BMCG 1013
DIFFERENTIAL EQUATIONS

FINITE DIFFERENCE METHOD (HYPERBOLIC EQUATION)

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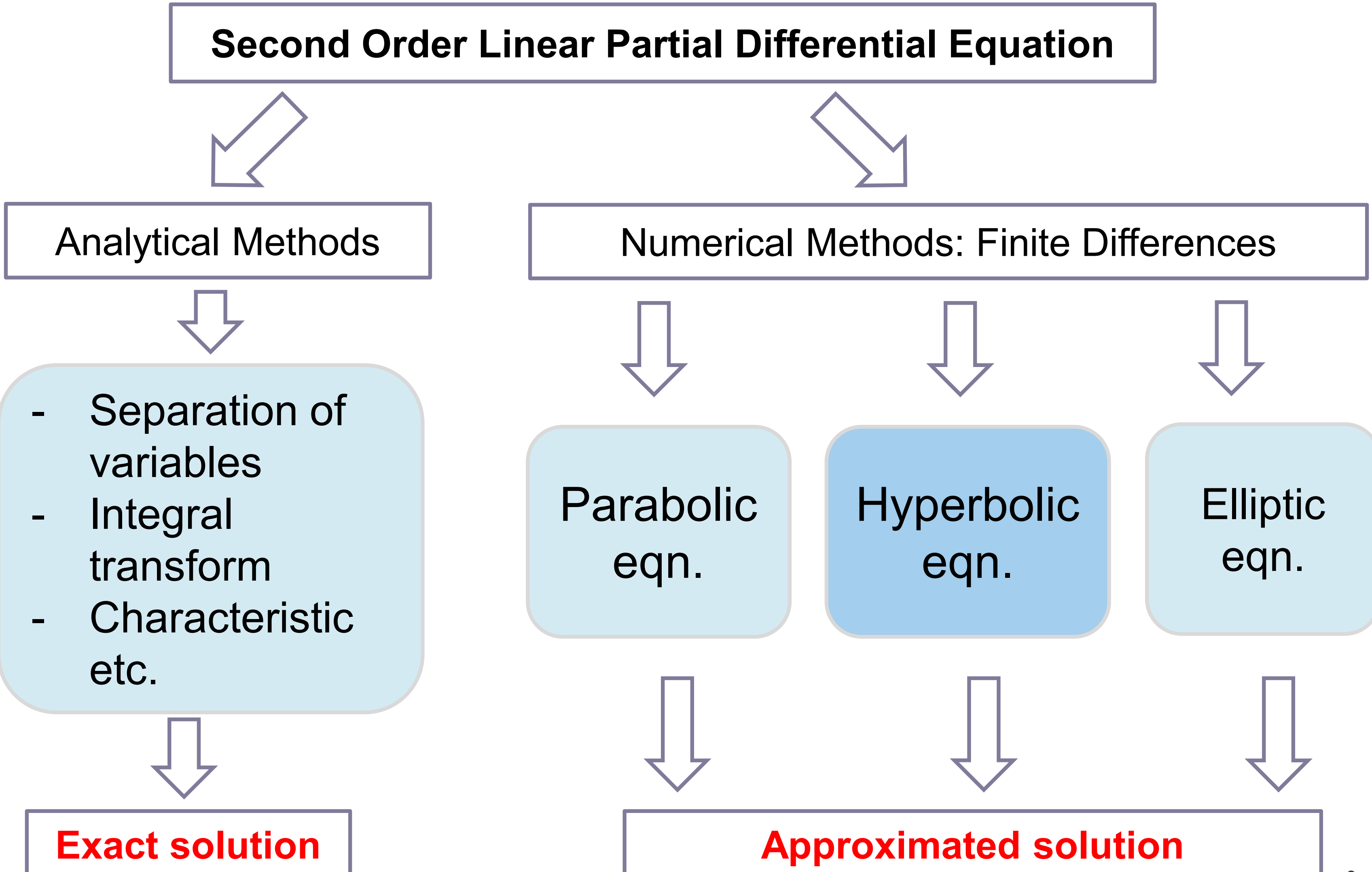
Lesson Outcomes

Upon completion of this lesson, students should be able to:

- solve hyperbolic equation using finite difference method
- apply finite difference method in wave equation


Second Order Linear Partial Differential Equation
Analytical Methods
Numerical Methods: Finite Differences
- Separation of variables
- Integral transform
- Characteristic
etc.
Parabolic eqn.
Hyperbolic eqn.
Elliptic eqn.
Exact solution
Approximated solution

Recall: Difference Formulas

Forward-Difference Formula:

$$\left(\frac{\partial u}{\partial t}\right)_{i,j} = \frac{u_{i,j+1} - u_{i,j}}{k} \qquad \left(\frac{\partial u}{\partial x}\right)_{i,j} = \frac{u_{i+1,j} - u_{i,j}}{h}$$

Backward-Difference Formula:

$$\left(\frac{\partial u}{\partial t}\right)_{i,j} = \frac{u_{i,j} - u_{i,j-1}}{k} \qquad \left(\frac{\partial u}{\partial x}\right)_{i,j} = \frac{u_{i,j} - u_{i-1,j}}{h}$$

Central-Difference Formula:

$$\left(\frac{\partial u}{\partial t}\right)_{i,j} = \frac{u_{i,j+1} - u_{i,j-1}}{2k} \qquad \left(\frac{\partial u}{\partial x}\right)_{i,j} = \frac{u_{i+1,j} - u_{i-1,j}}{2h}$$

$$\left(\frac{\partial^2 u}{\partial t^2}\right)_{i,j} = \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{k^2} \qquad \left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2}$$

5.4.3 Hyperbolic Equation – Wave Equation

Consider the **Wave Equation**:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < a, \qquad t > 0$$

with boundary conditions

$$u(0,t) = u(a,t) = 0, \qquad t > 0$$

and initial conditions

$$u(x,0) = f(x), \qquad 0 \le x \le a$$

$$\frac{\partial u}{\partial t}(x,0) = g(x), \qquad 0 \le x \le a$$

By central-difference formulas,

$$\left(\frac{\partial^2 u}{\partial t^2}\right)_{i,j} - c^2 \left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j} = 0$$

is approximated to

$$\frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{k^2} - c^2 \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} = 0$$

where $h = \Delta x$ and $k = \Delta t$.

Example 5.19:

Approximate the solution for the following wave equation

$$\frac{\partial^2 u}{\partial t^2} = 4\frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < 1, \qquad t > 0$$

with boundary conditions

$$u(0,t) = u(1,t) = 0, \qquad 0 \le t \le 1$$

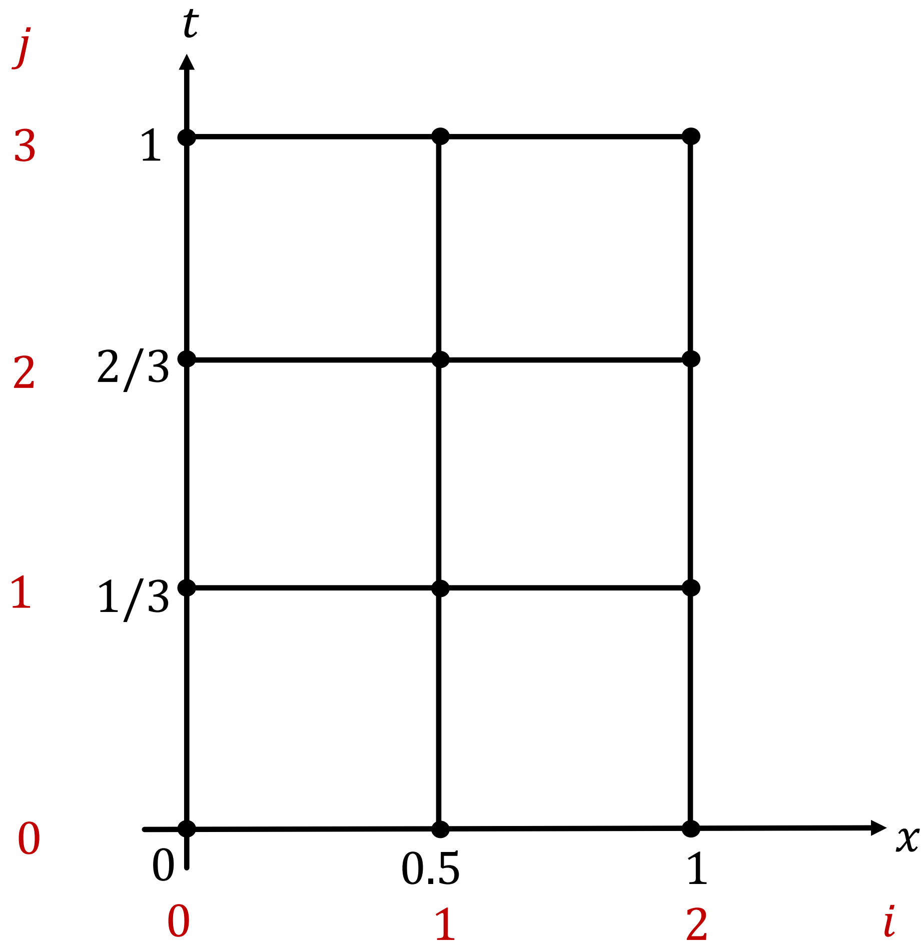
and initial conditions

$$u(x,0) = \sin 2\pi x\,, \qquad 0 \le x \le 1$$

$$\frac{\partial u}{\partial t}(x,0) = 4, \qquad 0 \le x \le 1$$

by using the central-difference formula. Given $h = 0.5$ and $k = \frac{1}{3}$.

Step 1: Sketch the grid points.



Recall:

$0 \leq x \leq 1, \qquad 0 \leq t \leq 1$

$h = 0.5$ and $k = 1/3$

Step 2: Compute boundary and initial values.

Recall:
$h = 0.5$ and $k = 1/3$

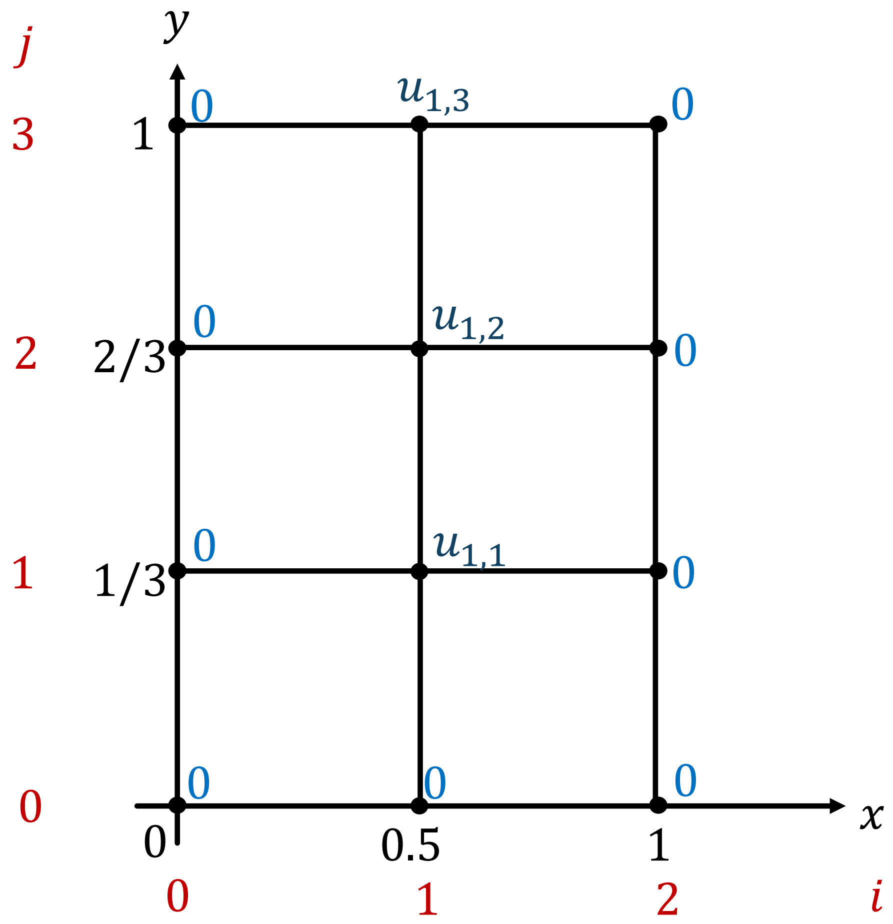
$$\boldsymbol{u(x,0) = \sin 2\pi x}, \qquad \boldsymbol{0 \le x \le 1}$$

$u(0,0) = 0$
$u(0.5,0) = 0$
$u(1,0) = 0$

$$\boldsymbol{u(0,t) = 0}, \qquad \boldsymbol{u(1,t) = 0}, \qquad \boldsymbol{0 \le t \le 1}$$

$u(0,0) = 0$	$u(1,0) = 0$
$u(0,0.3333) = 0$	$u(1,0.3333) = 0$
$u(0,0.6667) = 0$	$u(1,0.6667) = 0$
$u(0,1) = 0$	$u(1,1) = 0$

Step 3: Fill in the values into the grid.



Step 4: Obtain formula $u_{i,j+1}$ from difference formula.

By central-difference formulas,

$$\frac{\partial^2 u}{\partial t^2} = 4\frac{\partial^2 u}{\partial x^2} \quad \Rightarrow \quad \left(\frac{\partial^2 u}{\partial t^2}\right)_{i,j} - 4\left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j} = 0$$

Choose the required difference formula from the list:

Forward-Difference Formula:

$$\left(\frac{\partial u}{\partial t}\right)_{i,j} = \frac{u_{i,j+1} - u_{i,j}}{k} \qquad \left(\frac{\partial u}{\partial x}\right)_{i,j} = \frac{u_{i+1,j} - u_{i,j}}{h}$$

Backward-Difference Formula:

$$\left(\frac{\partial u}{\partial t}\right)_{i,j} = \frac{u_{i,j} - u_{i,j-1}}{k} \qquad \left(\frac{\partial u}{\partial x}\right)_{i,j} = \frac{u_{i,j} - u_{i-1,j}}{h}$$

Central-Difference Formula:

$$\left(\frac{\partial u}{\partial t}\right)_{i,j} = \frac{u_{i,j+1} - u_{i,j-1}}{2k} \qquad \left(\frac{\partial u}{\partial x}\right)_{i,j} = \frac{u_{i+1,j} - u_{i-1,j}}{2h}$$

$$\left(\frac{\partial^2 u}{\partial t^2}\right)_{i,j} = \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{k^2} \qquad \left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2}$$

Step 4: Obtain formula $u_{i,j+1}$ from difference formula.

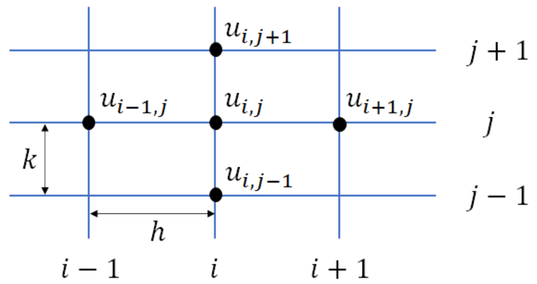
By central-difference formulas,

$$\left(\frac{\partial^2 u}{\partial t^2}\right)_{i,j} - 4\left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j} = 0$$

Recall:
$h = 0.5$ and $k = 1/3$

is approximated to

$$\frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{k^2} - 4\left(\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2}\right) = 0$$



$$u_{i,j+1} - 2u_{i,j} + u_{i,j-1} = \frac{4k^2}{h^2}\left(u_{i+1,j} - 2u_{i,j} + u_{i-1,j}\right)$$

$$u_{i,j+1} = \frac{4(1/3)^2}{(0.5)^2}\left(u_{i+1,j} - 2u_{i,j} + u_{i-1,j}\right) + 2u_{i,j} - u_{i,j-1}$$

$$u_{i,j+1} = \left(\frac{16}{9}\right)u_{i-1,j} - \left(\frac{14}{9}\right)u_{i,j} + \left(\frac{16}{9}\right)u_{i+1,j} - u_{i,j-1}$$

Step 4: Obtain formula $u_{i,j+1}$ from difference formula.

Recall:
$h = 0.5$ and $k = 1/3$

From the initial velocity,

$$\frac{\partial u}{\partial t}(x,0) = 4 \quad \xrightarrow{\text{discretized}} \quad \frac{\partial u}{\partial t}(x_i,0) = \left(\frac{\partial u}{\partial t}\right)_{i,j=0} = 4$$

By central-difference formulas,

$$\left(\frac{\partial u}{\partial t}\right)_{i,j} = \frac{u_{i,j+1} - u_{i,j-1}}{2k}$$

Choose the required difference formula from the list:

Central-Difference Formula:

$$\left(\frac{\partial u}{\partial t}\right)_{i,j} = \frac{u_{i,j+1} - u_{i,j-1}}{2k} \qquad \left(\frac{\partial u}{\partial x}\right)_{i,j} = \frac{u_{i+1,j} - u_{i-1,j}}{2h}$$

$$\left(\frac{\partial^2 u}{\partial t^2}\right)_{i,j} = \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{k^2} \qquad \left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2}$$

Step 4: Obtain formula $u_{i,j+1}$ from difference formula.

Recall:
$h = 0.5$ and $k = 1/3$

From the initial velocity,

$$\frac{\partial u}{\partial t}(x,0) = 4 \quad \xrightarrow{\text{discretized}} \quad \frac{\partial u}{\partial t}(x_i,0) = \left(\frac{\partial u}{\partial t}\right)_{i,j=0} = 4$$

By central-difference formulas,

$$\left(\frac{\partial u}{\partial t}\right)_{i,j} = \frac{u_{i,j+1} - u_{i,j-1}}{2k}$$

Hence,

$$\left(\frac{\partial u}{\partial t}\right)_{i,j=0} = \frac{u_{i,1} - u_{i,-1}}{2(1/3)} = 4$$

is approximated to

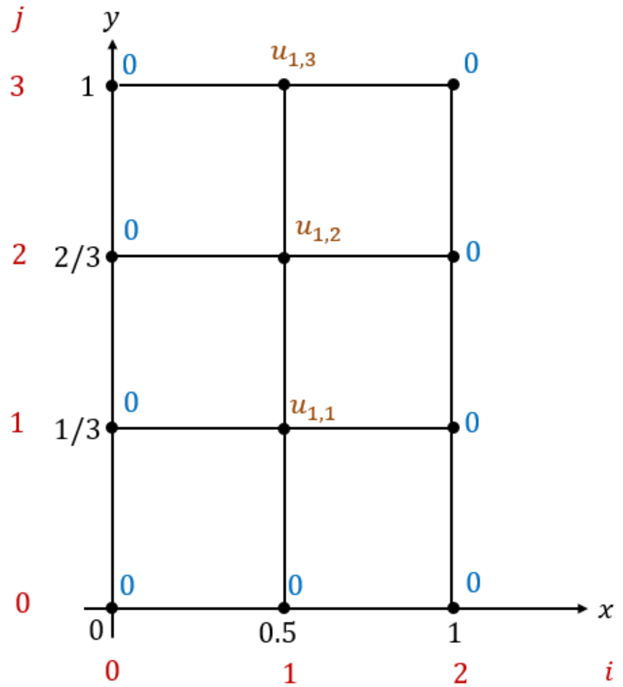
$$u_{i,1} - u_{i,-1} = 8/3$$

$$u_{i,-1} = u_{i,1} - 8/3$$

Step 5: Compute solutions for internal points $u_{i,j}$ (Refer to grid in Step 3).

$$u_{i,j+1} = \left(\frac{16}{9}\right) u_{i-1,j} - \left(\frac{14}{9}\right) u_{i,j} + \left(\frac{16}{9}\right) u_{i+1,j} - u_{i,j-1}$$

$$u_{i,-1} = u_{i,1} - 8/3$$



Let $i = 1, j = 0$,

$$u_{1,1} = \left(\frac{16}{9}\right) u_{0,0} - \left(\frac{14}{9}\right) u_{1,0} + \left(\frac{16}{9}\right) u_{2,0} - u_{1,-1}$$

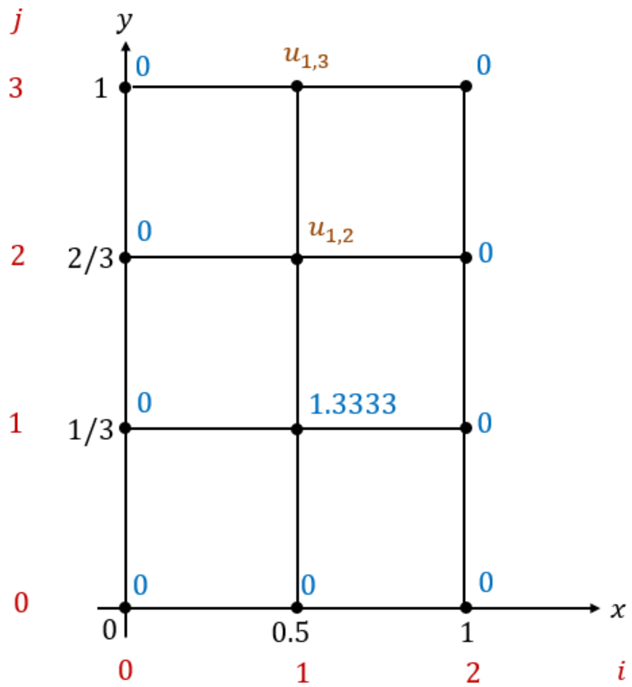
$$u_{1,1} = \left(\frac{16}{9}\right)(0) - \left(\frac{14}{9}\right)(0) + \left(\frac{16}{9}\right)(0) - \left(u_{1,1} - \frac{8}{3}\right)$$

$$2u_{1,1} = 2.6667$$

$$u_{1,1} = 1.3333$$

Step 5: Compute solutions for internal points $u_{i,j}$ (Refer to grid in Step 3).

$$u_{i,j+1} = \left(\frac{16}{9}\right)u_{i-1,j} - \left(\frac{14}{9}\right)u_{i,j} + \left(\frac{16}{9}\right)u_{i+1,j} - u_{i,j-1}$$



Let $i = 1, j = 1,$

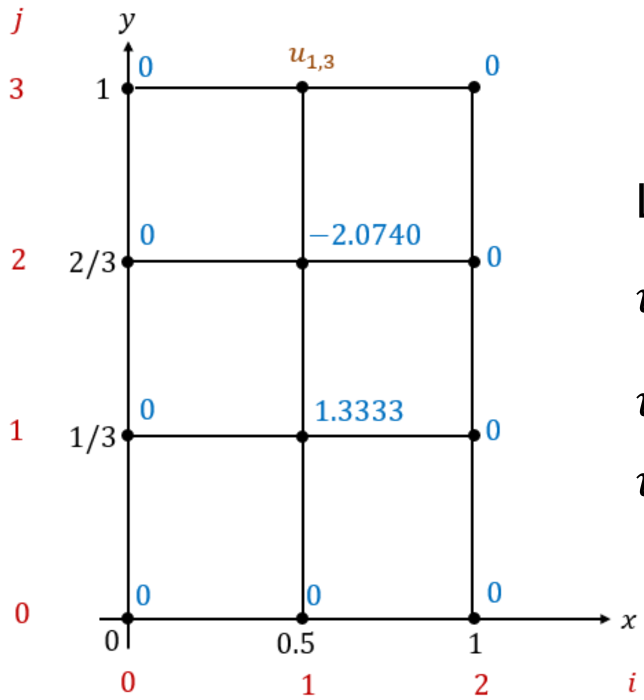
$$u_{1,2} = \left(\frac{16}{9}\right)u_{0,1} - \left(\frac{14}{9}\right)u_{1,1} + \left(\frac{16}{9}\right)u_{2,1} - u_{1,0}$$

$$u_{1,2} = \left(\frac{16}{9}\right)(0) - \left(\frac{14}{9}\right)(1.3333) + \left(\frac{16}{9}\right)(0) - 0$$

$$u_{1,2} = -2.0740$$

Step 5: Compute solutions for internal points $u_{i,j}$ (Refer to grid in Step 3).

$$u_{i,j+1} = \left(\frac{16}{9}\right)u_{i-1,j} - \left(\frac{14}{9}\right)u_{i,j} + \left(\frac{16}{9}\right)u_{i+1,j} - u_{i,j-1}$$



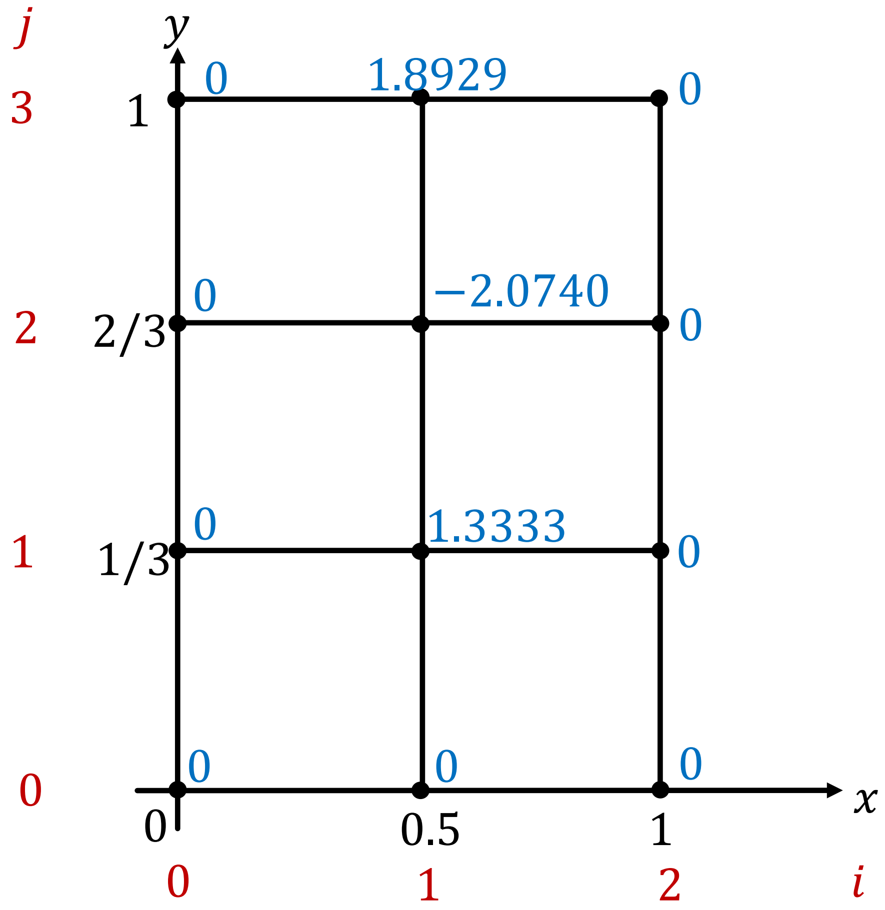
Let $i = 1, j = 2$,

$$u_{1,3} = \left(\frac{16}{9}\right)u_{0,2} - \left(\frac{14}{9}\right)u_{1,2} + \left(\frac{16}{9}\right)u_{2,2} - u_{1,1}$$

$$u_{1,3} = \left(\frac{16}{9}\right)(0) - \left(\frac{14}{9}\right)(-2.0740) + \left(\frac{16}{9}\right)(0) - 1.3333$$

$$u_{1,3} = 1.8929$$

Step 6: Fill in the values into the grid.



Example 5.20:

Approximate the solution for the following wave equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < 1, \qquad t > 0$$

with boundary conditions

$$u(0,t) = u(1,t) = 0, \qquad 0 \le t \le 1$$

and initial conditions

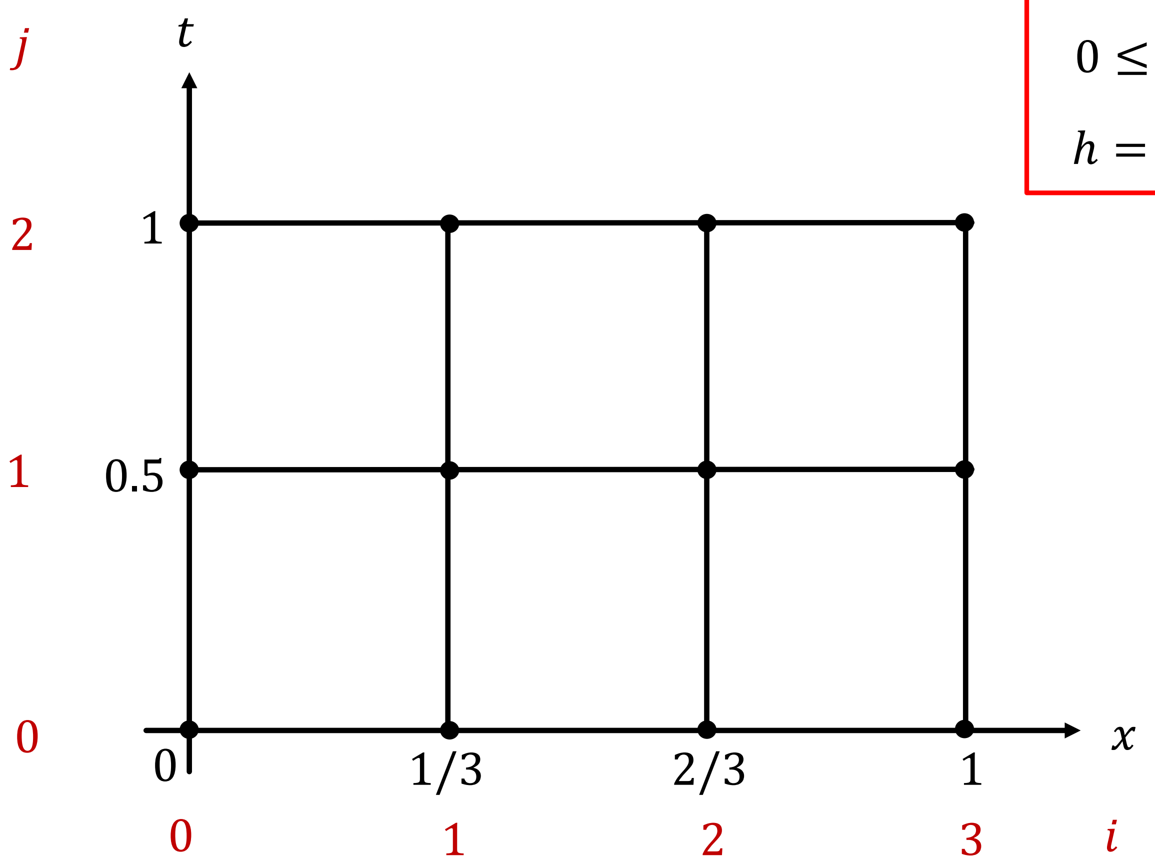
$$u(x,0) = x(x-1), \qquad 0 \le x \le 1$$

$$\frac{\partial u}{\partial t}(x,0) = 1, \qquad 0 \le x \le 1$$

by using the central-difference formula. Given $h = \frac{1}{3}$ and $k = 0.5$.

Solution:

Step 1: Sketch the grid points.



Recall:

$0 \leq x \leq 1, \qquad 0 \leq t \leq 1$

$h = 1/3$ and $k = 0.5$

Step 2: Compute boundary and initial values.

Recall:
$h = 1/3$ and $k = 0.5$

$\boldsymbol{u(x,0) = x(x-1), \qquad 0 \le x \le 1}$

$u(0,0) = 0$

$u(0.3333,0) = -0.2222$

$u(0.6667,0) = -0.2222$

$u(1,0) = 0$

$\boldsymbol{u(0,t) = 0, \qquad u(1,t) = 0, \qquad 0 \le t \le 1}$

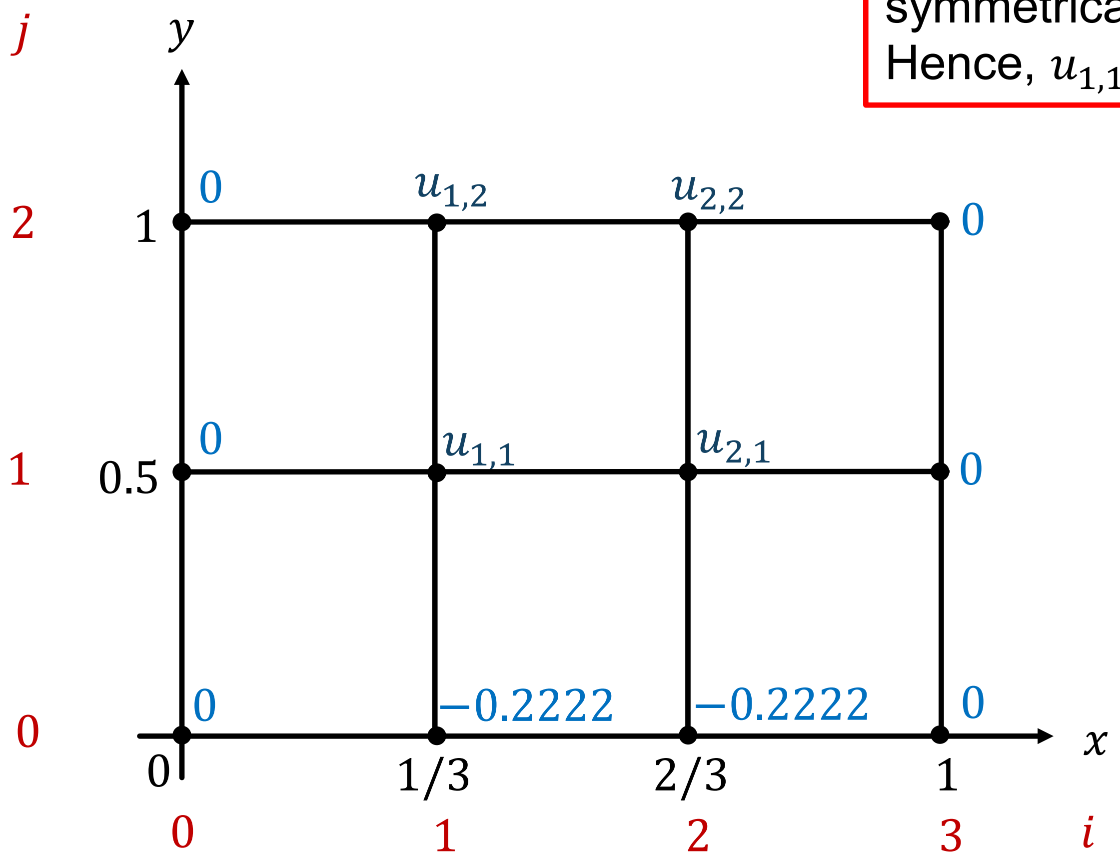
$u(0,0) = 0 \qquad u(1,0) = 0$

$u(0,0.5) = 0 \qquad u(1,0.5) = 0$

$u(0,1) = 0 \qquad u(1,1) = 0$

Step 3: Fill in the values into the grid.

Boundary and initial conditions are symmetrical about the line $x = 0.5$. Hence, $u_{1,1} = u_{2,1}$ and $u_{1,2} = u_{2,2}$



Step 4: Obtain formula $u_{i,j+1}$ from difference formula.

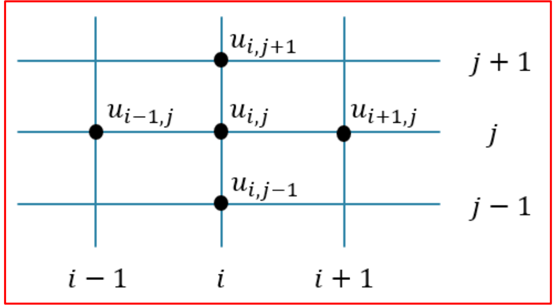
By central-difference formulas,

$$\left(\frac{\partial^2 u}{\partial t^2}\right)_{i,j} - \left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j} = 0$$

Recall:
$h = 1/3$ and $k = 0.5$

is approximated to

$$\frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{k^2} - \left(\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2}\right) = 0$$



$$u_{i,j+1} - 2u_{i,j} + u_{i,j-1} = \frac{k^2}{h^2}\left(u_{i+1,j} - 2u_{i,j} + u_{i-1,j}\right)$$

$$u_{i,j+1} = \frac{(0.5)^2}{(1/3)^2}\left(u_{i+1,j} - 2u_{i,j} + u_{i-1,j}\right) + 2u_{i,j} - u_{i,j-1}$$

$$u_{i,j+1} = 2.25u_{i-1,j} - 2.5u_{i,j} + 2.25u_{i+1,j} - u_{i,j-1}$$

Step 4: Obtain formula $u_{i,j+}$ from difference formula.

Recall:
$h = 1/3$ and $k = 0.5$

From the initial velocity,

$$\frac{\partial u}{\partial t}(x,0) = 1 \quad \xrightarrow{\text{discretized}} \quad \frac{\partial u}{\partial t}(x_i,0) = \left(\frac{\partial u}{\partial t}\right)_{i,j=0} = 1$$

By central-difference formulas,

$$\left(\frac{\partial u}{\partial t}\right)_{i,j} = \frac{u_{i,j+1} - u_{i,j-1}}{2k}$$

Hence,

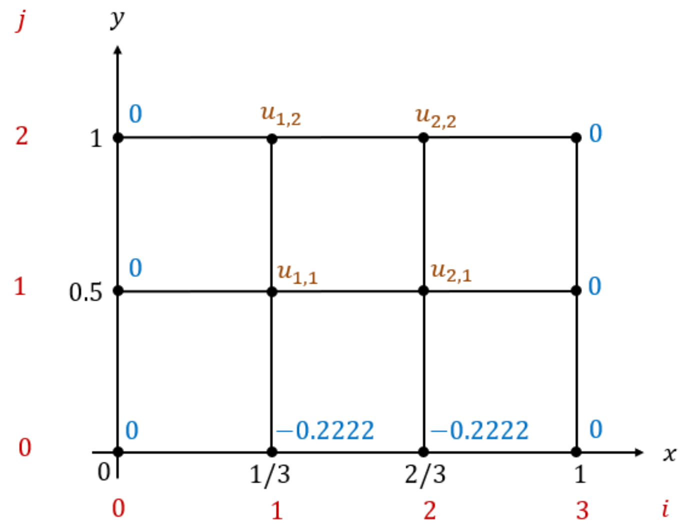
$$\left(\frac{\partial u}{\partial t}\right)_{i,j=0} = \frac{u_{i,1} - u_{i,-1}}{2(0.5)} = 1$$

is approximated to

$$u_{i,1} - u_{i,-1} = 1$$
$$u_{i,-1} = u_{i,1} - 1$$

Step 5: Compute solutions for internal points $u_{i,j}$ (Refer to grid in Step 3).

$$u_{i,j+1} = 2.25u_{i-1,j} - 2.5u_{i,j} + 2.25u_{i+1,j} - u_{i,j-1}$$
$$u_{i,-1} = u_{i,1} - 1$$



Boundary and initial conditions are symmetrical about the line $x = 0.5$.

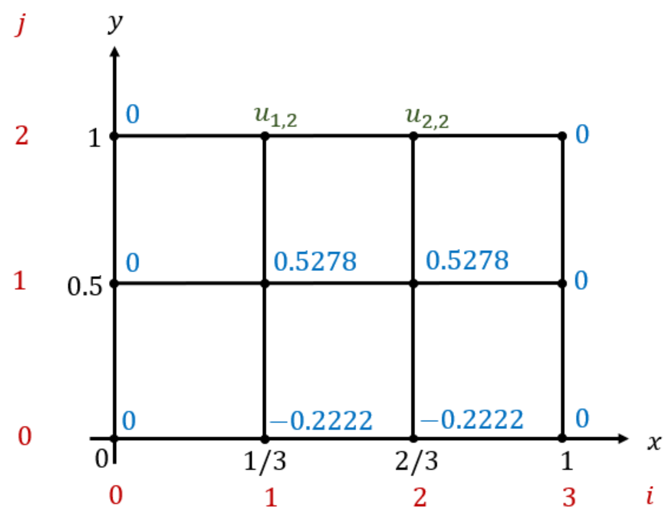
Hence,
$u_{1,1} = u_{2,1}$ and
$u_{1,2} = u_{2,2}$

Let $i = 1, j = 0$,

$$u_{1,1} = 2.25u_{0,0} - 2.5u_{1,0} + 2.25u_{2,0} - u_{1,-1}$$
$$u_{1,1} = 2.25(0) - 2.5(-0.2222) + 2.25(-0.2222) - (u_{1,1} - 1)$$
$$2u_{1,1} = 1.0556$$
$$u_{1,1} = 0.5278 \qquad \therefore u_{2,1} = 0.5278 \text{ due to symmetrical condition}$$

Step 5: Compute solutions for internal points $u_{i,j}$ (Refer to grid in Step 3).

$$u_{i,j+1} = 2.25u_{i-1,j} - 2.5u_{i,j} + 2.25u_{i+1,j} - u_{i,j-1}$$
$$u_{i,-1} = u_{i,1} - 1$$



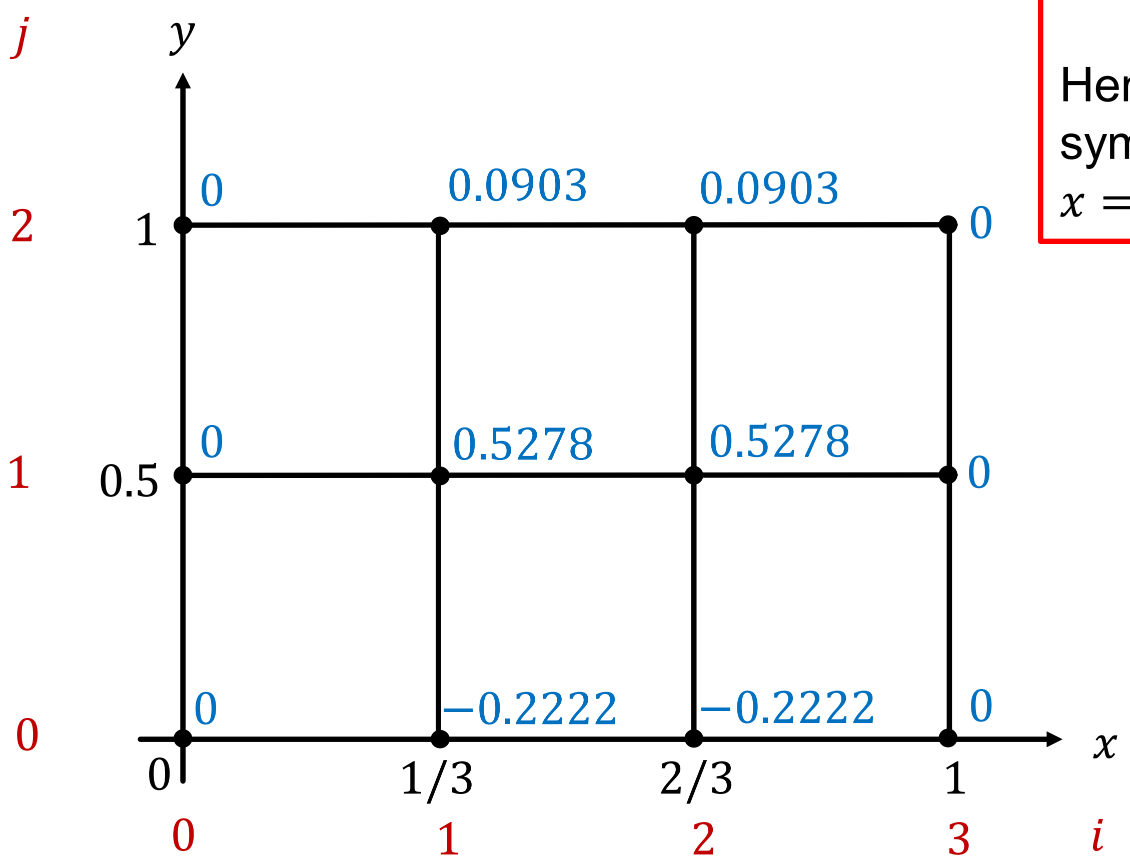
Boundary and initial conditions are symmetrical about the line $x = 0.5$.

Hence,
$u_{1,1} = u_{2,1}$ and
$u_{1,2} = u_{2,2}$

Let $i = 1, j = 1$,

$$u_{1,2} = 2.25u_{0,1} - 2.5u_{1,1} + 2.25u_{2,1} - u_{1,0}$$
$$u_{1,2} = 2.25(0) - 2.5(0.5278) + 2.25(0.5278) - (-0.2222)$$
$u_{1,2} = 0.0903$ $\therefore u_{2,2} = 0.0903$ due to symmetrical condition

Step 6: Fill in the values into the grid.



Boundary and initial conditions are symmetrical about the line $x = 0.5$.

Hence, all grid points are symmetrical about the line $x = 0.5$

Exercise 5.11:

1) Approximate the solution for the following wave equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < 1, \qquad t > 0$$

with boundary conditions

$$u(0,t) = u(1,t) = 0, \qquad 0 \le t \le 0.1$$

and initial conditions

$$u(x,0) = \sin \pi x\,, \qquad 0 \le x \le 1$$

$$\frac{\partial u}{\partial t}(x,0) = 0, \qquad 0 \le x \le 1$$

by using the central-difference formula. Given $h = 0.25$ and $k = 0.05$.

[Ans: $u_{0,0} = u_{0,1} = u_{0,2} = u_{4,0} = u_{4,1} = u_{4,2} = 0; u_{1,0} = u_{3,0} = 0.7071; u_{2,0} = 1;\ u_{1,1} = u_{3,1} = 0.6988; u_{2,1} = 0.9883; u_{1,2} = u_{3,2} = 0.6741; u_{2,2} = 0.9534$]

Exercise 5.11:

2) Approximate the solution for the following wave equation

$$\frac{\partial^2 u}{\partial t^2} = 9\frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < \pi, \qquad t > 0$$

with boundary conditions

$$u(0,t) = u(\pi,t) = 0, \qquad 0 \le t \le 1$$

and initial conditions

$$u(x,0) = x\cos\frac{x}{2}, \qquad 0 \le x \le \pi$$

$$\frac{\partial u}{\partial t}(x,0) = 2, \qquad 0 \le x \le \pi$$

by using the central-difference formula. Given $h = \frac{\pi}{4}$ and $k = 0.5$.

[Ans: $u_{0,0} = u_{0,1} = u_{0,2} = u_{4,0} = u_{4,1} = u_{4,2} = 0; u_{1,0} = 0.7256;$
$u_{2,0} = 1.1107;\ u_{3,0} = 0.9017; u_{1,1} = 1.1046;\ u_{2,1} = 1.0272; u_{3,1} = 0.6384;$
$u_{1,2} = -2.8279;\ u_{2,2} = -0.1922; u_{3,2} = -0.5353$]

References

1. S.C. Chapra and R.P. Canale. (2015). Numerical Methods for Engineers, 7th Edition. McGraw-Hill Education.

2. R.L. Burden, D.J. Faires and A.M. Burden. (2016). Numerical Analysis, 10th Edition. Cengage Learning.

Thank You

Questions & Answer?