying Plane</u>:

Spanned by Binormal vector, $\mathbf{B}$ and Tangent vector, $\mathbf{T}$.

Normal Vector $\mathbf{N}$ is normal to the rectifying plane.



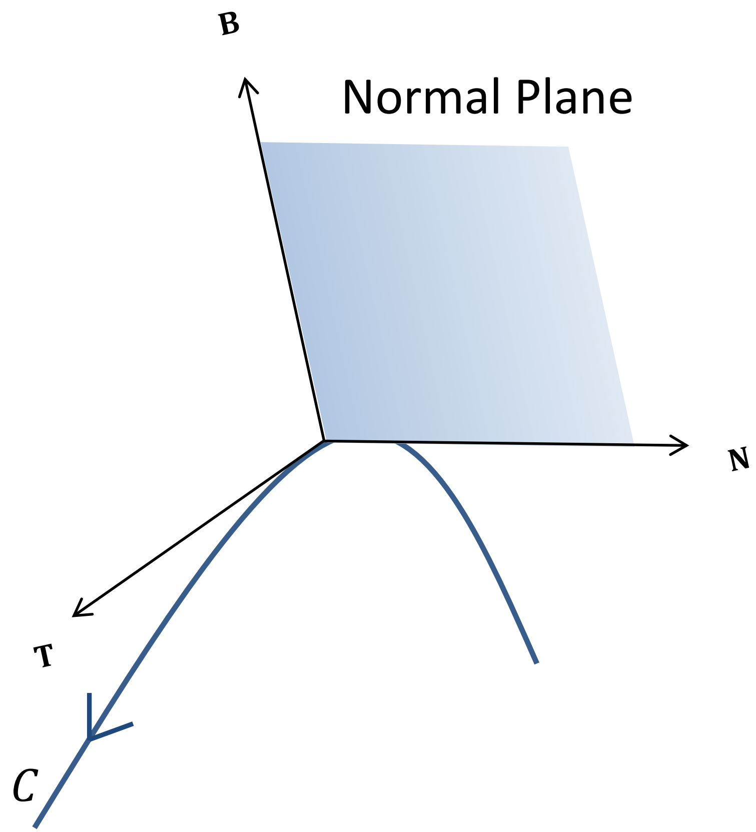
8.8 Unit Binormal Vector

Let $\mathbf{r}$ be a vector-valued function that represents a smooth curve, C.

<u>The Normal Plane</u>:

Spanned by Binormal vector, $\mathbf{B}$ and Normal vector, $\mathbf{N}$.

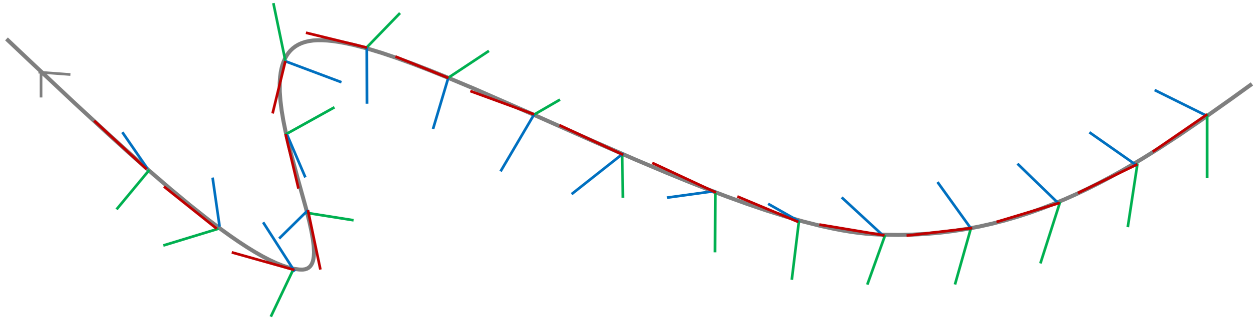
Tangent Vector $\mathbf{T}$ is normal to the normal plane.



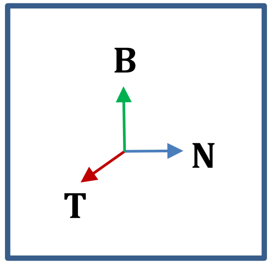
8.8 Unit Binormal Vector

Along a space curve, there are Frenet-Serret frames on the evenly-spaced points on the curve to show the direction of a moving particle. Tangent vector, $\mathbf{T}$, points in the moving direction, and Normal vector, $\mathbf{N}$, points to the inner side of the curve.

Imagine a boy is riding a roller coaster. The boy is sitting on the osculating plane, and is facing to the moving direction, $\mathbf{T}$. When the roller coaster turns to left-hand side, normal vector is pointing $90°$ to his left and binormal vector is pointing upwards from his head. For the case it turns to right, it means the boy is hanging downward where his hand is still pointing to his left.



Frenet-Serret frames along a curve



8.8 Unit Binormal Vector

Example 1:

Find the binormal vector for the curve traced out by $\mathbf{r}(t) = \langle \sin t, \cos t, 3t \rangle$. Next, compute binormal vector at $t = \pi$.

Solution:

$\mathbf{r}'(t) = \langle \cos t, -\sin t, 3 \rangle$

$\|\mathbf{r}'(t)\| = \sqrt{(\cos t)^2 + (-\sin t)^2 + (3)^2} = \sqrt{10}$

$\mathbf{T}(t) = \frac{1}{\sqrt{10}} \langle \cos t, -\sin t, 3 \rangle$

$\mathbf{T}'(t) = \frac{1}{\sqrt{10}} \langle -\sin t, -\cos t, 0 \rangle$

$\|\mathbf{T}'(t)\| = \frac{1}{\sqrt{10}} \sqrt{(-\sin t)^2 + (-\cos t)^2 + (0)^2} = \frac{1}{\sqrt{10}}$

$\mathbf{N}(t) = \langle -\sin t, -\cos t, 0 \rangle$

8.8 Unit Binormal Vector

Solution (cont.):

$$\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$$

$$= \frac{1}{\sqrt{10}}\begin{bmatrix} \cos t \\ -\sin t \\ 3 \end{bmatrix} \times \begin{bmatrix} -\sin t \\ -\cos t \\ 0 \end{bmatrix}$$

$$= \frac{1}{\sqrt{10}}\langle 3\cos t\,, -3\sin t\,, -1\rangle$$

Hence,

$$\mathbf{B}(\pi) = \frac{1}{\sqrt{10}}\langle 3\cos \pi\,, -3\sin \pi\,, -1\rangle = \frac{1}{\sqrt{10}}\langle -3, 0, -1\rangle.$$

8.8 Unit Binormal Vector

Example 2:

Find the binormal vector at $t = 0$ for the curve traced out by $\mathbf{r}(t) = \langle 2t, \cos 3t, \sin 3t \rangle$.

Solution:

$$\mathbf{r}'(t) = \langle 2, -3\sin 3t\,, 3\cos 3t \rangle$$

$$\|\mathbf{r}'(t)\| = \sqrt{(2)^2 + (-3\sin 3t)^2 + (3\cos 3t)^2} = \sqrt{13}$$

$$\mathbf{T}(t) = \frac{1}{\sqrt{13}} \langle 2, -3\sin 3t\,, 3\cos 3t \rangle$$

$$\mathbf{T}'(t) = \frac{1}{\sqrt{13}} \langle 0, -9\cos 3t\,, -9\sin 3t \rangle$$

$$\|\mathbf{T}'(t)\| = \frac{1}{\sqrt{13}} \sqrt{(0)^2 + (-9\cos 3t)^2 + (-9\sin 3t)^2} = \frac{9}{\sqrt{13}}$$

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|} = \langle 0, -\cos 3t\,, -\sin 3t \rangle$$

8.8 Unit Binormal Vector

Solution (cont.):

Substitute $t = 0$ into $\mathbf{T}(t)$ and $\mathbf{N}(t)$:

$$\mathbf{T}(0) = \frac{1}{\sqrt{13}}\langle 2, -3\sin 3(0), 3\cos 3(0)\rangle = \frac{1}{\sqrt{13}}\langle 2, 0, 3\rangle$$

$$\mathbf{N}(0) = \langle 0, -\cos 3(0), -\sin 3(0)\rangle = \langle 0, -1, 0\rangle$$

$$\mathbf{B}(0) = \mathbf{T}(0) \times \mathbf{N}(0)$$

$$= \frac{1}{\sqrt{13}}\begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} \times \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$$

$$= \frac{1}{\sqrt{13}}\langle 3, 0, -2\rangle$$

8.8 Unit Binormal Vector

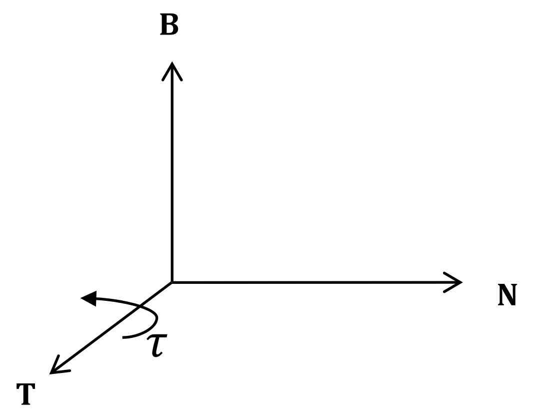
Exercise 8.11:

1) Find the binormal vector for the curve traced out by $\mathbf{r}(t) = \langle \sin 2t, \cos 2t, t \rangle$.
2) Find the binormal vector for the curve traced out by $\mathbf{r}(t) = \langle t, 3\sin t, 3\cos t \rangle$.
3) Find the binormal vector at $t = 1$ for the curve $\mathbf{r}(t) = \langle t, 2t, t^2 \rangle$.

[Ans: $\frac{1}{\sqrt{5}}\langle \cos 2t, -\sin 2t, -2 \rangle$; $\frac{1}{\sqrt{10}}\langle -3, \cos t, -\sin t \rangle$; $\left\langle \frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}}, 0 \right\rangle$]

8.9 Torsion

The **torsion** of a curve measures how much is the curve twisted out from a planar form. In other words, torsion measures how much is the osculating plane rotates around the tangent vector.



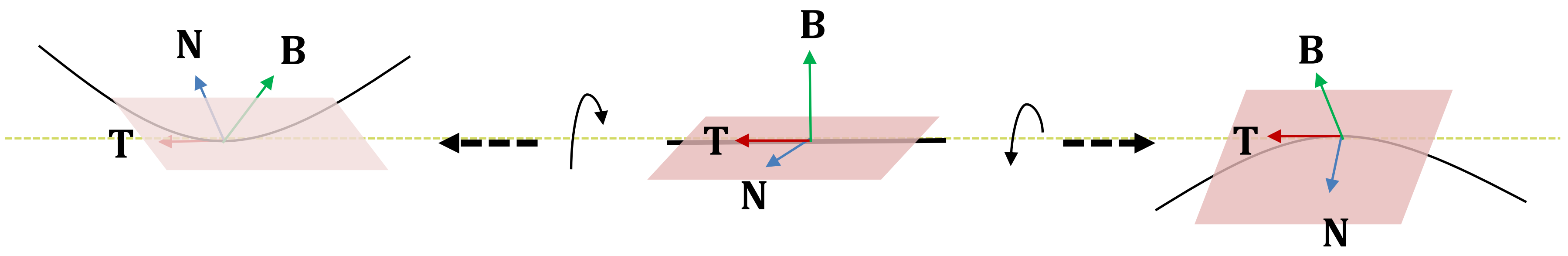
Torsion is determined both in magnitude and sign. The higher is the torsion magnitude, the more is the deviation of the curve from the osculating plane.

8.9 Torsion

Torsion measures the non-planarity of a space curve at a point P.

- A curve with zero torsion lies in a plane and is known as planar curve.
- For non-planar curve, the sign of torsion refers to the direction of rotation of the osculating plane at a particular point as follows:

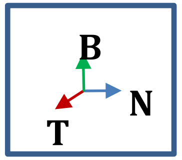
Case 2 | **Case 1** | Case 3



In case 2, the osculating plane rotates in right-handed screw, $\tau > 0$

In case 1, the osculating plane is parallel to xy-plane, $\tau = 0$

In case 3, the osculating plane rotates in left-handed screw, $\tau < 0$



8.9 Torsion

Derivation of formula of Torsion:

Compute $\mathbf{r}'''(t)$ on Eqn. (8.3),

$$\mathbf{r}'' = \frac{d^2s}{dt^2}\mathbf{T} + \left(\frac{ds}{dt}\right)^2 \kappa\mathbf{N} \qquad (8.3)$$

$$\mathbf{r}''' = \frac{d}{dt}\left[\frac{d^2s}{dt^2}\mathbf{T} + \left(\frac{ds}{dt}\right)^2 \kappa\mathbf{N}\right]$$

$$= \frac{d^3s}{dt^3}\mathbf{T} + \frac{d^2s}{dt^2}\frac{d\mathbf{T}}{dt} + \frac{d}{dt}\left[\left(\frac{ds}{dt}\right)^2 \kappa\right]\mathbf{N} + \left(\frac{ds}{dt}\right)^2 \kappa\frac{d\mathbf{N}}{dt}$$

$$= \frac{d^3s}{dt^3}\mathbf{T} + \frac{d^2s}{dt^2}\frac{d\mathbf{T}}{dt} + \frac{d}{dt}\left[\left(\frac{ds}{dt}\right)^2 \kappa\right]\mathbf{N} + \left(\frac{ds}{dt}\right)^2 \kappa\frac{d\mathbf{N}}{ds}\frac{ds}{dt}$$

$$= \frac{d^3s}{dt^3}\mathbf{T} + \frac{d^2s}{dt^2}\frac{d\mathbf{T}}{dt} + \frac{d}{dt}\left[\left(\frac{ds}{dt}\right)^2 \kappa\right]\mathbf{N} + \left(\frac{ds}{dt}\right)^2 \kappa(-\kappa\mathbf{T} + \tau\mathbf{B})\frac{ds}{dt}$$

(from Frenet-Serret formula)

$$= \alpha(t)\mathbf{T} + \beta(t)\mathbf{N} + \left(\frac{ds}{dt}\right)^3 \kappa\tau\mathbf{B}$$

Frenet-Serret formulas:

$$\frac{d\mathbf{T}}{ds} = \kappa\mathbf{N}$$

$$\frac{d\mathbf{N}}{ds} = -\kappa\mathbf{T} + \tau\mathbf{B}$$

$$\frac{d\mathbf{B}}{ds} = -\tau\mathbf{N}$$

8.9 Torsion

By taking dot product with $\mathbf{B}$,

$$\mathbf{B}\cdot\mathbf{r}''' = \mathbf{B}\cdot\left[\alpha(t)\mathbf{T} + \beta(t)\mathbf{N} + \left(\frac{ds}{dt}\right)^3 \kappa\tau\mathbf{B}\right]$$

$$= \|\mathbf{r}'\|^3\kappa\tau \qquad (8.11)$$

$$\frac{ds}{dt} = \|\mathbf{r}'(t)\|$$

since $\mathbf{B}$ is orthogonal to both $\mathbf{T}$ and $\mathbf{N}$, and $\mathbf{B}$ is a unit vector, i.e. $\mathbf{B}\cdot\mathbf{B} = 1$.

By substituting Eqn. (8.6) and Eqn. (8.7) into Eqn. (8.11),

$$\frac{\mathbf{r}'\times\mathbf{r}''}{\|\mathbf{r}'\times\mathbf{r}''\|}\cdot\mathbf{r}''' = \|\mathbf{r}'\|^3\frac{\|\mathbf{r}'\times\mathbf{r}''\|}{\|\mathbf{r}'\|^3}\tau$$

$$\kappa = \frac{\|\mathbf{r}'\times\mathbf{r}''\|}{\left(\frac{ds}{dt}\right)^3} = \frac{\|\mathbf{r}'\times\mathbf{r}''\|}{\|\mathbf{r}'\|^3} \qquad (8.6)$$

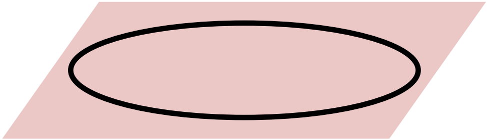
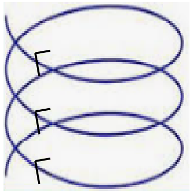
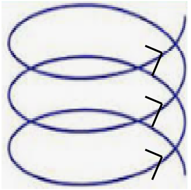
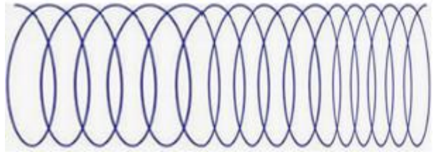
$$\mathbf{B} = \frac{\mathbf{r}'\times\mathbf{r}''}{\|\mathbf{r}'\times\mathbf{r}''\|} \qquad (8.7)$$

After cancellation of $\|\mathbf{r}'\|^3$ and rearrangement of equation, formula of Torsion is obtained:

$$\tau = \frac{(\mathbf{r}'\times\mathbf{r}'')\cdot\mathbf{r}'''}{\|\mathbf{r}'\times\mathbf{r}''\|^2} \qquad (8.12)$$

8.9 Torsion

Some Examples: Given b, c and d constants:

$\kappa = 0$ and $\tau = 0$ shows a line.	
$\kappa = c > 0$ and $\tau = 0$ shows a circle that lies in a plane.	
$\kappa = c > 0$ and $\tau = b > 0$ shows a right-handed screw helix. $\kappa = c > 0$ and $\tau = d < 0$ shows a left-handed screw helix.	 left-handed screw helix  right-handed screw helix
$\kappa = c > 0$ and $\tau > 0$ E.g. slinky. A stretched slinky has larger torsion compared to a compacted slinky.	 larger torsion smaller torsion

8.9 Torsion

Example 1:

Find the torsion for the curve traced out by

$$\mathbf{r}(t) = \langle \sin t\,, \cos t\,, 3t\rangle.$$

Solution:

$$\mathbf{r}'(t) = \langle \cos t\,, -\sin t\,, 3\rangle, \mathbf{r}''(t) = \langle -\sin t\,, -\cos t\,, 0\rangle, \mathbf{r}'''(t) = \langle -\cos t\,, \sin t\,, 0\rangle$$

$$\mathbf{r}' \times \mathbf{r}'' = \begin{bmatrix} \cos t \\ -\sin t \\ 3 \end{bmatrix} \times \begin{bmatrix} -\sin t \\ -\cos t \\ 0 \end{bmatrix} = \langle 3\cos t\,, -3\sin t\,, -1\rangle$$

$$\|\mathbf{r}' \times \mathbf{r}''\| = \sqrt{(3\cos t)^2 + (-3\sin t)^2 + (-1)^2} = \sqrt{10}$$

$$\tau = \frac{(\mathbf{r}' \times \mathbf{r}'') \cdot \mathbf{r}'''}{\|\mathbf{r}' \times \mathbf{r}''\|^2} = \frac{\langle 3\cos t\,, -3\sin t\,, -1\rangle \cdot \langle -\cos t\,, \sin t\,, 0\rangle}{(\sqrt{10})^2} = -\frac{3}{10}$$

8.9 Torsion

Example 2:

Find the torsion for the curve traced out by

$$\mathbf{r}(t) = \langle 2t, \cos 3t, \sin 3t \rangle.$$

Solution:

$\mathbf{r}'(t) = \langle 2, -3\sin 3t\,, 3\cos 3t \rangle, \mathbf{r}''(t) = \langle 0, -9\cos 3t\,, -9\sin 3t \rangle,$

$\mathbf{r}'''(t) = \langle 0{,}27\sin 3t\,, -27\cos 3t \rangle$

$$\mathbf{r}' \times \mathbf{r}'' = \begin{bmatrix} 2 \\ -3\sin 3t \\ 3\cos 3t \end{bmatrix} \times \begin{bmatrix} 0 \\ -9\cos 3t \\ -9\sin 3t \end{bmatrix} = \langle 27{,}18\sin 3t\,, -18\cos 3t \rangle$$

$$\|\mathbf{r}' \times \mathbf{r}''\| = \sqrt{(27)^2 + (18\sin 3t)^2 + (-18\cos 3t)^2} = \sqrt{1053}$$

$$\tau = \frac{(\mathbf{r}' \times \mathbf{r}'') \cdot \mathbf{r}'''}{\|\mathbf{r}' \times \mathbf{r}''\|^2} = \frac{\langle 27{,}18\sin 3t\,, -18\cos 3t \rangle \cdot \langle 0{,}27\sin 3t\,, -27\cos 3t \rangle}{(\sqrt{1053})^2}$$

$$= \frac{486}{1053} = \frac{6}{13}$$

8.9 Torsion

Exercise 8.12:

1) Find the torsion for the curve traced out by $\mathbf{r}(t) = \langle \sin 2t\,, \cos 2t, t \rangle$.

2) Find the torsion for the curve traced out by $\mathbf{r}(t) = \langle t, 3\sin t\,, 3\cos t \rangle$.

3) Find the torsion for the curve traced out by $\mathbf{r}(t) = \langle \cos t, \sin t\,, 3t \rangle$.

[Ans: $-\frac{2}{5}$; $-\frac{1}{10}$; $\frac{3}{10}$]