# ENGINEERING MATHEMATICS 1
# BMFG 1313
# DIFFERENTIATION

Nur Ilyana Anwar Apandi[1], Ser Lee Loh[2]

[1]ilyana@utem.edu.my , [2]slloh@utem.edu.my

# Learning Outcomes

Upon completion of this lesson, the student should be able to:

- Evaluate the differentiation for Trigonometry Functions
- Evaluate the differentiation for Exponential Functions
- Evaluate the differentiation for the Related Functions

# 6.1 Introduction of Differentiation

- Differentiation is used to solve many real-life problems.
- Derivatives are important in solving **Engineering and Science problems**, where they are commonly used to **model the behavior of moving objects**.
- The derivatives of $f(x)$, with respect to $x$ can be written as

$$f'(x) = \frac{d}{dx}[f(x)]$$

# 6.2 Differentiation of Trigonometry, Exponential and The Related Functions

*Differentiation of Trigonometry Functions*

$$\frac{d}{dx}[\sin x] = \cos x$$

$$\frac{d}{dx}[\cos x] = -\sin x$$

$$\frac{d}{dx}[\tan x] = \sec^2 x$$

$$\frac{d}{dx}[\sec x] = \sec x \tan x$$

$$\frac{d}{dx}[\operatorname{cosec} x] = -\operatorname{cosec} x \cot x$$

$$\frac{d}{dx}[\cot x] = -\operatorname{cosec}^2 x$$

$$\frac{d}{dx}[\sin f(x)] = f'(x) \cos f(x)$$

$$\frac{d}{dx}[\cos f(x)] = -f'(x) \sin f(x)$$

$$\frac{d}{dx}[\tan f(x)] = f'(x) \sec^2 f(x)$$

$$\frac{d}{dx}[\sec f(x)] = f'(x) \sec x \tan f(x)$$

$$\frac{d}{dx}[\operatorname{cosec} f(x)] = -f'(x) \operatorname{cosec} f(x) \cot f(x)$$

$$\frac{d}{dx}[\cot f(x)] = -f'(x) \operatorname{cosec}^2 f(x)$$

# 6.2 Differentiation of Trigonometry, Exponential and The Related Functions

*Differentiation of Exponential and The Related Functions:*

$$\frac{d}{dx}[e^x] = e^x$$

$$\frac{d}{dx}\left[e^{f(x)}\right] = f'(x)e^{f(x)}$$

$$\frac{d}{dx}[\ln x] = \frac{1}{x}$$

$$\frac{d}{dx}[\ln f(x)] = \frac{f'(x)}{f(x)}$$

where $f'(x) = \frac{d}{dx}[f(x)]$ for any $f(x)$.

# 6.2 Differentiation of Trigonometry, Exponential and The Related Functions

*Differentiation of Hyperbolic Functions*

$$\frac{d}{dx}[\sinh x] = \frac{d}{dx}\left[\frac{e^{x} - e^{-x}}{2}\right] = \frac{1}{2}(e^{x} + e^{-x}) = \cosh x$$

$$\frac{d}{dx}[\cosh x] = \frac{d}{dx}\left[\frac{e^{x} + e^{-x}}{2}\right] = \frac{1}{2}(e^{x} - e^{-x}) = \sinh x$$

$$\frac{d}{dx}[\tanh x] = \frac{d}{dx}\left[\frac{\sinh x}{\cosh x}\right] = \frac{\cosh x \cosh x - \sinh x \sinh x}{\cosh^2 x} = \frac{1}{\cosh^2 x} = \operatorname{sech}^2 x$$

# 6.2 Differentiation of Trigonometry, Exponential and The Related Functions

*Differentiation of Hyperbolic Functions (continue)*

$$\frac{d}{dx}[\operatorname{sech} x] = \frac{d}{dx}\left[\frac{1}{\cosh x}\right] = \frac{-\sinh x}{\cosh^2 x} = -\operatorname{sech} x \tanh x$$

$$\frac{d}{dx}[\operatorname{cosech} x] = \frac{d}{dx}\left[\frac{1}{\sinh x}\right] = \frac{-\cosh x}{\sinh^2 x} = -\operatorname{cosech} x \coth x$$

$$\frac{d}{dx}[\coth x] = \frac{d}{dx}\left[\frac{\cosh x}{\sinh x}\right] = \frac{\sinh x \sinh x - \cosh x \cosh x}{\sinh^2 x} = \frac{-1}{\sinh^2 x} = -\operatorname{cosech}^2 x$$

# 6.2 Differentiation of Trigonometry, Exponential and The Related Functions

*Differentiation of Inverse Trigonometry Functions*

For $|x| < 1$,

$$\frac{d}{dx}[\sin^{-1} x] = \frac{1}{\sqrt{1 - x^2}}$$

For $|x| < 1$,

$$\frac{d}{dx}[\cos^{-1} x] = \frac{-1}{\sqrt{1 - x^2}}$$

For any $x$,

$$\frac{d}{dx}[\tan^{-1} x] = \frac{1}{1 + x^2}$$

# 6.2 Differentiation of Trigonometry, Exponential and The Related Functions

*Differentiation of Inverse Hyperbolic Functions*

For any $x$,

$$\frac{d}{dx}[\sinh^{-1} x] = \frac{1}{\sqrt{1+x^2}}$$

For $x > 1$,

$$\frac{d}{dx}[\cosh^{-1} x] = \frac{1}{\sqrt{x^2-1}}$$

For $|x| < 1$,

$$\frac{d}{dx}[\tanh^{-1} x] = \frac{1}{1-x^2}$$

# 6.3 Basic Rules of Differentiation

## 6.3.1 Constant Multiplication Rule

$$\frac{d}{dx}[kf(x)] = k\frac{d}{dx}[f(x)] = kf'(x)$$

## 6.3.2 Sum Rule

$$\frac{d}{dx}[u(x) + v(x)] = \frac{d}{dx}[u(x)] + \frac{d}{dx}[u(x)] = u'(x) + v'(x)$$

# 6.3 Basic Rules of Differentiation

## 6.3.1 Constant Multiplication Rule

**Example:**

Find $\frac{dy}{dx}$ when $y = 7x^4$.

**Answer:**

Let $k = 7$ and $f(x) = x^4$. Hence $f'(x) = 4x^3$.

Thus,

$$\frac{dy}{dx} = \frac{d}{dx}[7x^4] = 7 \cdot 4x^3 = 28x^3$$

# 6.3 Basic Rules of Differentiation

## 6.3.2 Sum Rule

**Example:**

Find $\frac{dy}{dx}$ when $y = 4x^7 - e^{-x}$.

**Answer:**

Let $u(x) = 4x^7$ and $v(x) = -e^{-x}$, where $v'(x) = e^{-x}$ given that

$$\frac{d}{dx}[u(x) + v(x)] = u'(x) + v'(x) = 28x^6 + e^{-x}$$

# 6.3 Basic Rules of Differentiation

## 6.3.3 Product Rule

$$\frac{d}{dx}[uv] = uv' + vu'$$

## 6.3.4 Quotient Rule

$$\frac{d}{dx}\left[\frac{u}{v}\right] = \frac{vu' - uv'}{v^2}$$

Note that $u$ and $v$ are functions of $x$.

# 6.3 Basic Rules of Differentiation

## 6.3.3 Product Rule

**Example:**

Find $\frac{dy}{dx}$ for $y = e^{\pi x}x^4$.

**Answer:**

Let $u = e^{\pi x}$ and $v = x^4$. Such that $u' = \pi e^{\pi x}$ and $v' = 4x^3$. Thus

$$\frac{dy}{dx} = \frac{d}{dx}[uv] = uv' + vu' = 4e^{\pi x}x^3 + \pi x^4 e^{\pi x}$$

# 6.3 Basic Rules of Differentiation

## 6.3.3 Product Rule

**Example:**

Find $\frac{dy}{dx}$ for $y = x^3 \ln x$.

**Answer:**

Let $u = x^3$ and $v = \ln x$. Such that $u' = 3x^2$ and $v' = \frac{1}{x}$. Thus

$$\frac{dy}{dx} = \frac{d}{dx}[uv] = uv' + vu' = x^3 \cdot \frac{1}{x} + 3x^2 \ln x = x^2(1 + 3\ln x)$$

# 6.3 Basic Rules of Differentiation

## 6.3.4 Quotient Rule

**Example:**

Find $\frac{dy}{dx}$ for $y = \frac{x^4}{e^{\pi x}}$.

**Answer:**

Let $u = x^4$ and $v = e^{\pi x}$. Such that $u' = 4x^3$ and $v' = \pi e^{\pi x}$. Thus

$$\frac{dy}{dx} = \frac{vu' - uv'}{v^2} = \frac{4e^{\pi x}x^3 - \pi x^4 e^{\pi x}}{e^{2\pi x}} = \frac{4x^3 - \pi x^4}{e^{\pi x}}$$

# 6.3 Basic Rules of Differentiation

## 6.3.4 Quotient Rule

**Example:**

Find $\frac{dy}{dx}$ for $y = \frac{\cos x}{x-1}$.

**Answer:**

Let $u = \cos x$ and $v = x - 1$. Such that $u' = -\sin x$ and $v' = 1$. Thus

$$\frac{dy}{dx} = \frac{vu' - uv'}{v^2} = \frac{(x-1)(-\sin x) - \cos x\,(1)}{(x-1)^2} = \frac{(1-x)\sin x - \cos x}{(x-1)^2}$$

## Exercise 6.1:

Find $\frac{dy}{dx}$ for the following functions.

1) $y = x^4 e^{3x} + 3x^{-1}$

2) $y = e^{-2x}(\sin 3x - \cos 5x)$

3) $y = \frac{x+1}{x^4 - 3}$

4) $y = x \ln(4x^5 + 3)$

5) $y = \frac{\ln 2x}{6x-5}$

6) $y = (x+1)\cosh^{-1} x$

[Ans: $x^3 e^{3x}(3x+4) - \frac{3}{x^2}$; $e^{-2x}(3\cos 3x + 5\sin 5x - 2\sin 3x + 2\cos 5x)$; $\frac{-3x^4 - 4x^3 - 3}{(x^4-3)^2}$;

$\frac{20x^5}{4x^5+3} + \ln(4x^5+3)$; $\frac{6x-5-6x\ln 2x}{x(6x-5)^2}$; $\frac{x+1}{\sqrt{x^2-1}} + \cosh^{-1} x$]

# 6.3 Basic Rules of Differentiation

## 6.3.5 Chain Rule

If $w = f(x)$ and $y = g(w)$, then

$$\frac{dy}{dx} = \frac{dy}{dw}\frac{dw}{dx}$$

## 6.3.6 Extended Chain Rule

If $z = f(x)$, $w = g(z)$ and $y = h(w)$, then

$$\frac{dy}{dx} = \frac{dy}{dw}\frac{dw}{dz}\frac{dz}{dx}$$

# 6.3 Basic Rules of Differentiation

## 6.3.5 Chain Rule

**Example:**

Find $\frac{dy}{dx}$ when $y = (2x^2 + 5)^{-5}$.

**Answer:**

Let $w = 2x^2 + 5$ and $y = w^{-5}$.

Given that $\frac{dy}{dw} = -5w^{-6}$ and $\frac{dw}{dx} = 4x$.

Hence,

$$\frac{dy}{dx} = \frac{dy}{dw}\frac{dw}{dx} = (-5w^{-6})(4x) = -20xw^{-6}$$

$$= -20x(2x^2 + 5)^{-6}$$

# 6.3 Basic Rules of Differentiation

## 6.3.6 Chain Rule

**Example:**

Find $\frac{dy}{dx}$ when $y = e^{x^3+1}$.

**Answer:**

Let $y = e^w$, where $w = x^3 + 1$.

Hence $\frac{dy}{dw} = e^w$ and $\frac{dw}{dx} = 3x^2$.

Thus, $\frac{dy}{dx} = \frac{dy}{dw}\frac{dw}{dx} = (e^w)(3x^2) = 3x^2e^{x^3+1}$.

This implies, $\frac{d}{dx}[e^w] = e^w\frac{dw}{dx}$ for any $w$ as a function of $x$.

# 6.3 Basic Rules of Differentiation

## 6.3.6 Chain Rule

**Example:**

Find $\frac{dy}{dt}$ when $y = \sin(2\pi t + 1)$.

**Answer:**

Let $y = \sin u$, where $u = 2\pi t + 1$.

Hence, $\frac{dy}{du} = \cos u$ and $\frac{du}{dt} = 2\pi$.

Thus, $\frac{dy}{dt} = \frac{dy}{du}\frac{du}{dt} = (\cos u)(2\pi) = 2\pi\cos(2\pi t + 1)$ .

This implies, $\frac{d}{dx}[\sin u] = \cos u \frac{du}{dx}$ for any $u$ as a function of $x$.

# 6.3 Basic Rules of Differentiation

## 6.3.6 Extended Chain Rule

**Example:**

Find $\frac{dy}{dx}$ when $y = \ln(\tan^{-1}(x^2 + 3x - 4))$.

**Answer:**

Let $y = \ln w$, $w = \tan^{-1} z$, and $z = x^2 + 3x - 4$.

Hence, $\frac{dy}{dw} = \frac{1}{w}$, $\frac{dw}{dz} = \frac{1}{1+z^2}$ and $\frac{dz}{dx} = 2x + 3$.

Thus $\frac{dy}{dx} = \frac{dy}{dw}\frac{dw}{dz}\frac{dz}{dx} = \left(\frac{1}{w}\right)\left(\frac{1}{1+z^2}\right)(2x+3)$.

$$= \frac{2x+3}{(1+z^2)w} = \frac{2x+3}{[1+(x^2+3x-4)^2]\tan^{-1}(x^2+3x-4)}.$$

# Exercise 6.2:

Find $\frac{dy}{dx}$ for the following functions.

1) $y = (5x^2 - 3)^{\frac{1}{4}}$

2) $y = \cos^{-1}(4x + 3)$

3) $y = \tan^{-1}\frac{3x+2}{2x-1}$

4) $y = \ln(\sin(4x))$

5) $y = \mathrm{e}^{\sqrt{x^2-3x+5}}$

6) $y = \cos^2(1 - 3x)$

[Ans: $\frac{5}{2}x(5x^2 - 3)^{-\frac{3}{4}}$; $-\frac{4}{\sqrt{1-(4x+3)^2}}$; $-\frac{7}{(2x-1)^2+(3x+2)^2}$; $\frac{4\cos 4x}{\sin 4x}$; $\frac{(2x-3)e^{\sqrt{x^2-3x+5}}}{2\sqrt{x^2-3x+5}}$; $6\cos(1-3x)\sin(1-3x)$]

# 6.3 Basic Rules of Differentiation

## 6.3.7 Parametric Differentiation Rule

If $y = f(x)$ where $x = g(t)$ and $y = h(t)$ such that $t$ is a parameter, then

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

**Example:**

Find $\frac{dy}{dx}$ in terms of $t$ when $x = e^{4t}$ and $y = 2t^2$.

**Solution:**

$$\frac{dy}{dx} = \frac{dy}{dt} / \frac{dx}{dt} = \frac{4t}{4e^{4t}} = te^{-4t}$$

# 6.3 Basic Rules of Differentiation

## 6.3.8 Implicit Differentiation

Recall that, if $y = f(x)$ then $\frac{dy}{dx} = f'(x)$. However, not all function can be expressed in the explicit form of $y = f(x)$. There are cases when a function is in terms of $x$, the differentiation of $f(x)$ could turn to be more complex expression. For simplification of the solution, the implicit differentiation can be used to find the derivatives.

**Example:**

Find $\frac{dy}{dx}$ for $y^2 = 3x$.

**Solution:**

Differentiate each of the terms with respect to $x$,

$$\frac{d}{dx}(y^2) = \frac{d}{dx}(3x).$$

By applying chain rule,

$$\frac{d}{dy}(y^2)\frac{dy}{dx} = 3 \quad \Longrightarrow \quad 2y\frac{dy}{dx} = 3 \quad \text{and hence} \quad \frac{dy}{dx} = \frac{3}{2y}.$$

# 6.3 Basic Rules of Differentiation

## 6.3.8 Implicit Differentiation

**Example:**

Find $\frac{dy}{dx}$ for $x^2 + y^2 = 5x$.

**Solution:**

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = \frac{d}{dx}(5x)$$

$$2x + \frac{d}{dy}(y^2)\frac{dy}{dx} = 5$$

$$2x + 2y\frac{dy}{dx} = 5$$

Thus,

$$\frac{dy}{dx} = \frac{5 - 2x}{2y}.$$

# 6.3 Basic Rules of Differentiation

## 6.3.8 Implicit Differentiation

**Example:**

Find derivative of $xy = \sin(xy)$ with respect to $x$.

**Solution:**

By applying product rule,

$$\frac{d}{dx}(xy) = x\frac{d}{dx}(y) + y\frac{d}{dx}(x) = x\frac{dy}{dx} + y$$

Hence, differentiation of $\sin(xy)$ w.r.t. $x$ is given by

Similarly to $\frac{d}{dx}[\sin f(x)] = f'(x)\cos f(x)$

$$\frac{d}{dx}\sin(xy) = \left(x\frac{dy}{dx} + y\right)\cos(xy)$$

Thus,

$$x\frac{dy}{dx} + y = x\cos(xy)\frac{dy}{dx} + y\cos(xy) \quad \Rightarrow \quad \frac{dy}{dx} = -\frac{y\cos(xy)}{x\cos(xy)} = -\frac{y}{x}$$

# 6.3 Basic Rules of Differentiation

## 6.3.8 Implicit Differentiation

**Example:**

Find $\frac{dy}{dx}$ for $xy = e^{x+3y}$.

**Solution:**

By using Product Rule

Similarly to
$\frac{d}{dx}\left[e^{f(x)}\right] = f'(x)e^{f(x)}$

$$\frac{d}{dx}(xy) = \frac{d}{dx}\left(e^{x+3y}\right)$$

$$x\frac{dy}{dx} + y = \left(1 + 3\frac{dy}{dx}\right)e^{x+3y}.$$

$$x\frac{dy}{dx} + y = e^{x+3y} + 3e^{x+3y}\frac{dy}{dx}$$

$$\left(x - 3e^{x+3y}\right)\frac{dy}{dx} = e^{x+3y} - y$$

$$\frac{dy}{dx} = \frac{e^{x+3y} - y}{x - 3e^{x+3y}}$$

# 6.3 Basic Rules of Differentiation

## 6.3.8 Implicit Differentiation

**Example:**

Find $\frac{dy}{dx}$ for $\sin(2x + y) = xy - e^{x+3y}$.

**Solution:**

Similarly to
$\frac{d}{dx}[\sin f(x)] = f'(x)\cos f(x)$

Similarly to
$\frac{d}{dx}[e^{f(x)}] = f'(x)e^{f(x)}$

$$\frac{d}{dx}(\sin(2x + y)) = \frac{d}{dx}(xy) - \frac{d}{dx}(e^{x+3y})$$

$$(\cos(2x + y))\left(2 + \frac{dy}{dx}\right) = x\frac{dy}{dx} + y(1) - \left(1 + 3\frac{dy}{dx}\right)e^{x+3y}$$

Hence,

$$2\cos(2x + y) + \cos(2x + y)\frac{dy}{dx} = x\frac{dy}{dx} + y - e^{x+3y} - 3e^{x+3y}\frac{dy}{dx}$$

Rearrange it, $\frac{dy}{dx} = \frac{y - e^{x+3y} - 2\cos(2x+y)}{\cos(2x+y) - x + 3e^{x+3y}}$

## Exercise 6.3:

1) Find $\frac{dy}{dx}$ for the following functions.

a) $y = t^{\frac{1}{3}}, x = \sinh t$

b) $y = te^{-t}, x = \ln 2t$

c) $3y^2 + 4y - 2x - \cos x = 0$

d) $xy + 3x = (2 - x)\sin y$

e) $x - 4y = e^{2x+3y-1}$

f) $y = 3^{-x}$

[Ans: $\frac{1}{3t^{\frac{2}{3}}\cosh t}; te^{-t}(1-t); \frac{2-\sin x}{6y+4}; -\frac{\sin y+y+3}{x-(2-x)\cos y}; \frac{1-2e^{2x+3y-1}}{3e^{2x+3y-1}+4}; -3^{-x}\ln 3$]

## Exercise 6.3:

2) Find the equation of tangent, at the point (1,2) to the curve defined by

$$x^2 + y^2 - xy - 3 = 0.$$

3) Find the equation of normal line, at the point (1,-1) to the curve defined by

$$\frac{x}{y} + 4x = 3.$$

4) Find $\frac{dy}{dx}$ of the function below:

$$x = 2\cos 3\theta - 5\sin 4\theta$$
$$y = 4\cos 3\theta + 3\sin 4\theta$$

5) Find $\frac{dy}{dx}$ of $y = x^2 e^{3x} \sin 4x$.

[Ans: $y = 2; y = -\frac{1}{3}x - \frac{2}{3}; \frac{-6\sin 3\theta + 6\cos 4\theta}{-3\sin 3\theta - 10\cos 4\theta}; xe^{3x}[4x\cos 4x + (3x+2)\sin 4x]$]