ENGINEERING MATHEMATICS 1

BMFG 1313

VECTOR-VALUED FUNCTIONS (UNIT TANGENT AND UNIT NORMAL VECTOR)

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Lesson Outcomes

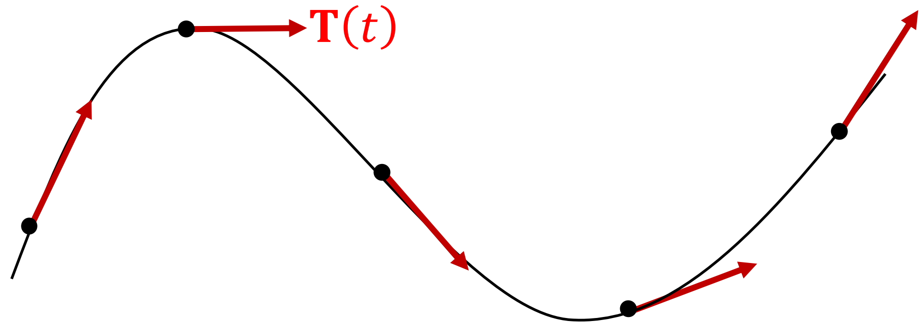
Upon completion of this lesson, the student should be able to:

- Evaluate the Unit Tangent Vector and Unit Normal Vector
- Evaluate the Integration of Vector-Valued Functions
- Evaluate motion along a Curve and Motion in Space

8.4 Unit Tangent Vector and Unit Normal Vector

Unit tangent vector is denoted by $\mathbf{T}(t)$. For some position vector, $\mathbf{r}(t)$, a tangent vector $\mathbf{T}(t)$ with magnitude of 1 unit given by

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}$$



The red-coloured arrow indicates the unit tangent vector, $\mathbf{T}(t)$, which is the direction of motion of a particle at each unit of time.

8.4 Unit Tangent Vector and Unit Normal Vector

Example:

Compute the unit tangent vector of $\mathbf{r}(t) = \langle -2\sin t\,, 2\cos t \rangle$.

Solution:

$$\mathbf{r}'(t) = \langle -2\cos t\,, -2\sin t \rangle$$

$$\|\mathbf{r}'(t)\| = \sqrt{(-2\cos t)^2 + (-2\sin t)^2}$$

$$= \sqrt{4\sin^2 t + 4\cos^2 t}$$

$$= \sqrt{4} = 2$$

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{\langle -2\cos t\,, -2\sin t \rangle}{2} = \langle -\cos t\,, -\sin t \rangle$$

8.4 Unit Tangent Vector and Unit Normal Vector

Example:

Compute the unit tangent vector of $\mathbf{r}(t) = \langle 3t, 3\cos t, 3\sin t \rangle$.

Solution:

$$\mathbf{r}'(t) = \langle 3, -3\sin t, 3\cos t \rangle$$

$$\|\mathbf{r}'(t)\| = \sqrt{(3)^2+(-3\sin t)^2+(3\cos t)^2}$$

$$= \sqrt{9 + 9\sin^2 t + 9\cos^2 t}$$

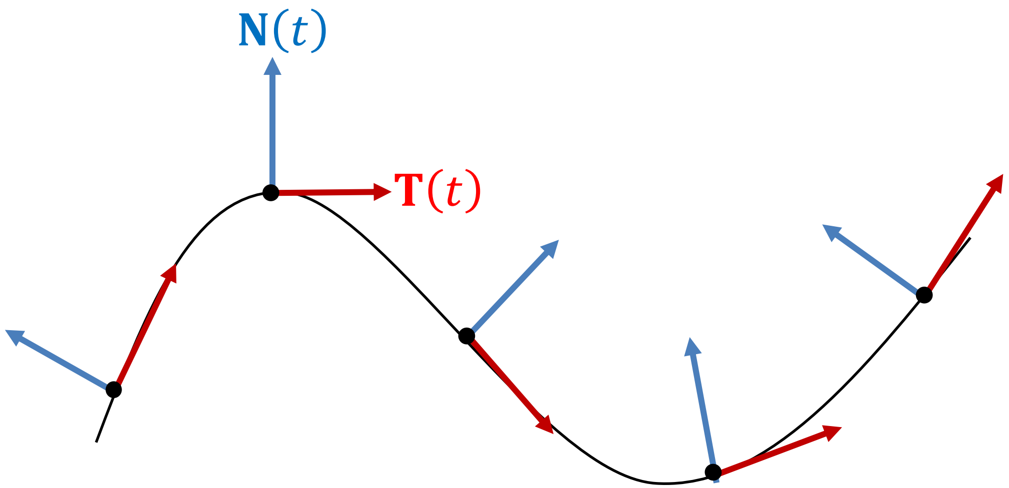
$$= \sqrt{9 + 9(1)} = \sqrt{18}$$

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} = \frac{\langle 3, -3\sin t, 3\cos t \rangle}{\sqrt{18}} = \left\langle \frac{3}{\sqrt{18}}, -\frac{3\sin t}{\sqrt{18}}, \frac{3\cos t}{\sqrt{18}} \right\rangle$$

8.4 Unit Tangent Vector and Unit Normal Vector

Unit normal vector of $\mathbf{r}(t)$ is denoted by $\mathbf{N}(t)$. $\mathbf{N}(t)$ is a unit vector which is perpendicular to its tangent vector, $\mathbf{T}(t)$ that is given by

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}$$



The red-coloured arrow indicates unit tangent vector, $\mathbf{T}(t)$,
which is the direction of motion of a particle while
the blue-coloured arrow indicates its respective unit normal vector, $\mathbf{N}(t)$.

8.4 Unit Tangent Vector and Unit Normal Vector

Example:

Compute the principal unit normal vector of $\mathbf{r}(t) = \langle 3t, 3\cos t, 3\sin t\rangle$.

Solution:

Based on previous example, $\mathbf{T}'(t) = \langle 0, -\frac{3\cos t}{\sqrt{18}}, -\frac{3\sin t}{\sqrt{18}}\rangle$ and

$$\|\mathbf{T}'(t)\| = \frac{1}{\sqrt{18}}\sqrt{(0)^2+(-3\cos t)^2+(-3\sin t)^2}$$

$$= \frac{1}{\sqrt{18}}\sqrt{9\cos^2 t + 9\sin^2 t} = \frac{1}{\sqrt{18}}\sqrt{9(1)} = \frac{3}{\sqrt{18}}.$$ Hence,

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|} = \frac{\frac{1}{\sqrt{18}}\langle 0, -3\cos t, -3\sin t\rangle}{\frac{3}{\sqrt{18}}} = \langle 0, -\cos t, -\sin t\rangle$$

8.4 Unit Tangent Vector and Unit Normal Vector

Example:

Compute the principal unit normal vector of a tangent vector,

$$\mathbf{T}(t) = \langle 2t, \ln t\,, t^2 \rangle.$$

Solution:

$$\mathbf{T}'(t) = \langle 2, \tfrac{1}{t}, 2t \rangle$$

$$\|\mathbf{T}'(t)\| = \sqrt{(2)^2 + \left(\tfrac{1}{t}\right)^2 + (2t)^2}$$

$$= \sqrt{4 + \tfrac{1}{t^2} + 4t^2} = \sqrt{\frac{4t^2+1+4t^4}{t^2}} = \frac{\sqrt{(1+2t^2)^2}}{t} = \frac{1+2t^2}{t}.$$ Hence,

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|} = \frac{\langle 2, \frac{1}{t}, 2t \rangle}{\frac{1+2t^2}{t}} = \frac{1}{1+2t^2}\langle 2t, 1, 2t^2 \rangle$$

Exercise 8.6:

Compute the unit tangent vector and principal unit normal vector for each of the following vectors.

1) $\mathbf{r}(t) = \langle 10, 3\cos t, 3\sin t\rangle,\ 0 \le t \le 2\pi.$

2) $\mathbf{r}(t) = \langle t^2, 4t, 4\ln t\rangle,\ t > 0.$

3) $\mathbf{r}(t) = \langle 2\sqrt{5}t, 2\sin 2t, 2\cos 2t\rangle$

[Ans: $\langle 0, -\sin t, \cos t\rangle, \langle 0, -\cos t, -\sin t\rangle$; $\frac{1}{t^2+2}\langle t^2, 2t, 2\rangle, \frac{1}{t^2+2}\langle 2t, 2-t^2, -2t\rangle$;
$\frac{1}{3}\langle \sqrt{5}, 2\cos 2t, -2\sin 2t\rangle, \langle 0, -\sin 2t, -\cos 2t\rangle$]

8.5 Integration of Vector-Valued Functions

8.5.1 Indefinite Integral

Let $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$ be a vector function,

$$\int \mathbf{r}(t)\,dt = \mathbf{R}(t) + \mathbf{C},$$

where $\mathbf{C}$ is an arbitrary constant vector.

8.5 Integration of Vector-Valued Functions

Example:

Evaluate the indefinite integral $\int \langle 2, t, \cos t \rangle \, dt$.

Solution:

$$\int \langle 2, t, \cos t \rangle \, dt = \left\langle 2t + c_1, \frac{1}{2}t^2 + c_2, \sin t + c_3 \right\rangle$$

$$= \left\langle 2t, \frac{1}{2}t^2, \sin t \right\rangle + \mathbf{C}$$

8.5 Integration of Vector-Valued Functions

Example:

Evaluate the indefinite integral $\int \langle 1, 2\cos 2t, -2\sin 2t \rangle \, dt$.

Solution:

$$\int \langle 1, 2\cos 2t, -2\sin 2t \rangle \, dt = \langle t + c_1, \sin 2t + c_2, \cos 2t + c_3 \rangle$$

$$= \langle t, \sin 2t, \cos 2t \rangle + \mathbf{C}$$

8.5 Integration of Vector-Valued Functions

8.5.2 Definite Integral

Let $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$, where f, g and h are integrable on the interval $[a, b]$.

$$\int_a^b \mathbf{r}(t)\,dt = \left[\int_a^b f(t)\,dt\right]\mathbf{i} + \left[\int_a^b g(t)\,dt\right]\mathbf{j} + \left[\int_a^b h(t)\,dt\right]\mathbf{k}$$

8.5 Integration of Vector-Valued Functions

Example:

Evaluate $\int_0^{\pi} \langle 1, 2\cos 2t\,, -2\sin 2t \rangle\, dt$

Solution:

$$\int_0^{\pi} \langle 1, 2\cos 2t\,, -2\sin 2t \rangle\, dt = \langle t, \sin 2t\,, \cos 2t \rangle \Big|_0^{\pi}$$

$$= \langle \pi, \sin 2\pi\,, \cos 2\pi \rangle - \langle 0, \sin 0\,, \cos 0 \rangle = \langle \pi, 0, 0 \rangle$$

8.5 Integration of Vector-Valued Functions

Example:

Evaluate $\int_1^3 \langle e^{2t}, t^{-1}, \sin 2t \rangle \, dt$

Solution:

$$\int_1^3 \langle e^{2t}, t^{-1}, \sin 2t \rangle \, dt = \left\langle \frac{1}{2} e^{2t}, \ln t\,, -\frac{1}{2} \cos 2t \right\rangle \Bigg|_1^3$$

$$= \left\langle \frac{1}{2} e^{6}, \ln 3\,, -\frac{1}{2} \cos 6 \right\rangle - \left\langle \frac{1}{2} e^{2}, \ln 1\,, -\frac{1}{2} \cos 2 \right\rangle = \langle 198.0199, 1.0986, 0.0024 \rangle$$

Exercise 8.7:

Evaluate

1) $\int \langle \cos 3t\, , \sin t\, , e^{4t} \rangle \, dt$

2) $\int \langle t \sin t\, , \sqrt{t+1}, t^3 \rangle \, dt$

3) $\int_1^4 \langle \sqrt{t}, 5 \rangle \, dt$

4) $\int_0^2 \langle \frac{4}{t+1}, e^{t-2}, te^t \rangle \, dt$

[Ans: $\left\langle \frac{\sin 3t}{3}, -\cos t\, , \frac{e^{4t}}{4} \right\rangle + \mathbf{c}$; $\left\langle -t\cos t + \sin t\, , \frac{2(t+1)^{\frac{3}{2}}}{3}, \frac{t^4}{4} \right\rangle + \mathbf{c}$; $\left\langle \frac{14}{3}, 15 \right\rangle$; $\langle 4\ln 3\, , 1 - e^{-2}, e^2 + 1 \rangle$]

8.5 Integration of Vector-Valued Functions

8.5.3 Arc Length

Arc length refers to the length of an arc, L on an interval $[c, d]$ that is divided into n equally subintervals. The points i on the arc or curve is denoted by P_i. By summing up the length of all line segments $|P_{i-1}P_i|$, the arch length is given by,

$$L = \lim_{n\to\infty} \sum_{i=1}^{n} |P_{i-1}P_i| = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{(x_i - x_{i-1})^2 + (y_i - y_{i-1})^2}$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{(\Delta x)^2 + (\Delta y)^2} = \lim_{n\to\infty} \sum_{i=1}^{n} \Delta x \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}.$$

Based on the definite integral definition and by approximating Δx close to zero, $\Delta x \approx dx$, the arc length can be expressed as

$$L = \int_{c}^{d} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$

8.5 Integration of Vector-Valued Functions

8.5.3 Arc Length

The arc length of indefinite integrals can be simplified using the Chain Rule as follows:

$$L = \int \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx \times \frac{dt}{dt} = \int \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \sqrt{\left(\frac{dx}{dt}\right)^2} dt$$

which also can be written as

$$L = \int \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} dt.$$

Given that a curve is defined parametrically by $x = f(t), y = g(t)$ and $z = h(t)$ where x, y and z can be differentiated with respect to t. Hence, the arc length on an interval $[c, d]$ can be expressed as

$$L = \int_c^d \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2}\, dt = \int_c^d \|\mathbf{r}'(t)\|\, dt.$$

8.5 Integration of Vector-Valued Functions

Example:

Find the arc length of $\mathbf{r}(t) = \langle \frac{1}{2}t^2, \frac{4}{3}t^{3/2}, 2t \rangle$, for $-1 \leq t \leq 2$.

Solution:

$$\mathbf{r}'(t) = \langle t, 2\sqrt{t}, 2 \rangle$$

$$\|\mathbf{r}'(t)\| = \sqrt{(t)^2 + \left(2\sqrt{t}\right)^2 + (2)^2}$$

$$= \sqrt{t^2 + 4t + 4} = \sqrt{(t+2)^2} = t + 2$$

$$L = \int_a^b \|\mathbf{r}'(t)\|\, dt = \int_{-1}^{2} t + 2\, dt = \frac{1}{2}t^2 + 2t\Big|_{-1}^{2}$$

$$= \left(\frac{1}{2}(4) + 4\right) - \left(\frac{1}{2} - 2\right) = \frac{15}{2}$$

8.5 Integration of Vector-Valued Functions

Example:

Find the arc length of $\mathbf{r}(t) = \langle 3t, 4\sin t, 4\cos t\rangle$, for $1 \le t \le 2$.

Solution:

$$\mathbf{r}'(t) = \langle 3, 4\cos t, -4\sin t\rangle$$

$$\|\mathbf{r}'(t)\| = \sqrt{(3)^2 + (4\cos t)^2 + (-4\sin t)^2}$$

$$= \sqrt{9 + 16\cos^2 t + 16\sin^2 t} = \sqrt{25} = 5$$

$$L = \int_a^b \|\mathbf{r}'(t)\|\,dt = \int_1^2 5\,dt = 5t\Big|_1^2 = 10$$

8.5 Integration of Vector-Valued Functions

Example:

Find the arc length of $\mathbf{r}(t) = \langle \frac{1}{2}t^2, \frac{4}{3}t^{3/2}, 2t \rangle$ from $(0,0,0)$ to $\left(8, \frac{32}{3}, 8\right)$.

Solution:

For $(0,0,0)$, this implies $\frac{1}{2}t^2 = 0, \frac{4}{3}t^{3/2} = 0, 2t = 0$. Hence $t = 0$.

For $\left(8, \frac{32}{3}, 8\right)$; this implies $\frac{1}{2}t^2 = 8, \frac{4}{3}t^{3/2} = \frac{32}{3}, 2t = 8$. Hence $t = 4$.

Based on the previous example,

$\mathbf{r}'(t) = \langle t, 2\sqrt{t}, 2 \rangle, \|\mathbf{r}'(t)\| = t + 2$

$$L = \int_a^b \|\mathbf{r}'(t)\|\, dt = \int_0^4 t + 2\, dt = \frac{1}{2}t^2 + 2t \Big|_0^4 = \left(\frac{1}{2}(16) + 8\right) = 16$$

8.5 Integration of Vector-Valued Functions

8.5.4 Arc Length Parameter with Base Point

The arc length of $\mathbf{r}(\tau)$ from $\tau = t_0$ to $\tau = t$ for $\mathbf{r}(\tau) = \langle f(\tau), g(\tau), h(\tau) \rangle$ is given by

$$L = \int_{t_0}^{t} \|\mathbf{r}'(\tau)\| \, d\tau$$

where $\|\mathbf{r}'(\tau)\| = \sqrt{[f'(\tau)]^2 + [g'(\tau)]^2 + [h'(\tau)]^2}$.

8.5 Integration of Vector-Valued Functions

8.5.4 Arc Length Parameter with Base Point

Example:

Find the arc length function of $\mathbf{r}(t) = \langle e^t, 2\sqrt{2}e^{t/2}, t\rangle$ from the point $(1, 2\sqrt{2}, 0)$.

Solution:

$$\mathbf{r}'(t) = \langle e^t, \sqrt{2}e^{t/2}, 1\rangle$$

$$\|\mathbf{r}'(t)\| = \sqrt{(e^t)^2 + \left(\sqrt{2}e^{t/2}\right)^2 + (1)^2} = \sqrt{e^{2t} + 2e^t + 1} = \sqrt{(e^t + 1)^2} = e^t + 1$$

From $\left(1, 2\sqrt{2}, 0\right) \Longrightarrow e^t = 1,\ 2\sqrt{2}e^{t/2} = 2\sqrt{2},\ \ t = 0$. This implies $t_0 = 0$.

Thus,

$$L = \int_{t_0}^{t} \|\mathbf{r}'(\tau)\|\, d\tau = \int_0^t e^\tau + 1\, d\tau = e^\tau + \tau\Big|_0^t = e^t + t - 1.$$

8.5 Integration of Vector-Valued Functions

8.5.4 Arc Length Parameter with Base Point

Example:

Find t such that the length of $\mathbf{r}(t) = \langle \sqrt{5}t, 2\sin t, 2\cos t \rangle$ from $(0,0,2)$ is 6π.

Solution:

Given that $L = \int_{t_0}^{t} \|\mathbf{r}'(\tau)\|\, d\tau = 6\pi$.

$\mathbf{r}'(t) = \langle \sqrt{5}, 2\cos t, -2\sin t \rangle$ and $\|\mathbf{r}'(t)\| = \sqrt{\left(\sqrt{5}\right)^2 + (2\cos t)^2 + (-2\sin t)^2} = 3.$

From $(0,0,2) \Longrightarrow \sqrt{5}t = 0, 2\sin t = 0$ and $2\cos t = 2$. This implies $t_0 = 0$.

Thus,

$$L = \int_{t_0}^{t} \|\mathbf{r}'(\tau)\|\, d\tau = \int_0^t 3\, d\tau = 3\tau\big|_0^t = 3t$$

Since $L = 6\pi$, hence $3t = 6\pi$ where $t = 2\pi$.

Exercise 8.8:

1) Find the arc length of the following curves.

a) $\mathbf{r}(t) = \langle 2\sin t\,, 5t, 2\cos t\rangle, \qquad -10 \le t \le 10.$

b) $\mathbf{r}(t) = \sqrt{2}t\,\mathbf{i} + e^{t}\,\mathbf{j} + e^{-t}\,\mathbf{k}, \qquad 0 \le t \le 1.$

2) Find the arc length $\mathbf{r}(t) = \langle t^2, \frac{2\sqrt{2}}{3}t^{3/2}, \frac{1}{2}t\rangle$ from $(1, \frac{2\sqrt{2}}{3}, \frac{1}{2})$ to $(81, 18\sqrt{2}, \frac{9}{2})$.

3) Find the arc length function of $\mathbf{r}(t) = \langle t^2, \frac{2}{3}\sqrt{12}\,t^{3/2}, 3t\rangle$ from the point $(1, \frac{2}{3}\sqrt{12}, 3)$.

4) Find t such that the length of $\mathbf{r}(t) = \langle 2\cos t\,, 2\sin t\,, \sqrt{5}t\rangle$ from $(2,0,0)$ is 3π.

[Ans: $20\sqrt{29}; e - e^{-1}; 84;\ t^2 + 3t - 4;\ \pi$]

8.6 Motion along a Curve and Motion in Space

8.6.1 Motion along a Curve

A motion along a curve for the two-dimensional (2D) vector-valued function at any time t, is given the following

i. Position vector, $\mathbf{r}(t) = \langle x(t), y(t) \rangle$,

ii. Velocity vector, $\mathbf{v}(t) = \mathbf{r}'(t)$,

iii. Speed, $s = \|\mathbf{v}(t)\|$,

iv. Acceleration: $\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t)$.

Note that $\mathbf{r}(t) = \int \mathbf{v}(t)\, dt$.

8.6 Motion along a Curve and Motion in Space

8.6.1 Motion along a Curve

Example:

Find the velocity and speed of a particle with position vector

$$\mathbf{r}(t) = \langle 3\cos t\,, 3\sin t \rangle.$$

Solution:

Velocity,

$\mathbf{v}(t) = \mathbf{r}'(t) = \langle -3\sin t\,, 3\cos t \rangle$

Speed,

$s = \|\mathbf{v}(t)\| = \sqrt{9\cos^2 t + 9\sin^2 t} = \sqrt{9(\sin^2 t + \cos^2 t)} = 3$

8.6 Motion along a Curve and Motion in Space

8.6.1 Motion along a Curve

Example:

Find the acceleration of a particle with position vector

$$\mathbf{r}(t) = \langle t + e^{-t} - 1, \frac{t^2}{2} + t \rangle.$$

Solution:

Based on the velocity,

$$\mathbf{v}(t) = \mathbf{r}'(t) = \langle 1 - e^{-t}, t + 1 \rangle$$

the acceleration is given by,

$$\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t) = \langle e^{-t}, 1 \rangle$$

8.6 Motion along a Curve and Motion in Space

8.6.1 Motion along a Curve

Example:

Given a velocity vector $\mathbf{v}(t) = \langle t+1, t^2 \rangle$ and an initial position $\mathbf{r}(0) = \langle 1, -1 \rangle$. Find the position vector for $t \geq 0$.

Solution:

$$\mathbf{r}(t) = \int \langle t+1, t^2 \rangle \, dt = \langle \tfrac{1}{2} t^2 + t + c_1, \tfrac{1}{3} t^3 + c_2 \rangle$$

$\mathbf{r}(0) = \langle c_1, c_2 \rangle$ and given $\mathbf{r}(0) = \langle 1, -1 \rangle$

This implies $c_1 = 1$ and $c_2 = -1$. Hence the position vector is given by

$$\mathbf{r}(t) = \langle \tfrac{1}{2} t^2 + t + 1, \tfrac{1}{3} t^3 - 1 \rangle \text{ for } t \geq 0$$

8.6 Motion along a Curve and Motion in Space

8.6.2 Motion in Space

A particle moves through the three-dimensional (3D) space at any time t, is given the following

i. Position vector, $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$,

ii. Velocity vector, $\mathbf{v}(t) = \mathbf{r}'(t)$,

iii. Speed, $s = \|\mathbf{v}(t)\|$,

iv. Acceleration: $\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t)$.

8.6.2 Motion in Space

Example:

Find the velocity and speed of a particle with position vector

$$\mathbf{r}(t) = \langle 3\sin t\,, 5\cos t\,, 4\sin t \rangle.$$

Solution:

Velocity, $\mathbf{v}(t) = \mathbf{r}'(t) = \langle 3\cos t\,, -5\sin t\,, 4\cos t \rangle$

Speed, $\|\mathbf{v}(t)\| = \sqrt{9\cos^2 t + 25\sin^2 t + 16\cos^2 t}$

$= \sqrt{25\sin^2 t + 25\cos^2 t} = 5$

8.6 Motion along a Curve and Motion in Space

8.6.2 Motion in Space

Example:

Find the acceleration of a particle with position vector

$$\mathbf{r}(t) = \langle 3\sin t, 5\cos t, 4\sin t\rangle.$$

Solution:

Based on the previous example, $\mathbf{r}'(t) = \langle 3\cos t, -5\sin t, 4\cos t\rangle$

Hence, acceleration, $\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t)$ is given by

$$\mathbf{r}''(t) = \langle -3\sin t, -5\cos t, -4\sin t\rangle$$

Exercise 8.9:

1) Find the acceleration for a particle with position vector

$$\mathbf{r}(t) = \langle \sin 2t\,, \cos 2t\,, \sin 2t \rangle.$$

[Ans: $\mathbf{a}(t) = \langle -4\sin 2t\,, -4\cos 2t\,, -4\sin 2t \rangle$]

2) Given an acceleration vector, $\mathbf{a}(t) = \langle 1, t \rangle$, initial velocity $\mathbf{v}(0) = \langle 2, -1 \rangle$ and initial position $\mathbf{r}(0) = \langle 0, 8 \rangle$, find the velocity and position vector for $t \geq 0$.

[Ans: $\mathbf{r}(t) = \langle \frac{t^2}{2} + 2t, \frac{t^3}{6} - t + 8 \rangle$]

3) Given an acceleration vector, $\mathbf{a}(t) = \langle t, e^{-t}, 1 \rangle$, initial velocity $\mathbf{v}(0) = \langle 0, 0, 1 \rangle$ and initial position $\mathbf{r}(0) = \langle 4, 0, 0 \rangle$, find the velocity and position vector for $t \geq 0$.

[Ans: $\mathbf{r}(t) = \langle \frac{t^3}{6} + 4, t + e^{-t} - 1, \frac{t^2}{2} + t \rangle$]