ENGINEERING MATHEMATICS 1
BMFG 1313
DIFFERENTIATION OF VECTOR-VALUED FUNCTIONS

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Lesson Outcomes

Upon completion of this lesson, the student should be able to:

- Compute operation of Vector-Valued functions
- Evaluate the differentiation of Vector-Valued functions

8.1 Introduction

A vector-valued function denoted by $\mathbf{r}$, is a function where the domain is a set of real numbers and the range is a set of vectors.

A vector-valued function or vector function may expresses or indicates the position of a moving particle at any particular of time, t.

A vector-valued function can be written as

$$\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$$

for some scalar functions f, g and h of t, which is called the component functions of $\mathbf{r}$.

8.1 Introduction

A vector-valued function or vector function may expresses or indicates the position of a moving particle at any particular of time, t, as shown in Figure 1.

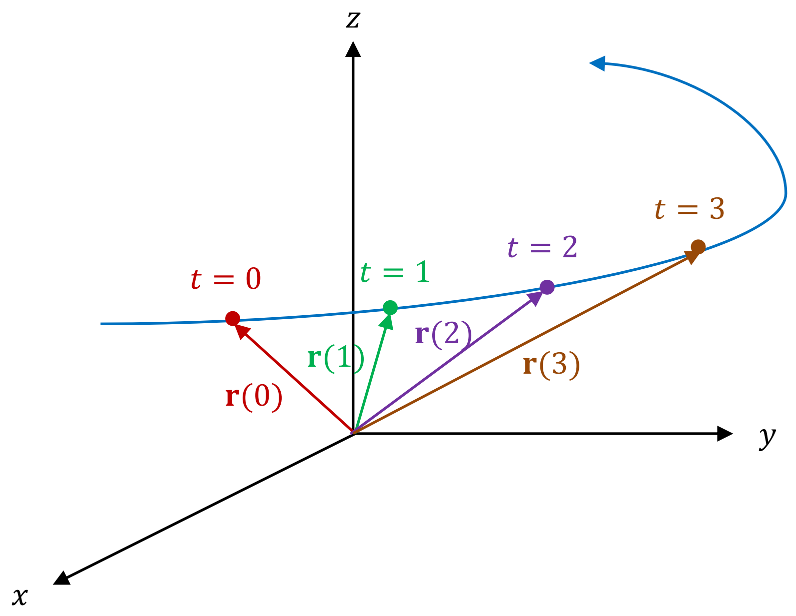

Figure 1: Vectors indicating a particle's position at several times

8.2 Operations of Vector-Valued Functions

Given two vector functions,

$$\mathbf{F}(t) = x_1(t)\mathbf{i} + y_1(t)\mathbf{j} + z_1(t)\mathbf{k}$$
$$\mathbf{G}(t) = x_2(t)\mathbf{i} + y_2(t)\mathbf{j} + z_2(t)\mathbf{k}$$

1) Vector Sum

$$\mathbf{F}(t) + \mathbf{G}(t) = [x_1(t) + x_2(t)]\mathbf{i} + [y_1(t) + y_2(t)]\mathbf{j} + [z_1(t) + z_2(t)]\mathbf{k}$$

2) Product of a scalar-valued function and a vector-valued function

$$f(t)\mathbf{F}(t) = f(t)x_1(t)\mathbf{i} + f(t)y_1(t)\mathbf{j} + f(t)z_1(t)\mathbf{k}$$

3) Dot Product

$$\mathbf{F}(t) \cdot \mathbf{G}(t) = x_1(t)x_2(t) + y_1(t)y_2(t) + z_1(t)z_2(t)$$

8.2 Operations of Vector-Valued Functions

Given two vector functions,

$$\mathbf{F}(t) = x_1(t)\mathbf{i} + y_1(t)\mathbf{j} + z_1(t)\mathbf{k} \text{ and } \mathbf{G}(t) = x_2(t)\mathbf{i} + y_2(t)\mathbf{j} + z_2(t)\mathbf{k}$$

4) Cross Product

$$\mathbf{F}(t) \times \mathbf{G}(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_1(t) & y_1(t) & z_1(t) \\ x_2(t) & y_2(t) & z_2(t) \end{vmatrix}$$

$$= \big(y_1(t)z_2(t) - y_2(t)z_1(t)\big)\mathbf{i} - \big(x_1(t)z_2(t) - x_2(t)z_1(t)\big)\mathbf{j}$$
$$+ \big(x_1(t)y_2(t) - x_2(t)y_1(t)\big)\mathbf{k}$$

5) Magnitude

$$\|\mathbf{F}(t)\| = \sqrt{[x_1(t)]^2 + [y_1(t)]^2 + [z_1(t)]^2}$$
$$\|f(t)\mathbf{F}(t)\| = |f(t)|\|\mathbf{F}(t)\|$$

8.2 Operations of Vector-Valued Functions

Example:

Given that $\mathbf{F}(t) = e^{-2t}\mathbf{i} - e^{3t}\mathbf{j} - t\mathbf{k}$ and $\mathbf{G}(t) = e^{-t}\mathbf{i} + e^{4t}\mathbf{j} - 4\mathbf{k}$.

Compute $e^t\mathbf{F}(t) + 2\mathbf{G}(t)$.

Solution:

$$\begin{aligned} e^t\mathbf{F}(t) + 2\mathbf{G}(t) &= e^t\langle e^{-2t}, -e^{3t}, -t\rangle + 2\langle e^{-t}, e^{4t}, -4\rangle \\ &= \langle e^{-t}, -e^{4t}, -te^{t}\rangle + \langle 2e^{-t}, 2e^{4t}, -8\rangle \\ &= \langle 3e^{-t}, e^{4t}, -te^{t} - 8\rangle \end{aligned}$$

8.2 Operations of Vector-Valued Functions

Example:

Given that $f(t) = t + 1$ and $\mathbf{F}(t) = \sin t\,\mathbf{i} + \cos t\,\mathbf{j} + \mathbf{k}$.

Compute $\|f(t)\mathbf{F}(t)\|$.

Solution:

$$\begin{aligned}\|f(t)\mathbf{F}(t)\| &= |f(t)|\,\|\mathbf{F}(t)\| \\ &= |(t+1)|\,\|\langle \sin t, \cos t, 1\rangle\| \\ &= (t+1)\sqrt{\sin^2 t + \cos^2 t + 1} \\ &= \sqrt{2}(t+1)\end{aligned}$$

8.2 Operations of Vector-Valued Functions

Example:

Given that $f(t) = -t^2$ and $\mathbf{F}(t) = 2\mathbf{i} + 2\sqrt{t}\mathbf{j} + t\mathbf{k}$.

Compute $\|f(t)\mathbf{F}(t)\|$.

Solution:

$$\begin{aligned}
\|f(t)\mathbf{F}(t)\| &= |f(t)|\|\mathbf{F}(t)\| \\
&= |-t^2|\|\langle 2,2\sqrt{t}, t\rangle\| \\
&= t^2\sqrt{4 + 4t + t^2} \\
&= t^2\sqrt{(t + 2)^2} \\
&= t^2(t + 2)
\end{aligned}$$

Exercise 8.1:

Given that $\mathbf{F}(t) = t^2\mathbf{i} + t\mathbf{j} - \sin t\,\mathbf{k}$ and $\mathbf{G}(t) = t\mathbf{i} + \frac{1}{t}\mathbf{j} - 5\mathbf{k}$. Find

1) $e^t\mathbf{F}(t)$

2) $\mathbf{F}(t) + 2\mathbf{G}(t)$

3) $t\mathbf{F}(t) - 3e^t\mathbf{G}(t)$

4) $t^2\mathbf{G}(t) + t^{-1}\mathbf{F}(t)$

5) $\|-2\mathbf{F}(t)\|$

[Ans: $\langle t^2e^t, te^t, -e^t \sin t\rangle$; $\left\langle t^2 + 2t, t + \frac{2}{t}, -\sin t - 10\right\rangle$;

$\left\langle t^3 - 3te^t, t^2 - \frac{3e^t}{t}, -t\sin t + 15e^t\right\rangle$; $\left\langle t^3 + t, t + 1, -5t^2 - \frac{\sin t}{t}\right\rangle$; $2\sqrt{t^4 + t^2 + \sin^2 t}$]

8.2.1 Dot Product

Dot product is used to measure the angle between two vectors. It is also needed to calculate projections of vector. The dot product of two vectors is a **scalar**. Hence, it is also known as scalar product.

Let $\mathbf{a}(t) = \langle a_1(t), a_2(t), a_3(t)\rangle$ and $\mathbf{b}(t) = \langle b_1(t), b_2(t), b_3(t)\rangle$. Hence

$$\begin{aligned}\mathbf{a}\cdot\mathbf{b} &= \langle a_1(t), a_2(t), a_3(t)\rangle \cdot \langle b_1(t), b_2(t), b_3(t)\rangle \\ &= a_1(t)b_1(t) + a_2(t)b_2(t) + a_3(t)b_3(t)\end{aligned}$$

Note that, $\mathbf{a}$ and $\mathbf{b}$ are orthogonal if $\mathbf{a}\cdot\mathbf{b} = 0$

8.2.1 Dot Product

Properties of dot product for a vector-valued function is the same as the constant vector:

Let $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ be vectors and k be a scalar.

- $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$ (Commutative property)
- $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$ (Distributive property)
- $(\mathbf{a} + \mathbf{b}) \cdot \mathbf{c} = \mathbf{a} \cdot \mathbf{c} + \mathbf{b} \cdot \mathbf{c}$ (Distributive property)
- $k(\mathbf{a} \cdot \mathbf{b}) = k\mathbf{a} \cdot \mathbf{b} = \mathbf{a} \cdot k\mathbf{b}$ (Associative property)
- $\mathbf{a} \cdot \mathbf{0} = \mathbf{0} \cdot \mathbf{a} = 0$
- $\mathbf{a} \cdot \mathbf{a} = \|\mathbf{a}\|^2$
- $\mathbf{a} \cdot \mathbf{b} = \|\mathbf{a}\|\|\mathbf{b}\|\cos\theta$

8.2.1 Dot Product

Example:

Find the dot product of $\mathbf{F}(t) = \langle \sin t, \cos t, \ln t \rangle$ and $\mathbf{G}(t) = \langle \cos t, \sin t, t \rangle$.

Solution:

$$\begin{aligned}\mathbf{F} \cdot \mathbf{G} &= \langle \sin t, \cos t, \ln t \rangle \cdot \langle \cos t, \sin t, t \rangle \\ &= \sin t \cos t + \cos t \sin t + t \ln t \\ &= 2 \sin t \cos t + t \ln t\end{aligned}$$

8.2.1 Dot Product

Example:

Given vectors $\mathbf{F}(t) = \langle -e^t, 3e^{2t}, e^{-2t} \rangle$ and $\mathbf{G}(t) = \langle k, e^{-t}, 2e^{3t} \rangle$. Find the value of k if vectors $\mathbf{F}$ and $\mathbf{G}$ are perpendicular.

Solution:

Given that $\mathbf{F}$ and $\mathbf{G}$ are perpendicular. Hence $\mathbf{F} \cdot \mathbf{G} = 0$. Thus

$$\mathbf{F} \cdot \mathbf{G} = \langle -e^t, 3e^{2t}, e^{-2t} \rangle \cdot \langle k, e^{-t}, 2e^{3t} \rangle = 0$$

$$-ke^t + 3e^t + 2e^t = 0$$

$$e^t(5-k) = 0$$

Since $e^t \neq 0$, therefore

$$5 - k = 0$$

$$\therefore k = 5$$

Exercise 8.2:

1) Find the dot product for each of the following pairs of vector valued functions.

a) $\mathbf{F}(t) = \langle -1, 9, -3 \rangle$ and $\mathbf{G}(t) = \langle -8, -3, 4 \rangle$

b) $\mathbf{F}(t) = \langle t^2, \sin t, \cos t \rangle$ and $\mathbf{G}(t) = \langle e^t, \sin t, \cos t \rangle$

c) $\mathbf{F}(t) = \langle t^{-2}, 2, -e^{2t} \rangle$ and $\mathbf{G}(t) = \langle t^5, -6, e^{-t} \rangle$

2) Show that $\mathbf{A}(t) = \langle 2t, t^3, t^2 \rangle$ and $\mathbf{B}(t) = \langle t^3, -3t, t^2 \rangle$ are orthogonal.

[Ans: $-31; t^2e^t + 1; t^3 - 12 - e^t$]

8.2.2 Cross Product

In this section, the second type of product of vectors is introduced, which is the *cross product*. The cross product of two vectors produces another **vector**.

$$\mathbf{a} \times \mathbf{b} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \times \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = [a_2 b_3 - a_3 b_2]\mathbf{i} - [a_1 b_3 - a_3 b_1]\mathbf{j} + [a_1 b_2 - a_2 b_1]\mathbf{k}$$

Note that, $\mathbf{a} \times \mathbf{b}$ is orthogonal to $\mathbf{a}$ and $\mathbf{b}$

8.2.2 Cross Product

Properties of cross product for a vector-valued function is the same as the constant vector:

Let $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ be vectors and k be a scalar.

- $\mathbf{a} \times \mathbf{b} = -(\mathbf{b} \times \mathbf{a})$
- $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$ (Distributive property)
- $(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}$ (Distributive property)
- $k(\mathbf{a} \times \mathbf{b}) = k\mathbf{a} \times \mathbf{b} = \mathbf{a} \times k\mathbf{b}$ (Associative property)
- $\mathbf{a} \times \mathbf{0} = \mathbf{0} \times \mathbf{a} = \mathbf{0}$
- $\mathbf{a} \times \mathbf{a} = \mathbf{0}$
- $\|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{a}\|\|\mathbf{b}\|\sin\theta$

8.2.2 Cross Product

Example:

Find the cross product of $\mathbf{F}(t) = \langle \sin t, \cos t, \ln t \rangle$ and $\mathbf{G}(t) = \langle \cos t, \sin t, t \rangle$.

Solution:

$$\mathbf{F} \times \mathbf{G} = \begin{bmatrix} \sin t \\ \cos t \\ \ln t \end{bmatrix} \times \begin{bmatrix} \cos t \\ \sin t \\ t \end{bmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \sin t & \cos t & \ln t \\ \cos t & \sin t & t \end{vmatrix}$$

$$= [t \cos t - \ln t \sin t]\mathbf{i} - [t \sin t - \ln t \cos t]\mathbf{j} + [\sin^2 t - \cos^2 t]\mathbf{k}$$

8.2.2 Cross Product

Example:

Find a vector which is perpendicular to $\mathbf{F}(t) = \langle t^2, e^{2t}, 4 \rangle$ and $\mathbf{G}(t) = \langle 7t, 2, e^{-t} \rangle$.

Solution:

$$\mathbf{F} \times \mathbf{G} = \begin{bmatrix} t^2 \\ e^{2t} \\ 4 \end{bmatrix} \times \begin{bmatrix} 7t \\ 2 \\ e^{-t} \end{bmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ t^2 & e^{2t} & 4 \\ 7t & 2 & e^{-t} \end{vmatrix}$$

$$= [(e^{2t})(e^{-t}) - (4)(2)]\mathbf{i} - [(t^2)(e^{-t}) - (4)(7t)]\mathbf{j} + [(t^2)(2) - (e^{2t})(7t)\,]\mathbf{k}$$

$$= [e^t - 8]\mathbf{i} - [t^2 e^{-t} - 28t]\mathbf{j} + [2t^2 - 7te^{2t}]\mathbf{k}$$

Exercise 8.3:

1) Find the cross product for each of the following pairs of vector valued functions.

a) $\mathbf{F}(t) = \langle -1,9,-3 \rangle$ and $\mathbf{G}(t) = \langle -8,-3,4 \rangle$

b) $\mathbf{F}(t) = \langle t^2, \sin t, \cos t \rangle$ and $\mathbf{G}(t) = \langle e^t, \sin t, \cos t \rangle$

c) $\mathbf{F}(t) = \langle t^{-2}, 2, -e^{2t} \rangle$ and $\mathbf{G}(t) = \langle t^5, -6, e^{-t} \rangle$

d) $\mathbf{F}(t) = \langle t+1, e^t, \sqrt{t} \rangle$ and $\mathbf{G}(t) = \langle \sin t, 2t, 1 \rangle$

2) Given $\mathbf{A}(t) = \langle 2t, t^3, t^2 \rangle$ and $\mathbf{B}(t) = \langle t^3, -3t, t^2 \rangle$, find the vector which is perpendicular to vectors $\mathbf{A}$ and $\mathbf{B}$.

[Ans: $\langle 27,28,75 \rangle$; $\langle 0, e^t \cos t - t^2 \cos t, t^2 \sin t - e^t \sin t \rangle$; $\langle 2e^{-t} - 6e^{2t}, -t^5 e^{2t} - t^{-2} e^{-t}, -6t^{-2} - 2t^5 \rangle$;
$\left\langle e^t - 2t^{\frac{3}{2}}, \sqrt{t} \sin t - t - 1, 2t^2 + 2t - e^t \sin t \right\rangle$; $\langle t^5 + 3t^3, t^5 - 2t^3, -6t^2 - t^6 \rangle$]

8.3 Differentiation of Vector Valued Function

Differentiation of vector-valued functions $\mathbf{r}(t)$ is somehow one applies the rules of differentiation to the individual components of $\mathbf{r}$.

Derivative and Tangent Vector

Let

$$\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k},$$

where $f, g,$ and h are differentiable functions on (a, b). Then $\mathbf{r}$ has a derivative on (a, b) and

$$\mathbf{r}'(t) = f'(t)\mathbf{i} + g'(t)\mathbf{j} + h'(t)\mathbf{k}.$$

Note that $\mathbf{r}'(t)$ is a **tangent vector** at the point corresponding to $\mathbf{r}(t)$, such that $\mathbf{r}'(t) \neq 0$.

8.3 Differentiation of Vector Valued Function

Example:

Compute the derivative of $\mathbf{F}(t) = \langle 2\cos t, 4\sin t, 5t \rangle$.

Solution:

$$\mathbf{F}'(t) = \langle \frac{d}{dt}(2\cos t), \frac{d}{dt}(4\sin t), \frac{d}{dt}(5t) \rangle$$
$$= \langle -2\sin t, 4\cos t, 5 \rangle.$$

8.3 Differentiation of Vector Valued Function

Example:

Compute the acceleration of the given path:

$$\mathbf{F}(t) = \langle \ln t\,, t^3, 5t + e^t \rangle.$$

Solution:

Velocity, $\mathbf{F}'(t) = \langle \frac{d}{dt}(\ln t), \frac{d}{dt}(t^3), \frac{d}{dt}(5t + e^t) \rangle$

$$= \langle \frac{1}{t}, 3t^2, 5 + e^t \rangle.$$

Acceleration, $\mathbf{F}''(t) = \langle \frac{d}{dt}(\frac{1}{t}), \frac{d}{dt}(3t^2), \frac{d}{dt}(5 + e^t) \rangle$

$$= \langle -\frac{1}{t^2}, 6t, e^t \rangle.$$

8.3 Differentiation of Vector Valued Function

Example:

Compute the speed of the given path:

$$\mathbf{F}(t) = \langle 1, \sqrt{2}t, t^2 \rangle.$$

Solution:

Speed, $\|\mathbf{F}(t)\| = \sqrt{(1)^2 + \left(\sqrt{2}t\right)^2 + (t^2)^2}$

$$= \sqrt{1 + 2t^2 + t^4}$$

$$= \sqrt{(1 + t^2)^2}$$

$$= 1 + t^2$$

Exercise 8.4:

1) Compute the derivative of the following position vector valued functions.

a) $\mathbf{F}(t) = \langle e^{2t}, 4e^{t}, te^{t} \rangle$

b) $\mathbf{F}(t) = \langle t^{-2}, 2, -e^{2t} \rangle$

c) $\mathbf{F}(t) = \langle t + 1, e^{t}, \sqrt{t} \rangle$

d) $\mathbf{F}(t) = \langle t^{4}, \sqrt{t + 1}, \frac{3}{t^{2}} \rangle$

2) Calculate the velocity, speed and acceleration of the paths given as follows.

a) $\mathbf{F}(t) = \langle 3t - 5, 2t + 7 \rangle$

b) $\mathbf{F}(t) = \langle 5 \cos t, 3 \sin t \rangle$

c) $\mathbf{F}(t) = \langle t \sin t, t \cos t, t^{2} \rangle$

d) $\mathbf{F}(t) = \langle e^{t}, e^{2t}, 2e^{t} \rangle$

[Ans 1): $\langle 2e^{2t}, 4e^{t}, e^{t} + te^{t} \rangle$; $\langle -2t^{-3}, 0, -2e^{2t} \rangle$; $\left\langle 1, e^{t}, \frac{1}{2} t^{-\frac{1}{2}} \right\rangle$; $\left\langle 4t^{3}, \frac{1}{2}(t+1)^{-\frac{1}{2}}, -6t^{-3} \right\rangle$]

[Ans 2): $\sqrt{13}$; $\sqrt{9 + 16 \sin^{2} t}$; $\sqrt{1 + 5t^{2}}$; $e^{t}\sqrt{5 + 4e^{2t}}$]

8.3.1 Derivative Rules

Let $\mathbf{u}$ and $\mathbf{v}$ be differentiable vector-valued functions and f be a differentiable scalar-valued function and let $\mathbf{c}$ be a constant vector. The following rules apply.

- $\frac{d}{dt}(\mathbf{c}) = \mathbf{0}$ (Constant Rule)
- $\frac{d}{dt}\big(\mathbf{u}(t) + \mathbf{v}(t)\big) = \mathbf{u}'(t) + \mathbf{v}'(t)$ (Sum Rule)
- $\frac{d}{dt}\big(f(t)\mathbf{u}(t)\big) = f'(\mathrm{t})\mathbf{u}(t) + f(t)\mathbf{u}'(t)$ (Product Rule)
- $\frac{d}{dt}\big(\mathbf{u}(t) \cdot \mathbf{v}(t)\big) = \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t)$ (Dot Product Rule)
- $\frac{d}{dt}\big(\mathbf{u}(t) \times \mathbf{v}(t)\big) = \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t)$ (Cross Product Rule)

8.3.1 Derivative Rules

Example:

Compute $\frac{d}{dt}\big(f(t)\mathbf{u}(t)\big)$ where $f(t) = 2t$ and $\mathbf{u}(t) = \langle e^{2t}, 3t, \sin t\rangle$ by using product rule.

Solution:

$$\begin{aligned}\frac{d}{dt}\big(f(t)\mathbf{u}(t)\big) &= f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t)\\ &= 2\langle e^{2t}, 3t, \sin t\rangle + 2t\langle 2e^{2t}, 3, \cos t\rangle\\ &= \langle 2e^{2t}, 6t, 2\sin t\rangle + \langle 4te^{2t}, 6t, 2t\cos t\rangle\\ &= \langle 2e^{2t} + 4te^{2t}, 12t, 2\sin t + 2t\cos t\rangle\end{aligned}$$

8.3.1 Derivative Rules

Example:

Compute $\frac{d}{dt}\big(\mathbf{u}(t)\cdot\mathbf{v}(t)\big)$ where $\mathbf{u}(t) = \langle \sin t\,, \cos t\,, 2t\rangle$ and $\mathbf{v}(t) = \langle \cos t\,, \sin t\,, 3t\rangle$ by using dot product rule.

Solution:

$$
\begin{aligned}
\frac{d}{dt}\big(\mathbf{u}(t)\cdot\mathbf{v}(t)\big) &= \mathbf{u}'(t)\cdot\mathbf{v}(t) + \mathbf{u}(t)\cdot\mathbf{v}'(t) \\
&= \langle \cos t\,, -\sin t\,, 2\rangle \cdot \langle \cos t\,, \sin t\,, 3t\rangle \\
&\quad + \langle \sin t\,, \cos t\,, 2t\rangle \cdot \langle -\sin t\,, \cos t\,, 3\rangle \\
&= \cos^2 t - \sin^2 t + 6t + (-\sin^2 t) + \cos^2 t + 6t \\
&= 2\cos^2 t - 2\sin^2 t + 12t
\end{aligned}
$$

8.3.1 Derivative Rules

Example:

Compute $\frac{d}{dt}\left(\mathbf{u}(t) \times \mathbf{v}(t)\right)$ where $\mathbf{u}(t) = \langle e^{2t}, \cos t\,, 4t \rangle$ and $\mathbf{v}(t) = \langle e^{2t}, \sin t\,, 3t \rangle$ by using cross product rule.

Solution:

$$\frac{d}{dt}\left(\mathbf{u}(t) \times \mathbf{v}(t)\right) = \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t)$$

$$= \begin{bmatrix} 2e^{2t} \\ -\sin t \\ 4 \end{bmatrix} \times \begin{bmatrix} e^{2t} \\ \sin t \\ 3t \end{bmatrix} + \begin{bmatrix} e^{2t} \\ \cos t \\ 4t \end{bmatrix} \times \begin{bmatrix} 2e^{2t} \\ \cos t \\ 3 \end{bmatrix}$$

$$= \langle -3t \sin t - 4 \sin t\,, -6te^{2t} + 4e^{2t}, 2e^{2t} \sin t + e^{2t} \sin t \rangle$$

$$+ \langle 3 \cos t - 4t \cos t\,, -3e^{2t} + 8te^{2t}, e^{2t} \cos t - 2e^{2t} \cos t \rangle$$

$$= \langle -(3t + 4) \sin t + (3 - 4t) \cos t\,, e^{2t}(2t + 1), e^{2t}(3 \sin t - \cos t) \rangle$$

Exercise 8.5:

1) Compute the following derivatives.

a) $\frac{d}{dt}\left(2t^3\mathbf{u}(t)\right)$ given $\mathbf{u}(t) = t\mathbf{i} - t^2\mathbf{j} - t^3\mathbf{k}$

b) $\frac{d}{dt}\left(\sqrt{t^2 - 1}\,\langle t, 1, 2t\rangle\right)$

2) Let $\mathbf{F}(t) = (\sin t)\mathbf{i} + 4t^2\mathbf{k}$, $\mathbf{G}(t) = (-\cos t)\mathbf{i} + 2\mathbf{j} + (2\mathrm{t} - 1)\mathbf{k}$ and $f(t) = e^{2t}$.Find

a) $\frac{d}{dt}[f(t)\mathbf{F}(t)]$

b) $\frac{d}{dt}[\mathbf{F}(t) \cdot \mathbf{G}(t)]$

c) $\frac{d}{dt}[\mathbf{F}(t) - \mathbf{G}(t)]$

[Ans 1): $2\mathrm{t}^3\langle 4, -5t, -6t^2\rangle$; $(t^2 - 1)^{-\frac{1}{2}}\langle 2t^2 - 1, t, 4t^2 - 2\rangle$]

2): $\langle e^{2t}(2\sin t + \cos t), 0, 8e^{2t}t(t + 1)\rangle$; $-\cos^2 t + \sin^2 t + 24t^2 - 8t$; $\langle \cos t - \sin t\,, 0, 2(4t - 1)\rangle$]