ENGINEERING MATHEMATICS 1
BMFG 1313
INTEGRATION

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Lesson Outcomes

Upon completion of this lesson, the student should be able to:

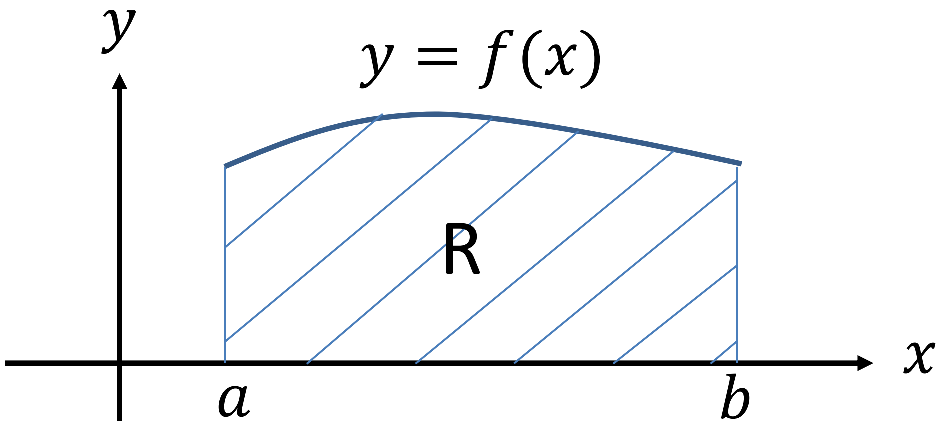
- Evaluate Integration by Substitution
- Evaluate Integration by Parts
- Evaluate Integration involving Trigonometric Functions
- Evaluate Integration involving Exponential Functions

7.1 Introduction of Integration

Integration is anti-derivatives.

Integration is used to find **areas, volumes, central points** etc. Integration is a **direct relationship between two variables** that can be found from the relationship involving the **rate of change** of the two variables.

The simplest application: Area $R = \int_a^b f(x)\, dx$



7.1 Introduction of Integration

Integration of Trigonometry Functions

$$\int \cos x \, dx = \sin x + c$$

$$\int \sin x \, dx = -\cos x + c$$

$$\int \sec^2 x \, dx = \tan x + c$$

$$\int \sec x \tan x \, dx = \sec x + c$$

$$\int \operatorname{cosec} x \cot x \, dx = -\operatorname{cosec} x + c$$

$$\int \operatorname{cosec}^2 x \, dx = -\cot x + c$$

Note that c is a constant of indefinite integration.

7.1 Introduction of Integration

Integration of Exponential, Hyperbolic and The Related Functions:

$$\int e^x \, dx = e^x + c$$

$$\int \frac{1}{x} dx = \ln x + c$$

$$\int \cosh x \, dx = \sinh x + c$$

$$\int \sinh x \, dx = \cosh x + c$$

$$\int \operatorname{sech}^2 x \, dx = \tanh x + c$$

7.2 Integration by Substitution

Example:

Evaluate $\int \sec^2(3x-5)\,dx$ using integration by substitution.

Solution:

Let $u = 3x - 5$. Given that $\frac{du}{dx} = 3$. Thus, $\frac{1}{3}du = dx$.

This implies $\int \sec^2(3x-5)\,dx = \int \sec^2 u \cdot \frac{1}{3}du$

$$= \frac{1}{3}\int \sec^2 u\ du = \frac{1}{3}\tan u + c$$

$$= \frac{1}{3}\tan(3x-5) + c$$

7.2 Integration by Substitution

Example:

Find the indefinite integrals of $\frac{1}{2+\sqrt{x}}$.

Solution:

Let $u = \sqrt{x}$ and $u^2 = x.$ Given that $2u\,\frac{du}{dx} = 1$. Thus, $2u\,du = dx$

This implies $\int \frac{1}{2+\sqrt{x}}\,dx = \int \frac{1}{2+u}\,2u\,du$

$$= \int \frac{2u}{2+u}\,du = \int 2 - \frac{4}{2+u}\,du$$

$$= 2u - 4\ln|2+u| + c$$

$$= 2\sqrt{x} - 4\ln|2+\sqrt{x}| + c$$

7.2 Integration by Substitution

Example:

By using integration of substitution, evaluate $\int_2^4 \frac{x}{x^2+7}\,dx$.

Solution:

Let $u = x^2 + 7$. Given that $\frac{du}{dx} = 2x, \frac{1}{2}du = x\,dx$

when $x = 2, u = 11$ and when $x = 4, u = 23$

This implies, $\int_2^4 \frac{x}{x^2+7}\,dx = \int_{11}^{23} \frac{1}{u} \cdot \frac{1}{2}\,du = \frac{1}{2}\int_{11}^{23} \frac{1}{u}\,du$

$$= \frac{1}{2} \ln u \Big|_{11}^{23}$$

$$= \frac{1}{2}(\ln 23 - \ln 11)$$

$$= 0.3688$$

7.2 Integration by Substitution

Example:

Evaluate $\int 2e^{1-2t}\,dt$.

Solution:

Let $u = 1 - 2t$. Given that $\frac{du}{dt} = -2$. Thus, $-du = 2dt$.

This implies $\int e^{2-t}\,dt = -\int e^{u} \cdot du$

$$= -e^{u} + c$$
$$= -e^{1-2t} + c$$

7.2 Integration by Substitution

Example:

By using integration of substitution, evaluate $\int \frac{1}{\sqrt{1+x}+2} dx$.

Solution:

Let $u = \sqrt{1+x} + 2$ such that $(u-2)^2 = 1 + x$.

Given that $\frac{du}{dx}(2u-4) = 1$ and $dx = (2u-4)du$.

This implies, $\int \frac{1}{\sqrt{1+x}+2} dx = \int \frac{2u-4}{u} du = \int 2 - \frac{4}{u} du = 2u - 4 \ln u + c.$

Thus, $\int \frac{1}{\sqrt{1+x}+2} dx = 2\sqrt{1+x} + 4 - 4 \ln\left|\sqrt{1+x} + 2\right| + c.$

7.2 Integration by Substitution

If $f(x)$ contains the function as shown in table below, integration by substitution by using the trigonometry function may be useful to solve the integration.

If $f(x)$ contains	Use substitution
$\sqrt{a^2 - x^2}$	$x = \text{a} \sin\theta$
$\sqrt{a^2 + x^2}$	$x = \text{a} \tan\theta$
$\sqrt{x^2 - a^2}$	$x = \text{a} \sec\theta$

7.2 Integration by Substitution

Example:

Find the indefinite integrals of $\sqrt{4-x^2}$.

Solution:

Let $x = 2\sin\theta$.

Given that $x^2 = 4\sin^2\theta$ and $\frac{dx}{d\theta} = 2\cos\theta$.

Hence $\sqrt{4-x^2} = \sqrt{4-4\sin^2\theta} = 2\cos\theta$ and $dx = 2\cos\theta\, d\theta$.

Thus implies, $\int \sqrt{4-x^2}\, dx = \int \sqrt{4-4\sin^2\theta} \cdot 2\cos\theta\, d\theta$

7.2 Integration by Substitution

Solution (continue):

$$\int \sqrt{4-x^2}\,dx = 2\int \sqrt{4-4\sin^2\theta}\cos\theta\,d\theta$$

$$= \int 2\cos\theta\sqrt{4(1-\sin^2\theta)}\,d\theta$$

Use identity $\cos 2\theta = 2\cos^2\theta - 1$

$$= \int 2\cos\theta\sqrt{4(\cos^2\theta)}\,d\theta$$

$$= \int 2\cos\theta\,(2\cos\theta)d\theta \;= \int 4\cos^2\theta\,d\theta$$

$$= \int 4\left(\frac{\cos 2\theta+1}{2}\right)d\theta = 2\int \cos 2\theta + 1\,d\theta$$

$$= 2\left(\frac{\sin 2\theta}{2}+\theta\right)+c$$

Use identity $\sin 2\theta = 2\sin\theta\cos\theta$

since $x = 2\sin\theta$, hence $\cos\theta = \sqrt{4-x^2}$

$$= \sin 2\theta + 2\theta + c$$

From $x = 2\sin\theta$, we obtain $\theta = \sin^{-1}\frac{x}{2}$

$$=\frac{1}{2}x\sqrt{4-x^2} + 2\sin^{-1}\frac{x}{2} + c$$

7.2 Integration by Substitution

Example:

Find the indefinite integrals of $x\sqrt{x^2-2}$.

Solution:

Let $x = \sqrt{2}\sec\theta$.

Given that $x^2 = 2\sec^2\theta$ and $\frac{dx}{d\theta} = \sqrt{2}\sec\theta\tan\theta$.

Hence $\sqrt{x^2-2} = \sqrt{2\sec^2\theta - 2}$ and $dx = \sqrt{2}\sec\theta\tan\theta\, d\theta$.

Thus implies, $\int x\sqrt{x^2-2}\,dx = \int \sqrt{2}\sec\theta\sqrt{2(\sec^2\theta - 1)} \cdot \sqrt{2}\sec\theta\tan\theta\, d\theta$

7.2 Integration by Substitution

Solution (continue):

$\int x\sqrt{x^2-2}\,dx = \int 2\sqrt{2}\sec\theta\tan\theta \cdot \sec\theta\tan\theta\, d\theta = \int 2\sqrt{2}\ \sec^2\theta \cdot \tan^2\theta d\theta$

By using substitution

Let $u = \tan\theta$ and $\frac{du}{d\theta} = \ \sec^2\theta$ given that $du\ = \sec^2\theta d\theta$.

$$\int 2\sqrt{2}\ \sec^2\theta \cdot \tan^2\theta d\theta = 2\sqrt{2}\int u^2\,du = \frac{2\sqrt{2}}{3}u^3 + c = \frac{2\sqrt{2}}{3}\tan^3\theta + c$$

From $x = \sqrt{2}\sec\theta \Longrightarrow \cos\theta = \frac{\sqrt{2}}{x}$, it gives $\tan\theta = \sqrt{\frac{x^2-2}{2}}$.

Hence, $\int x\sqrt{x^2-2}\,dx = \frac{2\sqrt{2}}{3}\left(\frac{x^2-2}{2}\right)^{\frac{3}{2}} + c = \frac{1}{3}(x^2-2)^{\frac{3}{2}} + c$

7.2 Integration by Substitution

Exercise 7.1:

Evaluate each of the following functions by using integration by substitution.

1) $\int \frac{12x^2+16}{x^3+4x}\,dx$

2) $\int \frac{4x+6}{(x^2+3x+7)^4}\,dx$

3) $\int \frac{x}{\sqrt{1-3x^2}}\,dx$

[Ans: 1) $4\ln|x^3+4x|+C$; 2) $-\frac{2}{3(x^2+3x+7)^3}+C$; 3) $-\frac{1}{3}\sqrt{1-3x^2}+c$]

7.2 Integration by Substitution

Exercise 7.1:

Evaluate each of the following functions by using integration by substitution.

4) $\int_0^2 2x^2 \sqrt[4]{5x^3+4}\ dx$

5) $\int_1^2 \frac{2x^3+3}{(x^4+6x)^3} dx$

6) $\int_0^{\pi} (2x-3)\sin(3x^2-9x)\ dx$

[Ans:4) 11.4843; 5) 0.0048; 6) 0.2553;]

7.2 Integration by Substitution

Exercise 7.1:

Evaluate each of the following functions by using integration by substitution.

7) $\int x\sqrt{5x-2}\ dx$

8) $\int \frac{x^2-1}{x^3-3x+2}\ dx$

9) $\int \cos x\,(\sin^3 x)\ dx$

10) $\int \tan x\ dx$

11) $\int \frac{2}{3+\sqrt{x+1}}\ dx$

12) $\int \sqrt{1-x^2}\ dx$

[Ans: 7) $\frac{2}{125}(5x-2)^{\frac{5}{2}}+\frac{4}{75}(5x-2)^{\frac{3}{2}}+c$; 8) $\frac{1}{3}\ln|x^3-3x+2|+c$; 9) $\frac{1}{4}\sin^4 x+c$; 10) $-\ln|\cos x|+c$;

11) $4\sqrt{x+1}-12\ln\left|3+\sqrt{x+1}\right|+c$; 12) $\frac{1}{2}x\sqrt{1-x^2}+\frac{1}{2}\sin^{-1}x+c$]

7.3 Integration by Parts

Based on the product rule of differentiation,

$$\frac{d}{dx}[uv] = u\frac{dv}{dx} + v\frac{du}{dx}$$
$$= uv' + vu'$$

$$uv = \int uv'\,dx + \int vu'\,dx$$

Hence, formula for Integration by Parts is given as

$$\int uv'\,dx = uv - \int vu'\,dx$$

Alternatively:

$$\frac{d}{dx}[uv] = u\frac{dv}{dx} + v\frac{du}{dx}$$

$$uv = \int u\,dv + \int v\,du$$

Hence,

$$\int u\,dv = uv - \int v\,du$$

Note that choosing u is based on Rules **L-PeT** (i.e. **L**ogarithm, **P**olynomial, **e**xponential and **T**rigonometry)

7.3 Integration by Parts

Example:

Find the indefinite integrals of xe^{2x}.

Solution:

By using integration by parts,

$$\int u\,dv = uv - \int v\,du$$

$$u = x, \qquad \frac{dv}{dx} = e^{2x}$$

$$\frac{du}{dx} = 1, \qquad v = \frac{1}{2}e^{2x}.$$

Hence, $\int xe^{2x}\,dx = \frac{1}{2}xe^{2x} - \int \frac{1}{2}e^{2x}\,dx$

$$= \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + c$$

7.3 Integration by Parts

Example:

Find the indefinite integrals of $x^2 \sin x$.

Solution:

By using integration by parts,

$$\int u\,dv = uv - \int v\,du$$

$$u = x^2, \qquad \frac{dv}{dx} = \sin x$$

$$\frac{du}{dx} = 2x, \qquad v = -\cos x.$$

Hence, $\int x^2 \sin x\,dx = -x^2 \cos x - \int -2x \cos x\,dx$

Now, to solve $\int 2x \cos x\,dx$,

$$u = 2x, \qquad \frac{dv}{dx} = \cos x$$

$$\frac{du}{dx} = 2, \qquad v = \sin x$$

7.3 Integration by Parts

Example:

Find the indefinite integrals of $x^2 \sin x$.

Solution (continue):

$$\int u\,dv = uv - \int v\,du$$

$$\int x^2 \sin x\,dx = -x^2 \cos x - \int -2x \cos x\,dx$$

$$= -x^2 \cos x + \int 2x \cos x\,dx$$

$$= -x^2 \cos x + 2x \sin x - \int 2 \sin x\,dx$$

$$= -x^2 \cos x + 2x \sin x - 2(-\cos x) + c$$

$$= -x^2 \cos x + 2x \sin x + 2 \cos x + c$$

7.4 Integration by Parts (Tabular Method)

Tabular method is an alternative way to the method of Integration by Parts,

$$\int uv'\,dx = uv - \int vu'\,dx$$

Sign $+/-$	Differentiate u	Integrate v'
$+$	u ①	v'
$-$	u'	② v
$+$ ⋮ (Alternate signs started from +)	⋮ (Repeating differentiate the function from one row to the next row)	⋮ (Repeating integrate the function from one row to the next row)

Question: How to decide the u and v' of the integrand?

7.4 Integration by Parts (Tabular Method)

There are a lot of type of functions which can be solved by integration by parts or tabular method. Among of them, the most common used functions are the **product functions of**

Case 1: Polynomial and exponential functions

Case 2: Polynomial and $\sin\theta$ or $\cos\theta$ functions

Case 3: Exponential and $\sin\theta$ or $\cos\theta$ functions

7.4 Integration by Parts (Tabular Method)

For case 1 and case 2, choose the polynomial function as u and let the exponential function or $\sin\theta$ or $\cos\theta$ be v'. Let say $\int uv'\,dx$,

Sign $+/-$	Differentiate u	Integrate v'
$+$	u	v'
$-$	u'	v
$\vdots$	$\vdots$	$\vdots$
$+$	0	V

(Repeating differentiate the polynomial function until zero)

(Repeating integrate the function from one row to the next row and stop at the same row of zero from column of u)

Find the product functions according to the colored arrows.
The answer will be the linear combination of all the product functions.

7.4 Integration by Parts (Tabular Method)

For case 3, one is free to choose exponential, $\sin\theta$ or $\cos\theta$ to be either u or v'.

Sign $+/-$	Differentiate u	Integrate v'
$+$	u	v'
$-$	u'	v
$+$	u''	$\int v'\,dx$

(Differentiate function u twice) (Integrate function v' twice)

Find all product functions according to the colored arrows.
For the arrows pointing from left to right, let the linear combination of these product functions be g. Also, let the product function in the last row (from red arrows) be h. The integration will be

$$\int uv'\,dx = g + \int h\,dx$$

where $\int h\,dx = k\int uv'\,dx$, k is a constant value. Lastly, solve for $\int uv'\,dx$.

7.4 Integration by Parts (Tabular Method)

Example:

Find the indefinite integrals of $x^2 \sin 3x$.

Solution:

Construct Table of Tabular Method

Sign $+/-$	Differentiate	Integrate
$+$	x^2	$\sin 3x$
$-$	$2x$	$-\frac{1}{3}\cos 3x$
$+$	2	$-\frac{1}{9}\sin 3x$
$-$	0	$\frac{1}{27}\cos 3x$

$$\int x^2 \sin 3x \, dx = -\frac{1}{3}x^2 \cos 3x + \frac{2}{9}x \sin 3x + \frac{2}{27}\cos 3x + c$$

7.4 Integration by Parts (Tabular Method)

Example:

Find the indefinite integrals of xe^{2x}.

Solution:

Sign $+/-$	Differentiate	Integrate
$+$	x	e^{2x}
$-$	1	$e^{2x}/2$
$+$	0	$e^{2x}/4$

From the Table,

$$\int xe^{2x}\,dx = \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + c.$$

7.4 Integration by Parts (Tabular Method)

Example:

Find the indefinite integrals of $e^{\pi x}\sin x$.

Solution:

Sign $+/-$	Differentiate	Integrate
$+$	$e^{\pi x}$	$\sin x$
$-$	$\pi e^{\pi x}$	$-\cos x$
$+$	$\pi^2 e^{\pi x}$	$-\sin x$

From the Table,

$$\int e^{\pi x}\sin x\,dx = -e^{\pi x}\cos x + \pi e^{\pi x}\sin x - \pi^2\int e^{\pi x}\sin x\,dx$$

Rearrange the equation, such that

$$(1+\pi^2)\int e^{\pi x}\sin x\,dx = -e^{\pi x}\cos x + \pi e^{\pi x}\sin x$$

Thus, $\int e^{\pi x}\sin x\,dx = \frac{1}{1+\pi^2}(-e^{\pi x}\cos x + \pi e^{\pi x}\sin x) + c.$

7.3 Integration by Parts

Exercise 7.2:

Find the indefinite integrals of the following functions.

1) $x^3 e^{-3x}$

2) $x^2 \cos 4x$

3) $(3x - 2) \cos 2\pi x$

4) $(x^2 + 1) \sin \frac{5\pi x}{3}$

[Ans: 1) $-\frac{1}{3}x^3 e^{-3x} - \frac{1}{3}x^2 e^{-3x} - \frac{2}{9}x e^{-3x} - \frac{2}{27}e^{-3x} + c$; 2) $\frac{1}{4}x^2 \sin 4x + \frac{1}{8}x \cos 4x - \frac{1}{32}\sin 4x + c$;

3) $\frac{(3x-2)}{2\pi}\sin 2\pi x + \frac{3}{4\pi^2}\cos 2\pi x + c$; 4) $-\frac{3}{5\pi}(x^2+1)\cos\frac{5\pi x}{3} + \frac{18x}{25\pi^2}\sin\frac{5\pi x}{3} + \frac{54}{125\pi^3}\cos\frac{5\pi x}{3} + c$;]

7.3 Integration by Parts

Exercise 7.2:

Find the indefinite integrals of the following functions.

5) $e^{2x}\sin 3x$

6) $e^{-x}\cos 2x$

7) $x\ln x$

[Ans: 5) $-\frac{3}{13}e^{2x}\cos 3x+\frac{2}{13}e^{2x}\sin 3x+c$;

6) $\frac{2}{5}e^{-x}\sin 2x-\frac{1}{5}e^{-x}\cos 2x+c$; 7) $\frac{x^2}{2}\ln x-\frac{x^2}{4}+c$;]