ENGINEERING MATHEMATICS 1
BMFG 1313
NUMERICAL INTEGRATION

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Lesson Outcomes

Upon completion of this lesson, the student should be able to:

- Solve the definite integral using Trapezoidal, Simpson's 1/3 and Simpson's 3/8 method
- Calculate area under a curve

Numerical Integration

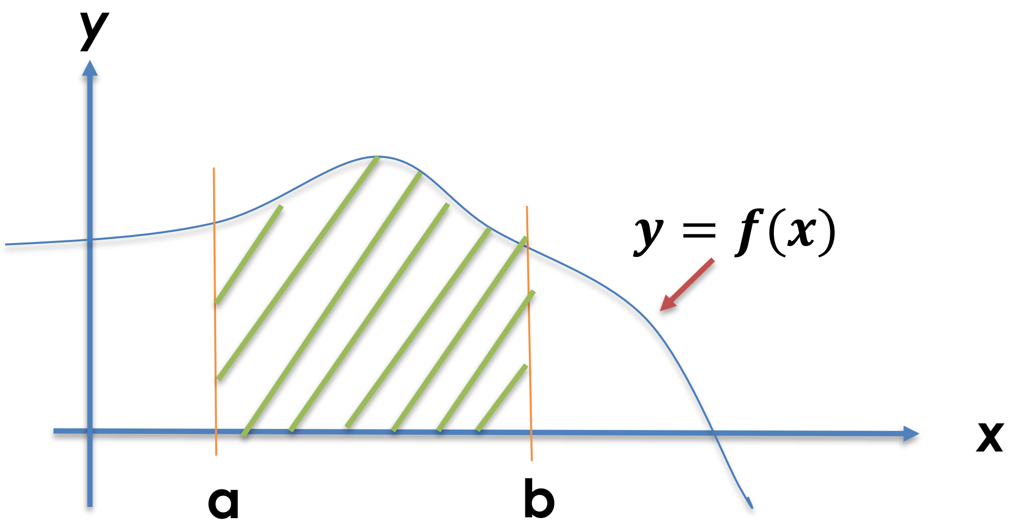
(Estimating the numerical value of a definite integral)

- **Trapezoidal Rule** → Approximating area under graph by approximate it to a **trapezoid** → **Approximated solution**
- **Simpson's 1/3 Rule** → Approximating the area under **quadratic polynomial** → **Approximated solution**
- **Simpson's 3/8 Rule** → Approximating the area under **cubic polynomial** → **Approximated solution**

7.5 Introduction to Numerical Integration

- In calculus, integration is the reverse of differentiation.
- It represents area under a curve.

$$I = \int_a^b f(x) \; dx$$



7.5 Introduction to Numerical Integration

When we need numerical integration?

Situation 1

Analytical integration may be impossible, e.g.

$$\int_0^1 e^{x^2}\,dx$$

or complicated, e.g.

$$\int_0^3 e^{\frac{1}{x}} \sin\sqrt{\ln(3x + x^2)}\;dx$$

Situation 2

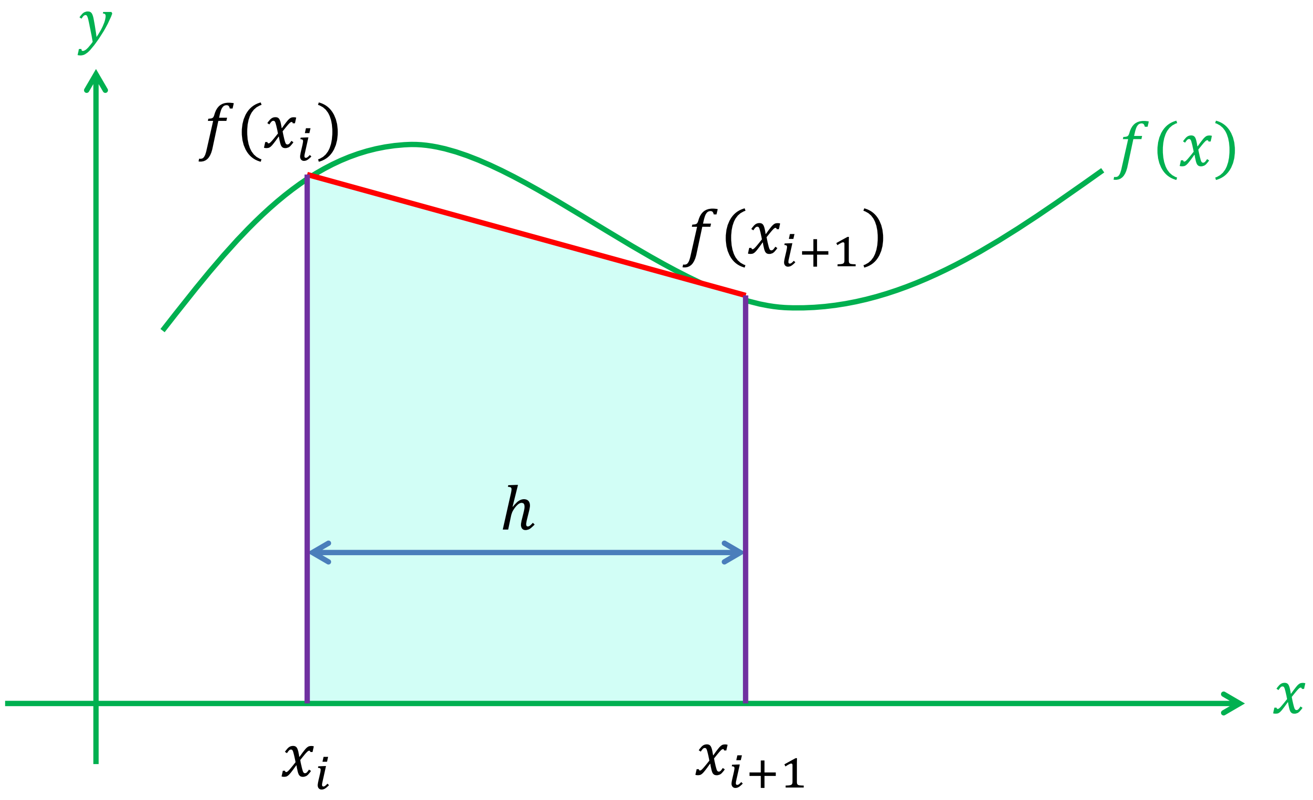
Given tabulated data, e.g.

x	1.0	1.5	2.0
$f(x)$	3.4	5.7	7.3

Estimate

$$\int_{1.0}^{2.0} [f(x)]^3\,dx$$

7.6 Trapezoidal Rule

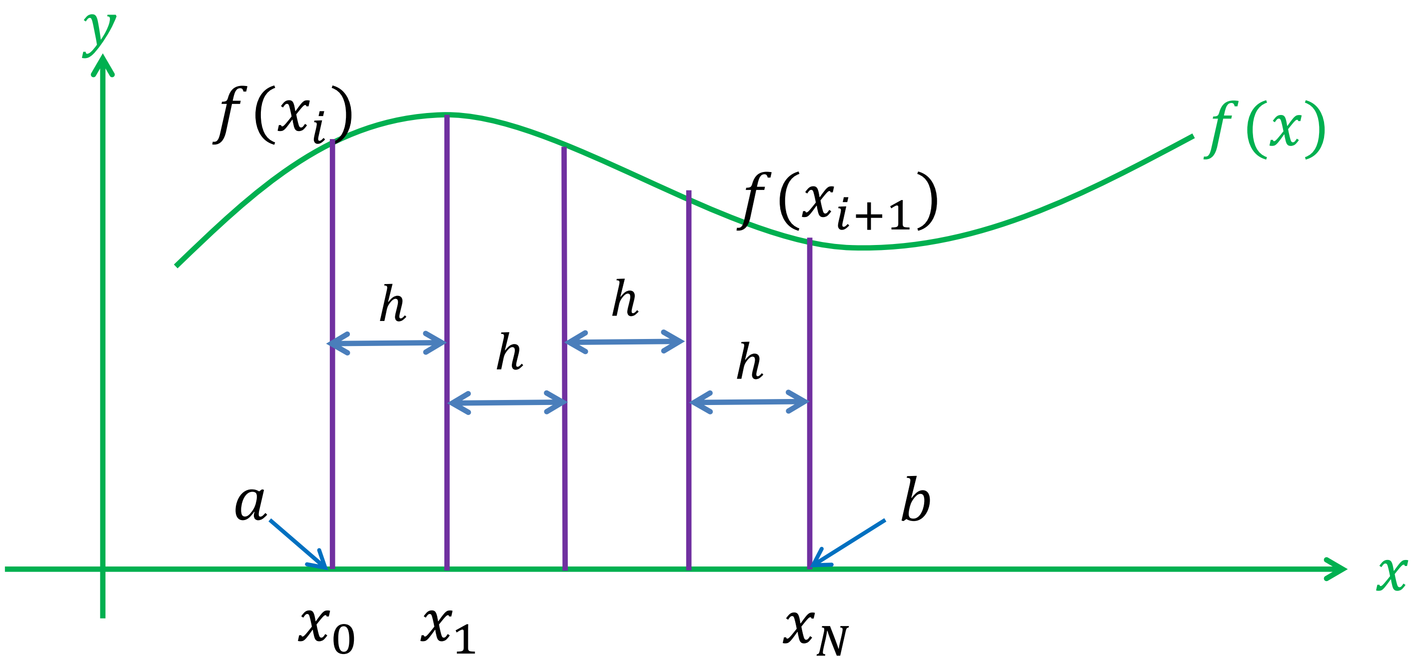


$$\text{Area} = \int_{x_i}^{x_{i+1}} f(x)\,dx \approx \frac{h}{2}[f_i + f_{i+1}]$$

⇐ Area of trapezoid

7.6 Trapezoidal Rule

When the interval $[a, b]$ is divided into N equal strips of width h such that $x_i = a + ih, i = 0,1,2,3, \dots, N$ and $b = a + Nh$,



$$\int_{x_0}^{x_N} f(x)\,dx = \int_{x_0}^{x_1} f(x)\,dx + \int_{x_1}^{x_2} f(x)\,dx + \cdots + \int_{x_{N-1}}^{x_N} f(x)\,dx$$

7.6 Trapezoidal Rule

$$\int_{x_0}^{x_N} f(x)\,dx = \int_{x_0}^{x_1} f(x)\,dx + \int_{x_1}^{x_2} f(x)\,dx + \cdots + \int_{x_{N-1}}^{x_N} f(x)\,dx$$

$$= \frac{h}{2}[f_0 + f_1] + \frac{h}{2}[f_1 + f_2] + \cdots + \frac{h}{2}[f_{N-1} + f_N]$$

Simplify and yields **Trapezoidal Composite Rule**:

$$\int_{x_0}^{x_N} f(x)\,dx = \frac{h}{2}[f_0 + f_N + 2(f_1 + f_2 + \cdots f_{N-1})]$$

7.6 Trapezoidal Rule

Example 1:

Given the following data:

x	1.2	1.3	1.4	1.5	1.6	1.7
$f(x)$	2.2	2.1	4.7	6.2	5.5	5.0

1) Use Trapezoidal Rule to estimate the area, $A = \int_{1.2}^{1.7} f(x)\, dx$. Let number of intervals be $N = 5$.

2) Use Trapezoidal Rule to estimate the volume,

$$V = \pi \int_{1.2}^{1.7} [f(x)]^2\, dx$$

with $h = 0.1$.

7.6 Trapezoidal Rule

Solution:

1) $A = \int_{1.2}^{1.7} f(x)\,dx, N = 5.$

i	x_i	H&T $f(xi)$	$f(xi)$
0	1.2	2.2	
1	1.3		2.1
2	1.4		4.7
3	1.5		6.2
4	1.6		5.5
5	1.7	5.0	
	Sum	**7.2**	**18.5**

Trapezoidal Rule

$$A = \int_{1.2}^{1.7} f(x)\,dx$$

$$= \frac{h}{2}[f_0 + f_5 + 2(f_1 + f_2 + f_3 + f_4)]$$

$$= \frac{0.1}{2}[7.2 + 2(18.5)]$$

$$= 2.21$$

7.6 Trapezoidal Rule

Solution:

2) $V = \pi \int_{1.2}^{1.7} [f(x)]^2 \; dx, h = 0.1.$

i	x_i	$f(xi)$	H&T $[f(xi)]^2$	$[f(xi)]^2$
0	1.2	2.2	4.84	
1	1.3	2.1		4.41
2	1.4	4.7		22.09
3	1.5	6.2		38.44
4	1.6	5.5		30.25
5	1.7	5.0	25.0	
	Sum		**29.84**	**95.19**

Trapezoidal Rule

$$V = \pi \int_{1.2}^{1.7} [f(x)]^2 \, dx$$

$$= \frac{h}{2}[f_0 + f_5 + 2(f_1 + f_2 + f_3 + f_4)]\pi$$

$$= \frac{0.1}{2}[29.84 + 2(95.19)]\,\pi$$

$$= 34.5921$$

7.6 Trapezoidal Rule

Example 2:

Use the Trapezoidal Rule to estimate the following definite integral with $h = 0.1$.

$$\int_0^{0.8} \sqrt{1 + x + x^2}\, dx$$

7.6 Trapezoidal Rule

Solution:

x_i	H&T $f(xi)$	$f(xi)$
0	1.0000	
0.1		1.0536
0.2		1.1136
0.3		1.1790
0.4		1.2490
0.5		1.3229
0.6		1.4000
0.7		1.4799
0.8	1.5620	
Sum	**2.5620**	**8.7978**

$$\int_0^{0.8} \sqrt{1+x+x^2}\, dx,\ \ h = 0.1.$$

Trapezoidal Rule

$$\int_0^{0.8} \sqrt{1+x+x^2}\, dx$$

$$= \frac{0.1}{2}[2.5620 + 2(8.7978)]$$

$$= 1.0079$$

7.6 Trapezoidal Rule

Exercise 7.3

Use the Trapezoidal Rule to estimate the following definite integral

1) $\int_1^2 xe^{\sqrt{x}}\,dx,\ \ h = 0.2$

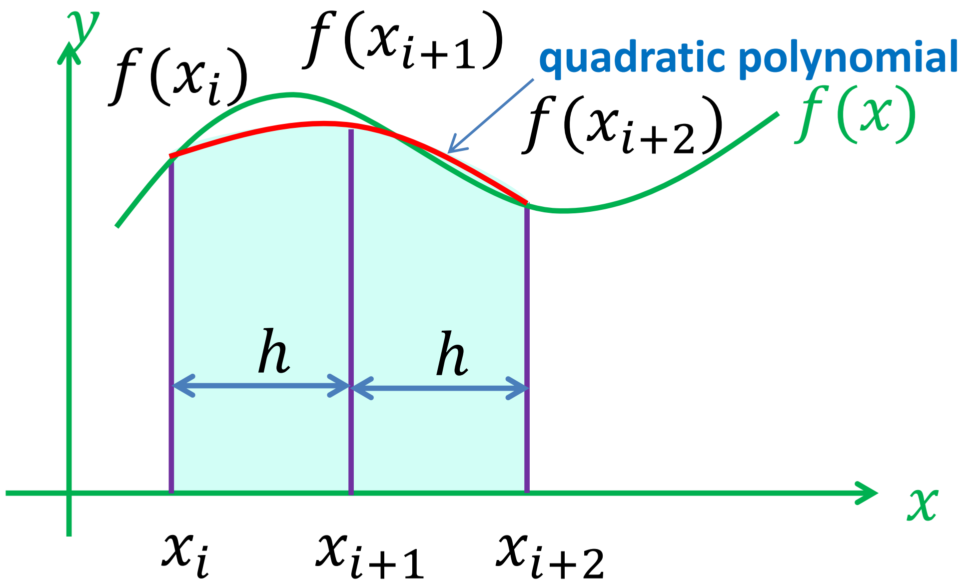
2) $\int_1^3 \frac{1}{x+2}\,dx,\ \ N = 4$

3) $\int_0^{\frac{\pi}{2}} \frac{\sin x}{x}\,dx,\ \ h = \frac{\pi}{8}$

[Ans: 5.2372; 0.5123; 1.3655]

7.7 Simpson's 1/3 Rule

Simpson's 1/3 rule approximating the integral of a function using **quadratic polynomials** (i.e., parabolic arcs instead of the straight line segments used in the Trapezoidal Rule).



Also known as Simpson's rule

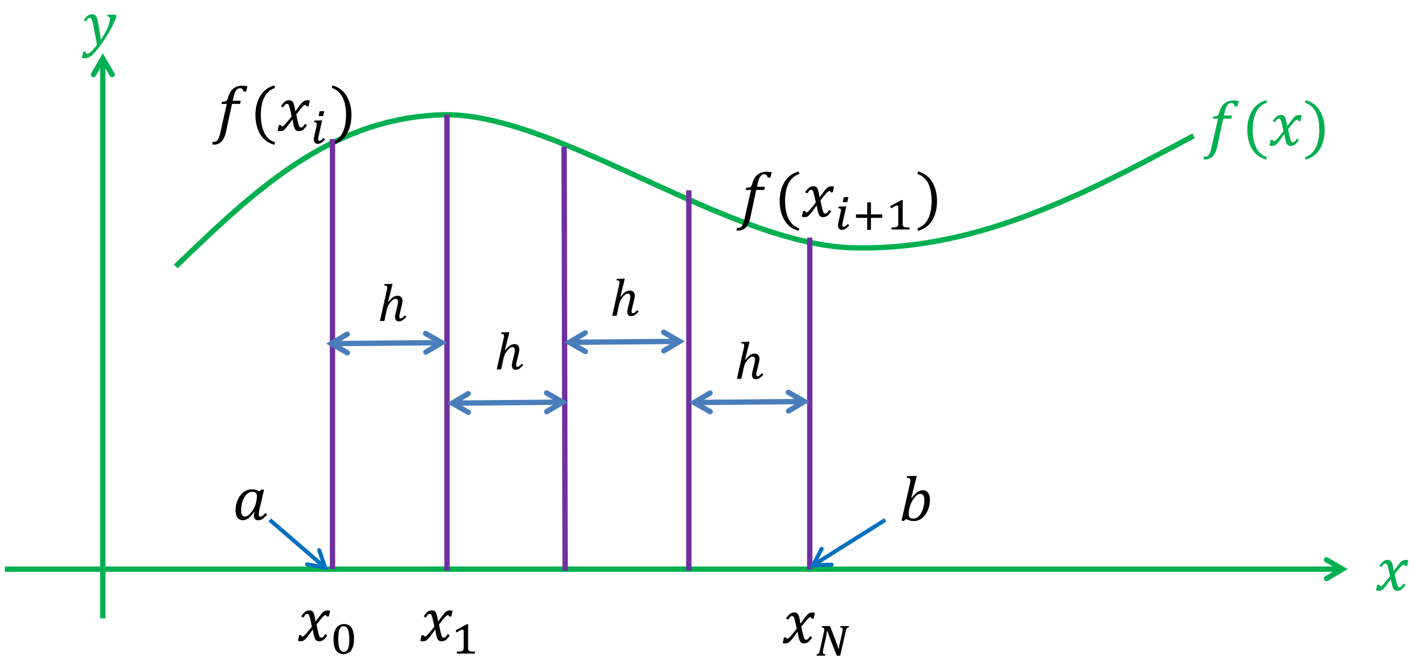
From Lagrange Interpolation

$$\text{Area} = \int_{x_i}^{x_{i+2}} f(x)\,dx \approx \frac{h}{3}[f_i + 4f_{i+1} + f_{i+2}]$$

7.7 Simpson's 1/3 Rule

Multiple of 2

When the interval $[a, b]$ is divided into N equal strips of width h such that $x_i = a + ih, i = 0,1,2,3, \dots, N$ and $b = a + Nh$,



$$\int_{x_0}^{x_N} f(x)\,dx = \int_{x_0}^{x_2} f(x)\,dx + \int_{x_2}^{x_4} f(x)\,dx + \cdots + \int_{x_{N-2}}^{x_N} f(x)\,dx$$

7.7 Simpson's 1/3 Rule

$$\int_{x_0}^{x_N} f(x)\,dx = \int_{x_0}^{x_2} f(x)\,dx + \int_{x_2}^{x_4} f(x)\,dx + \cdots + \int_{x_{N-2}}^{x_N} f(x)\,dx$$

$$= \frac{h}{3}[f_0 + 4f_1 + f_2] + \frac{h}{3}[f_2 + 4f_3 + f_4] +$$

$$\ldots + \frac{h}{3}[f_{N-2} + 4f_{N-1} + f_N]$$

Simplify and yields **Simpson's Composite Rule**:

$$\int_{x_0}^{x_N} f(x)\,dx = \frac{h}{3}[f_0 + f_N + 4(f_1 + f_3 + f_5 + \cdots f_{N-1})$$

$$+2(f_2 + f_4 + f_6 + \cdots f_{N-2})]$$

7.7 Simpson's 1/3 Rule

Derivation of Formulas:

Formula of Simpson's 1/3 rule can be derived by approximating $f(x)$ to a second order polynomial using Newton's interpolation.

Let $x_i = a$ and $x_{i+2} = b$, consider the points $(a, f(a)), \left(\frac{a+b}{2}, f\left(\frac{a+b}{2}\right)\right), (b, f(b))$

$$f(x) = b_0 + b_1(x-a) + b_2(x-a)\left(x - \frac{a+b}{2}\right) \quad (1)$$

where

$$b_0 = f(a)$$

$$b_1 = \frac{f\left(\frac{a+b}{2}\right) - f(a)}{\frac{a+b}{2} - a} = \frac{2\left[f\left(\frac{a+b}{2}\right) - f(a)\right]}{b-a}$$

$$b_2 = \frac{\frac{f(b) - f\left(\frac{a+b}{2}\right)}{b - \frac{a+b}{2}} - \frac{f\left(\frac{a+b}{2}\right) - f(a)}{\frac{a+b}{2} - a}}{b-a} = \frac{2\left[f(a) - 2f\left(\frac{a+b}{2}\right) + f(b)\right]}{(b-a)^2}$$

7.7 Simpson's 1/3 Rule

Derivation of Formulas (cont.):

Integrate Eqn (1):

Area = $\int_a^b f(x)\,dx \approx \int_a^b b_0 + b_1(x-a) + b_2(x-a)\left(x - \frac{a+b}{2}\right) dx$

$$= \left[b_0 x + b_1\left(\frac{x^2}{2} - ax\right) + b_2\left(\frac{x^3}{3} - \frac{(3a+b)x^2}{4} + \frac{a(a+b)x}{2}\right)\right]_a^b$$

$$= b_0(b-a) + b_1\left(\frac{b^2-a^2}{2} - a(b-a)\right) + b_2\left(\frac{b^3-a^3}{3} - \frac{(3a+b)(b^2-a^2)}{4} + \frac{a(a+b)(b-a)}{2}\right)$$

Substitute the values of b_0, b_1 and b_2 into equation above:

Area = $\int_a^b f(x)\,dx \approx \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$

$$= \frac{h}{3}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

$$= \frac{h}{3}[f(x_i) + 4f(x_{i+1}) + f(x_{i+2})]$$

This formula can also be derived using Lagrange interpolation with the similar procedure.

7.7 Simpson's 1/3 Rule

Example:

Given the following data:

x	2	3	4	5	6
$f(x)$	2.35	4.62	5.84	4.20	1.58

Let $h = 1$, use Simpson's Rule to estimate

$$\int_2^6 f(x)\,dx$$

7.7 Simpson's 1/3 Rule

Solution:

$\int_2^6 f(x)\,dx, h = 1.$

i	x_i	H&T $f(xi)$	Odd $f(xi)$	Even $f(xi)$
0	2	2.35		
1	3		4.62	
2	4			5.84
3	5		4.2	
4	6	1.58		
	Sum	**3.93**	**8.82**	**5.84**

Simpson's Rule

$$\int_2^6 f(x)\,dx$$

$$= \frac{h}{3}[f_0 + f_4 + 4(f_1 + f_3) + 2f_2]$$

$$= \frac{1}{3}[3.93 + 4(8.82) + 2(5.84)]$$

$$= 16.9633$$

7.7 Simpson's 1/3 Rule

Exercise 7.4:

Approximate

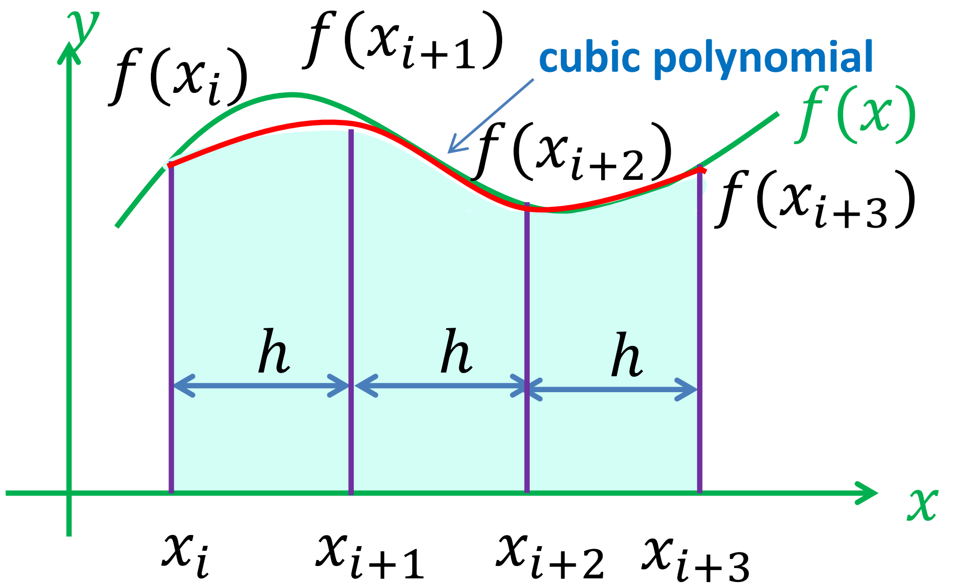
$$\int_0^2 x^2 e^{-x^2}\, dx$$

by using Simpson's Rule with $h = 0.25$

[Ans: 0.4227]

7.8 Simpson's 3/8 Rule

Simpson's 3/8 rule approximating the integral of a function using **cubic polynomials** (i.e., cubic arcs)



Also known as Simpson's Second Rule

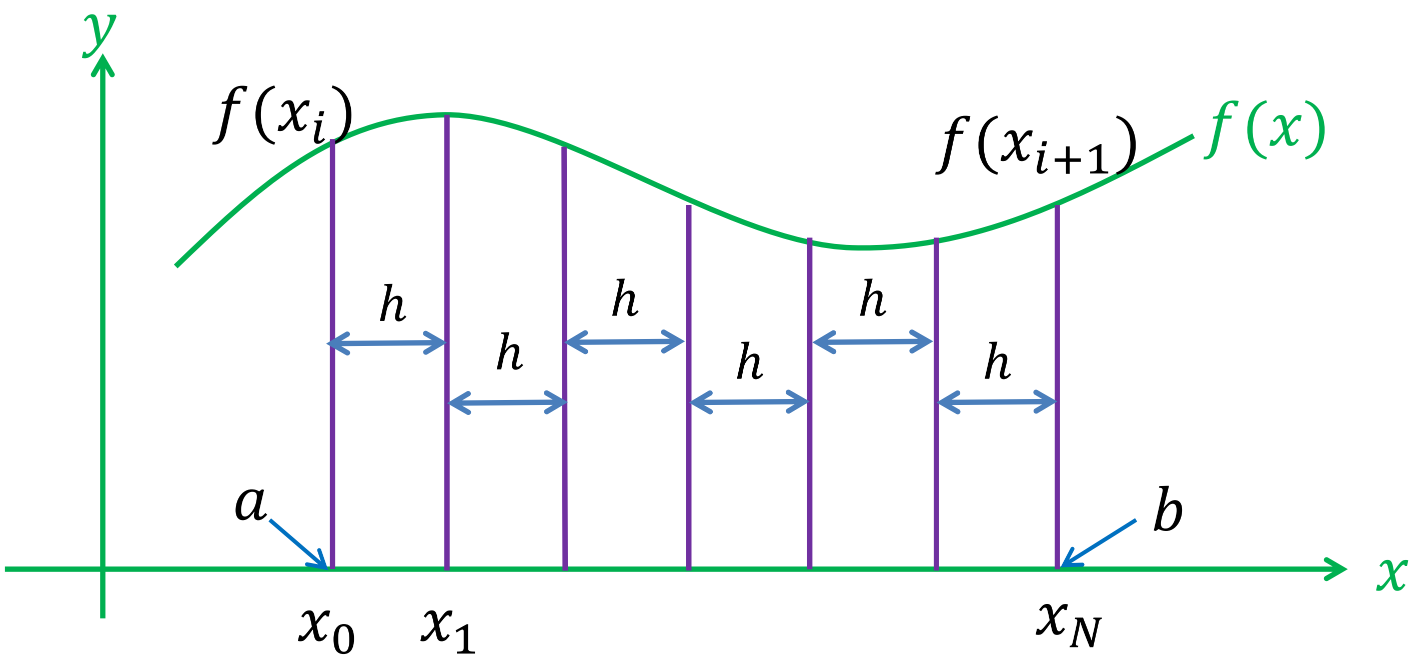
From Lagrange Interpolation

$$\text{Area} = \int_{x_i}^{x_{i+3}} f(x)\,dx \approx \frac{3h}{8}\left[f_i + 3f_{i+1} + 3f_{i+2} + f_{i+3}\right]$$

7.8 Simpson's 3/8 Rule

Multiple of 3

When the interval $[a, b]$ is divided into N equal strips of width h such that $x_i = a + ih, i = 0,1,2,3, \dots, N$ and $b = a + Nh$,



$$\int_{x_0}^{x_N} f(x)\,dx = \int_{x_0}^{x_3} f(x)\,dx + \int_{x_3}^{x_6} f(x)\,dx + \cdots + \int_{x_{N-3}}^{x_N} f(x)\,dx$$

7.8 Simpson's 3/8 Rule

$$\int_{x_0}^{x_N} f(x)\,dx = \int_{x_0}^{x_3} f(x)\,dx + \int_{x_3}^{x_6} f(x)\,dx + \cdots + \int_{x_{N-3}}^{x_N} f(x)\,dx$$

$$= \frac{3h}{8}[f_0 + 3f_1 + 3f_2 + f_3] + \frac{3h}{8}[f_3 + 3f_4 + 3f_5 + f_6] +$$

$$\ldots + \frac{3h}{8}[f_{N-3} + 3f_{N-2} + 3f_{N-1} + f_N]$$

Simplify and yields **Simpson's 3/8 Composite Rule**:

$$\int_{x_0}^{x_N} f(x)\,dx = \frac{3h}{8}[f_0 + f_N + 2(f_3 + f_6 + f_9 + \cdots)$$

$$+3(f_1 + f_2 + f_4 + \cdots)]$$

7.8 Simpson's 3/8 Rule

Example 1:

Given the following data:

x	2.0	2.2	2.4	2.6	2.8	3.0	3.2
$f(x)$	1	0.5	0.333	0.25	0.2	0.167	0.143

Use Simpson's 3/8 Rule to estimate $\int_2^{3.2} f(x)\,dx$ with

1) $h = 0.2$
2) $h = 0.4$

If $f(x) = \frac{1}{5x-9}$, compute absolute error for each of the cases.

7.8 Simpson's 3/8 Rule

Solution:

1) $\int_2^{3.2} f(x)\,dx, h = 0.2.$

i	x_i	H&T $f(x)$	x3 $f(x)$	$f(x)$
0	2.0	1		
1	2.2			0.5
2	2.4			0.333
3	2.6		0.25	
4	2.8			0.2
5	3.0			0.167
6	3.2	0.143		
	Sum	**1.143**	**0.25**	**1.2**

Simpson's 3/8 Rule

$$\int_2^{3.2} f(x)\,dx$$

$$= \frac{3h}{8}[f_0 + f_6 + 2(f_3) + 3(f_1 + f_2 + f_4 + f_5)]$$

$$= \frac{0.6}{8}[1.143 + 2(0.25) + 3(1.2)]$$

$$= 0.3932$$

7.8 Simpson's 3/8 Rule

Solution:

2) $\int_2^{3.2} f(x)\,dx, h = 0.4.$

i	x	H&T $f(x)$	x3 $f(x)$	$f(x)$
0	2.0	1		
1	2.4			0.333
2	2.8			0.2
3	3.2	0.143		
	Sum	**1.143**		**0.533**

Simpson's 3/8 Rule

$$\int_2^{3.2} f(x)\,dx$$

$$= \frac{3h}{8}[f_0 + f_3 + 3(f_1 + f_2)]$$

$$= \frac{1.2}{8}[1.143 + 3(0.533)]$$

$$= 0.4113$$

7.8 Simpson's 3/8 Rule

Solution:

Error

$$\int_{2}^{3.2} f(x)\,dx = \int_{2}^{3.2} \frac{1}{5x-9}\,dx$$

$$= \left[\frac{1}{5}\ln(5x-9)\right]_{2}^{3.2} = 0.3892$$

	Estimated value	Absolute error
$h = 0.2$	0.3932	0.0040
$h = 0.4$	0.4113	0.0221

Conclusion: $h = 0.2$ gives better approximation

7.8 Simpson's 3/8 Rule

Example 2:

Use Simpson's 3/8 Rule to approximate

1) $\int_{1}^{4} \frac{x}{x^2+4} \, dx$ with $h = 0.5$

2) $\int_{-1}^{5} x^3 e^{-x} \, dx$ with $N = 9$

7.8 Simpson's 3/8 Rule

Solution:

1) $\int_1^4 \frac{x}{x^2+4} dx, h = 0.5.$

i	x_i	H&T $f(xi)$	x3 $f(xi)$	$f(xi)$
0	1	0.2000		
1	1.5			0.24
2	2			0.25
3	2.5		0.2439	
4	3			0.2308
5	3.5			0.2154
6	4	0.2000		
Sum		**0.4000**	**0.2439**	**0.9362**

Simpson's 3/8 Rule

$$\int_1^4 \frac{x}{x^2+4} dx$$

$$= \frac{3h}{8}[f_0 + f_6 + 2(f_3) + 3(f_1 + f_2$$

$$+ f_4 + f_5)]$$

$$= \frac{1.5}{8}[0.4 + 2(0.2439) + 3(0.9362)]$$

$$= 0.6931$$

7.8 Simpson's 3/8 Rule

Solution:

2) $\int_{-1}^{5} x^3 e^{-x}\, dx, N = 9.$

i	x_i	H&T $f(xi)$	x3 $f(xi)$	$f(xi)$
0	-1.0000	-2.7183		
1	-0.3333			-0.0517
2	0.3333			0.0265
3	1.0000		0.3679	
4	1.6667			0.8744
5	2.3333			1.2319
6	3.0000		1.3443	
7	3.6667			1.2601
8	4.3333			1.0679
9	5.0000	0.8422		
	Sum	**-1.8760**	**1.7121**	**4.4092**

Simpson's 3/8 Rule

$$\int_{-1}^{5} x^3 e^{-x}\, dx$$

$$= \frac{3h}{8}[f_0 + f_9 + 2(f_3 + f_6)$$

$$+ 3(f_1 + f_2 + f_4 + f_5 + f_7 + f_8)]$$

$$= \frac{2}{8}[-1.8760 + 2(1.7121)$$

$$+ 3(4.4092)]$$

$$= 3.6940$$

7.8 Simpson's 3/8 Rule

Exercise 7.5:

Use Simpson's 3/8 Rule to estimate the following definite integral

1) $\int_0^{\pi} \sqrt{(\cos x)^2 + 3}\, dx,\ N = 12$

2) $\int_2^4 \frac{1}{\left(1-\sqrt{x}\right)^3} dx,\ N = 9$

3) $\int_1^2 \frac{1}{\left(1+\sqrt{x}\right)^3} dx,\ N = 10$

[Ans: 5.8699; -7.6705; cannot be solved by Simpson's 3/8 rule]