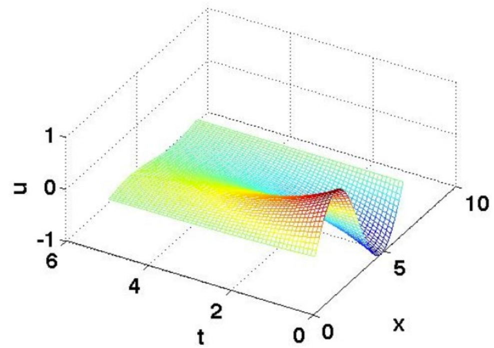
BMCG 1013
DIFFERENTIAL EQUATIONS

APPLICATION OF PARTIAL DIFFERENTIAL EQUATIONS



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Lesson Outcomes

Upon completion of this lesson, students should be able to:

- solve heat equation on the heated rod problems.
- solve wave equation on the vibrating spring problems.

5.3 Applications of PDE

5.3.1 The Heat Equation

In this section, we will study the heat flow problem in a long uniform rod. In general, this problem depends on the initial distribution of temperature and the physical properties of rod. The physical properties of rod refers to the thermal diffusivity of the material where it measures the rate of transfer of heat of the material from the hot end to the cold end.

By using the method of separation of variables, the solution function $u(x,t)$ that represents temperature at a point x along the rod at time t can be determined.

Heat Conduction

A uniform homogeneous rod or metal bar is made up of infinite numbers of molecules that are interconnected by cohesive force. When heat is applied to one end of rod, the heat tends to transfer due to heat flows from molecule to molecule. Heat moves rapidly in a substance with high thermal diffusivity.

In the following section, the heat transfer of the heated rod with zero temperature at the endpoints will be discussed. Graphical interpretation will be illustrated in the next slide.

Graphical Interpretation

Given a rod with length L with the setup and condition as shown in Figure 5.1, we aim to find the temperature of the rod at position point x after some time t.

Boundary condition:
$u(0,t) = 0$
Always zero temperature at this end point

Boundary condition:
$u(L,t) = 0$
Always zero temperature at this end point

Initial condition:
$u(x,0) = f(x)$
At $t = 0$, temperature of rod at position x is set up as $f(x)$

Figure 5.1: Graphical interpretation of experimental setup for heat conduction

For heat equation, the function $u(x,t)$ represents temperature at a point x along the rod at time t.

For example:

$u(0,5) = 10$ means at the time $t = 5$, the temperature at the point $x = 0$ is 10℃. (The unit of measurements of length and time depend on the unit of measurement of thermal diffusivity)

Theorem: The heated rod with zero temperature at the endpoints

The partial differential equation

$$u_t = ku_{xx} \quad , \quad 0 < x < L, \quad t > 0$$

for a rod of length L and thermal diffusivity of k, with boundary conditions

$$u(0,t) = u(L,t) = 0, \quad t > 0$$

and initial condition

$$u(x,0) = f(x), \qquad 0 < x < L$$

has the series solution

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) e^{-\frac{n^2\pi^2 kt}{L^2}} \tag{5.1}$$

where b_n is the Fourier series coefficients obtained by

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx, \qquad n = 1,2,3,\dots \tag{5.2}$$

5.3.1 The Heat Equation

Example 5.13:

Solve the following PDE for the heat transfer of a metal bar with zero temperature at both ends by method of separation of variables:

$$u_t = \frac{1}{9} u_{xx} \quad , \quad 0 < x < 2\pi, \quad t > 0.$$

$$u(0, t) = u(2\pi, t) = 0; \; u(x, 0) = x(2\pi - x)$$

Next, by using the first three non-zero terms of the series, compute the temperature of the metal bar at the point $x = \pi$ when $t = 1$.

(Give your answer correct to 4 decimal places)

Solution:

Let $u(x,t) = X(x)T(t)$, the heat equation becomes

$$XT' = \frac{1}{9}X''T \tag{5.3}$$

Separate the variables, we have

$$\frac{X''}{X} = \frac{9T'}{T} = p \tag{5.4}$$

where constant p is known as separation constant.

From Eqn. (5.4),

$$\frac{X''}{X} = p \quad \text{and} \quad \frac{9T'}{T} = p \tag{5.5}$$

Hence X and T must satisfy

$$X'' - pX = 0 \;\; \text{and} \;\; T' - \frac{1}{9}pT = 0 \tag{5.6}$$

Since $u(x,t) = X(x)T(t)$ and $T(t) \neq 0$, the boundary conditions become

(5.7)

$$u(0,t) = X(0)T(t) = 0 \Rightarrow X(0) = 0$$

$$u(2\pi,t) = X(2\pi)T(t) = 0 \Rightarrow X(2\pi) = 0$$

Now, we need to consider three possible cases for p.

Case (1): $p = 0$.

From Eqn (5.6), (substitute $p = 0$ into $X'' - pX = 0$)

$$X'' = 0$$

$$X' = A$$

$$\therefore X(x) = Ax + B \qquad (5.8)$$

By substituting boundary conditions (5.7) into Eqn. (5.8),

$X(0) = 0,\ X(0) = A(0) + B$

$$\therefore B = 0$$

$X(2\pi) = 0,\ X(2\pi) = A(2\pi) + B$

$$0 = A(2\pi) + 0$$

$$A = 0$$

From Eqn. (5.8), we found that $X(x) = 0$ and this implies

$$u(x,t) = X(x)T(t) = 0.$$

This case gives a trivial solution. Hence, we omit this case.

Case (2): $p > 0$.

Let $p = \lambda^2$ and $\lambda > 0$. Substitute $p = \lambda^2$ into Eqn. (5.6) leads to ODE

$$X'' - \lambda^2 X = 0$$

Characteristic equation:

$$m^2 - \lambda^2 = 0$$

$$m = \pm\lambda$$

$$\therefore X(x) = Ae^{\lambda x} + Be^{-\lambda x} \qquad (5.9)$$

$X(0) = 0,\ X(0) = Ae^{\lambda(0)} + Be^{-\lambda(0)}$

$$\therefore 0 = A + B$$

$$A = -B$$

$X(2\pi) = 0,\ 0 = Ae^{2\pi\lambda} + Be^{-2\pi\lambda}$

$$0 = -Be^{2\pi\lambda} + Be^{-2\pi\lambda}$$

$$0 = B(-e^{2\pi\lambda} + e^{-2\pi\lambda})$$

Since $-e^{2\pi\lambda} + e^{-2\pi\lambda} \neq 0$, $B = 0$ and then $A = 0$. From Eqn. (5.9),

$$X(x) = 0 \quad \Rightarrow \quad u(x,t) = X(x)T(t) = 0$$

This case gives a trivial solution. Hence, we omit this case.

Case (3): $p < 0$.

Let $p = -\lambda^2$ and $\lambda > 0$. Substitute $p = -\lambda^2$ into Eqn (5.6) leads to ODE

$$X'' + \lambda^2 X = 0 \text{ and } T' + \frac{1}{9}\lambda^2 T = 0$$

Characteristic equation:

$$m^2 + \lambda^2 = 0 \quad \Rightarrow \quad m = \pm\lambda i$$

$$\therefore X(x) = A\cos\lambda x + B\sin\lambda x \tag{5.10}$$

$X(0) = 0,\ X(0) = A\cos 0 + B\sin 0,\quad X(2\pi) = 0,\ X(2\pi) = A\cos 2\pi\lambda + B\sin 2\pi\lambda$

$$\therefore 0 = A \qquad\qquad 0 = B\sin 2\pi\lambda$$

$$B \neq 0,\ \therefore \sin 2\pi\lambda = 0$$

Given $\sin n\pi = 0, n = 1,2,3, \ldots$
So, $\sin 2\pi\lambda = 0$ implies $2\pi\lambda = n\pi$

$$2\pi\lambda = n\pi,\ n = 1,2,3, \ldots$$

$$\lambda_n = \frac{n}{2},\ n = 1,2,3, \ldots \tag{5.11}$$

$$\therefore X_n(x) = B_n \sin\lambda_n x$$

$$= B_n \sin\frac{nx}{2},\quad n = 1,2,3, \ldots \tag{5.12}$$

$$T' + \frac{1}{9}\lambda^2 T = 0$$

$$\frac{dT}{dt} = -\frac{1}{9}\lambda^2 T$$

$$\int \frac{1}{T}\, dT = \int -\frac{1}{9}\lambda^2 \, dt$$

$$\ln T = -\frac{1}{9}\lambda^2 t + k$$

$$T = e^{-\frac{1}{9}\lambda^2 t + k}$$

$$= Ce^{-\frac{1}{9}\lambda^2 t}$$

Let $C = e^k$

From Eqn. (5.11), $\lambda_n = \frac{n}{2}$

$$T_n(t) = C_n e^{-\frac{1}{9}\lambda_n^2 \, t} = C_n e^{-\frac{n^2}{36}t}, \qquad n = 1,2,3, \dots \qquad (5.13)$$

Hence,

$$u_n(x,t) = X_n(x) T_n(t)$$

$$= \left(B_n \sin\frac{nx}{2}\right) C_n e^{-\frac{n^2}{36}t}$$

Let $D_n = B_n C_n$

$$= D_n (\sin\frac{nx}{2}) e^{-\frac{n^2}{36}t}, \; n = 1,2,3, \dots \qquad (5.14)$$

If $u_1, u_2, \ldots, u_\infty$ are linearly independent solutions to a PDE, then the linear combination of all $u_1, u_2, \ldots, u_\infty$ are also a solution to the PDE

By applying superposition principle on Eqn. (5.14),

$$u(x,t) = \sum_{n=1}^{\infty} D_n (\sin \frac{nx}{2}) e^{-\frac{n^2}{36}t} \tag{5.15}$$

Apply initial condition, $u(x,0) = x(2\pi - x)$, on Eqn. (5.15),

$$x(2\pi - x) = \sum_{n=1}^{\infty} D_n \sin \frac{nx}{2} \tag{5.16}$$

Recall from Fourier Sine Series:

$$f(x) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L}$$

where

$$B_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} \, dx$$

From Eqn. (5.16), by half-range expansion for $f(x)$ on $0 < x < 2\pi$,

$$D_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$$

$$= \frac{2}{2\pi}\int_0^{2\pi} x(2\pi - x)\sin\frac{nx}{2}\,dx$$

$$= \frac{1}{\pi}\left[-\frac{2(2\pi x - x^2)}{n}\cos\frac{nx}{2} + \frac{4(2\pi - 2x)}{n^2}\sin\frac{nx}{2} - \frac{16}{n^3}\cos\frac{nx}{2}\right]_0^{2\pi}$$

$$= \frac{1}{\pi}\left[\left(-\frac{16}{n^3}\cos n\pi\right) - \left(-\frac{16}{n^3}\right)\right]$$

$$= \frac{16}{n^3\pi}[1 - \cos n\pi]$$

($\cos n\pi = (-1)^n$)

$$= \frac{16}{n^3\pi}[1 - (-1)^n]$$

Tabular Method:

	Diff.	Integrate
+	$2\pi x - x^2$	$\sin\frac{nx}{2}$
−	$2\pi - 2x$	$-\frac{2}{n}\cos\frac{nx}{2}$
+	-2	$-\frac{4}{n^2}\sin\frac{nx}{2}$
−	0	$\frac{8}{n^3}\cos\frac{nx}{2}$

By substituting D_n into Eqn. (5.15), the temperature function is given by

$$u(x,t) = \sum_{n=1}^{\infty} D_n (\sin \frac{nx}{2}) e^{-\frac{n^2}{36}t}$$

$$= \sum_{n=1}^{\infty} \frac{16}{n^3 \pi} [1 - (-1)^n] (\sin \frac{nx}{2}) e^{-\frac{n^2}{36}t}$$

$$= \frac{32}{\pi} \sum_{n\ odd}^{\infty} \frac{1}{n^3} (\sin \frac{nx}{2}) e^{-\frac{n^2}{36}t}$$

Finally,

$$u(\pi,1) = \frac{32}{\pi} \sum_{n\ odd}^{\infty} \frac{1}{n^3} (\sin \frac{n\pi}{2}) e^{-\frac{n^2}{36}(1)}$$

$$= \frac{32}{\pi} \left(\frac{1}{1^3} \left(\sin \frac{\pi}{2} \right) e^{-\frac{(1)^2}{36}} + \frac{1}{3^3} \left(\sin \frac{3\pi}{2} \right) e^{-\frac{(3)^2}{36}} + \frac{1}{5^3} \left(\sin \frac{5\pi}{2} \right) e^{-\frac{(5)^2}{36}} + \cdots \right)$$

$$\approx \frac{32}{\pi} (0.9726 - 0.0288 + 0.004 \ldots)$$

$$\approx 9.6542°\text{C}$$

Example 5.14:

Solve the following heat equation by using method of separation of variables:

$$u_t = 4u_{xx} \quad , \quad 0 < x < \pi, \quad t > 0.$$

$$u(0,t) = u(\pi,t) = 0; \; u(x,0) = 2\sin 3x$$

Solution:

Let $u(x,t) = X(x)T(t)$, the heat equation becomes

$$XT' = 4X''T \tag{5.17}$$

Separate the variables, we have

$$\frac{X''}{X} = \frac{T'}{4T} = p \tag{5.18}$$

where constant p is known as separation constant.

From Eqn. (5.18),

$$\frac{X''}{X} = p \quad \text{and} \quad \frac{T'}{4T} = p \tag{5.19}$$

Hence X and T must satisfy

$$X'' - pX = 0 \ \text{ and } \ T' - 4pT = 0 \tag{5.20}$$

Since $u(x,t) = X(x)T(t)$ and $T(t) \neq 0$, the boundary conditions become

$$\begin{aligned} u(0,t) &= X(0)T(t) = 0 \Rightarrow X(0) = 0 \\ u(\pi,t) &= X(\pi)T(t) = 0 \Rightarrow X(\pi) = 0 \end{aligned} \tag{5.21}$$

Now, we need to consider three possible cases for p.

Case (1): $p = 0$.

From Eqn (5.20), (substitute $p = 0$ into $X'' - pX = 0$)

$$X'' = 0$$

$$X' = A$$

$$\therefore X(x) = Ax + B \tag{5.22}$$

By substituting boundary conditions from Eqn. (5.21) into Eqn. (5.22),

$X(0) = 0,\ X(0) = A(0) + B$

$$\therefore B = 0$$

$X(\pi) = 0,\ X(\pi) = A(\pi) + B$

$$0 = A(\pi) + 0$$

$$A = 0$$

From Eqn. (5.22), we found that $X(x) = 0$ and this implies

$$u(x,t) = X(x)T(t) = 0.$$

This case gives a trivial solution. Hence, we omit this case.

Case (2): $p > 0$.

Let $p = \lambda^2$ and $\lambda > 0$. Substitute $p = \lambda^2$ into Eqn (5.20) leads to ODE

$$X'' - \lambda^2 X = 0$$

Characteristic equation:

$$m^2 - \lambda^2 = 0$$

$$m = \pm\lambda$$

$$\therefore X(x) = Ae^{\lambda x} + Be^{-\lambda x} \qquad (5.23)$$

$X(0) = 0,\ X(0) = Ae^{\lambda(0)} + Be^{-\lambda(0)}$

$$\therefore 0 = A + B$$

$$A = -B$$

$X(\pi) = 0,\ 0 = Ae^{\pi\lambda} + Be^{-\pi\lambda}$

$$0 = -Be^{\pi\lambda} + Be^{-\pi\lambda}$$

$$0 = B(-e^{\pi\lambda} + e^{-\pi\lambda})$$

Since $-e^{\pi\lambda} + e^{-\pi\lambda} \neq 0$, $B = 0$ and then $A = 0$. From Eqn. (5.23),

$$X(x) = 0 \quad \Rightarrow \quad u(x,t) = X(x)T(t) = 0$$

This case gives a trivial solution. Hence, we omit this case.

Case (3): $p < 0$.

Let $p = -\lambda^2$ and $\lambda > 0$. Substitute $p = -\lambda^2$ into Eqn (5.20) leads to ODE

$$X'' + \lambda^2 X = 0 \text{ and } T' + 4\lambda^2 T = 0$$

Characteristic equation:

$$m^2 + \lambda^2 = 0 \quad \Rightarrow \quad m = \pm\lambda i$$

$$\therefore X(x) = A\cos\lambda x + B\sin\lambda x \tag{5.24}$$

$X(0) = 0,\ X(0) = A\cos 0 + B\sin 0,\quad X(\pi) = 0,\ X(\pi) = A\cos\pi\lambda + B\sin\pi\lambda$

$$\therefore 0 = A \qquad\qquad 0 = B\sin\pi\lambda$$

$$B \neq 0, \therefore \sin\pi\lambda = 0$$

Given $\sin n\pi = 0, n = 1,2,3,\ldots$
So, $\sin\pi\lambda = 0$ implies $\pi\lambda = n\pi$

$$\pi\lambda = n\pi,\ n = 1,2,3,\ldots$$

$$\lambda_n = n,\ n = 1,2,3,\ldots \tag{5.25}$$

$$\therefore X_n(x) = B_n\sin\lambda_n x$$

$$= B_n\sin nx\,,\quad n = 1,2,3,\ldots \tag{5.26}$$

$$T' + 4\lambda^2 T = 0$$

$$\frac{dT}{dt} = -4\lambda^2 T$$

$$\int \frac{1}{T}\,dT = \int -4\lambda^2\,dt$$

$$\ln T = -4\lambda^2 t + k$$

From Eqn. (5.25), $\lambda_n = n$

$$T = e^{-4\lambda^2 t + k}$$

Let $C = e^k$

$$= Ce^{-4\lambda^2 t}$$

$$T_n(t) = C_n e^{-4\lambda_n^2 t} = C_n e^{-4n^2 t}, \qquad n = 1,2,3,\ldots \qquad (5.27)$$

Hence,

$$u_n(x,t) = X_n(x)T_n(t)$$

$$= (B_n \sin nx)C_n e^{-4n^2 t}$$

Let $D_n = B_n C_n$

$$= D_n(\sin nx)e^{-4n^2 t},\ n = 1,2,3,\ldots \qquad (5.28)$$

If $u_1, u_2, \ldots, u_\infty$ are linearly independent solutions to a PDE, then the linear combination of all $u_1, u_2, \ldots, u_\infty$ are also a solution to the PDE

By applying superposition principle on Eqn. (5.28),

$$u(x,t) = \sum_{n=1}^{\infty} D_n (\sin nx) e^{-4n^2 t} \tag{5.29}$$

Apply initial condition, $u(x,0) = 2\sin 3x$, on Eqn. (5.29),

$$2\sin 3x = \sum_{n=1}^{\infty} D_n \sin nx \tag{5.30}$$

Recall from Fourier Sine Series:

$$f(x) = \sum_{n=1}^{\infty} B_n \sin \frac{n\pi x}{L}$$

where

$$B_n = \frac{2}{L}\int_0^L f(x) \sin \frac{n\pi x}{L}\, dx$$

From Eqn. (5.30), the half-range expansion for $f(x)$ on $0 < x < \pi$ is

$$D_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$$

$$= \frac{2}{\pi}\int_0^\pi 2\sin 3x \sin nx\ dx$$

$$\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$$

$$= \frac{2}{\pi}\int_0^\pi \cos(3x - nx) - \cos(3x + nx)\ dx$$

$$= \frac{2}{\pi}\left[\frac{\sin(3-n)x}{3-n} - \frac{\sin(3+n)x}{3+n}\right]_0^\pi, \qquad n \neq 3$$

$$= \frac{2}{\pi}\left[\left(\frac{\sin(3-n)\pi}{3-n} - \frac{\sin(3+n)\pi}{3+n}\right) - (0-0)\right], \qquad n \neq 3 \qquad (5.31)$$

$$= 0, \qquad n \neq 3$$

From Eqn. (5.31),

$$D_n = \frac{2}{\pi}\left[\frac{\sin(3-n)\,\pi}{3-n} - \frac{\sin(3+n)\pi}{3+n}\right], \qquad n \neq 3$$

When $n = 3$,

$$D_3 = \frac{2}{\pi}\left[\lim_{n\to 3}\frac{\sin(3\pi - n\pi)}{3-n} - \frac{\sin 6\pi}{6}\right]$$

$$= \frac{2}{\pi}\left[\lim_{n\to 3}\frac{-\pi\cos(3\pi - n\pi)}{-1} - 0\right]$$

$$= \frac{2}{\pi}(\pi)$$

$$D_3 = 2$$

Apply L'Hopital's Rule

Alternative Way to find D_n for this question:

From Eqn. (5.30),

$$2\sin 3x = \sum_{n=1}^{\infty} D_n \sin nx$$

By expanding the summation,

$$2\sin 3x = D_1 \sin x + D_2 \sin 2x + D_3 \sin 3x + D_4 \sin 4x + \cdots$$

By comparing coefficients of both sides,

$$D_3 = 2 \text{ and } D_1 = D_2 = D_4 = \cdots = 0$$

Hence, the only possible D_n for $u(x,t)$ is D_3 when $n = 3$.

By substituting D_n into Eqn. (5.29), the temperature function is given by

$$u(x,t) = \sum_{n=1}^{\infty} D_n (\sin nx) e^{-4n^2 t}$$

$$= D_3 (\sin 3x) e^{-36t}$$

$$= 2e^{-36t} \sin 3x$$

Exercise 5.8:

1) Solve the following heat equation by method of separation of variables.

$$5u_t = u_{xx} \quad , \quad 0 < x < 10, \quad t > 0.$$

$$u(0,t) = u(10,t) = 0; \; u(x,0) = \sin \pi x$$

[Ans: $(\sin \pi x)e^{-\frac{\pi^2 t}{5}}$]

2) Suppose that a rod of length 30 (cm) with $k = 0.09\text{cm}^2/s$ is heated and its initial temperature is $u(x,0) = 30$. At time $t = 0$, its lateral surface is insulated and its two ends are imbedded in ice at 0℃ $(u(0,t) = u(L,t) = 0)$. By method of separation of variables, calculate the rod's temperature at its midpoint after 20 minutes for the following case.

[Ans: $u(15,1200) = 11.6859$℃]

5.3.2 The Wave Equation

The wave equation usually describes mechanical waves, such as water wave, sound wave and seismic waves. Its application can be found in electromagnetics, fluid dynamics and acoustics.

Consider a string, stretched between two fixed points, is being initially plucked to the form of $f(x)$ to start vibrate and the string moves in a direction perpendicular to the x-axis as shown in Figure 5.2. The vertical displacement of a point at x at time t is represented by $u(x,t)$.

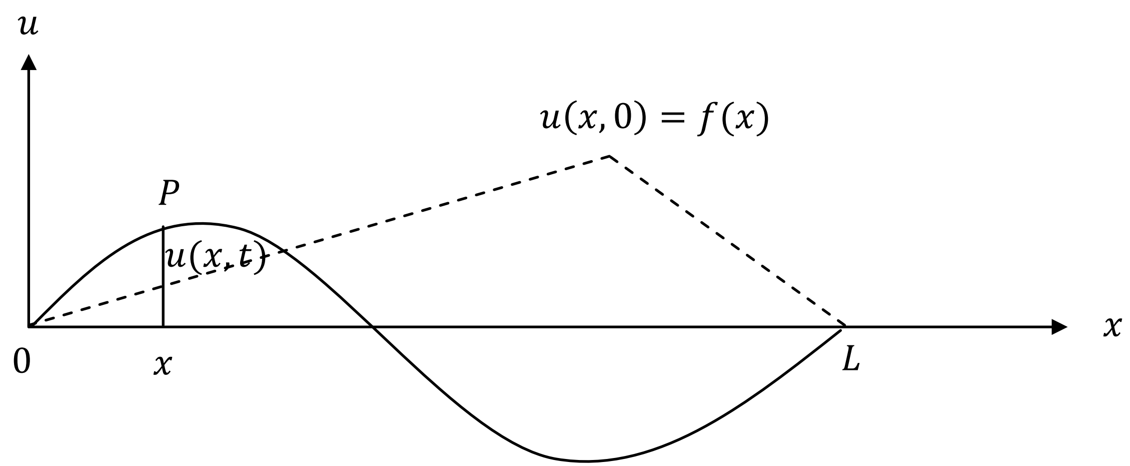


Figure 5.2: Displacement of vibrating string $u(x,t)$ at time t

Theorem: The vibrating string with an initial velocity

The partial differential equation

$$u_{tt} = c^2 u_{xx} \quad , \quad 0 < x < L, \quad t > 0$$

where c is the physical constant (a ratio of string's tension to its density) with boundary conditions

$$u(0,t) = u(L,t) = 0, \quad t > 0$$

and initial conditions

$$u(x,0) = f(x), \quad u_t(x,0) = g(x), \quad 0 < x < L$$

has the series solution

$$u(x,t) = \sum_{n=1}^{\infty} \left\{ a_n \cos\left(\frac{n\pi ct}{L}\right) + b_n \sin\left(\frac{n\pi ct}{L}\right) \right\} \sin\left(\frac{n\pi x}{L}\right) \qquad (5.32)$$

where a_n and b_n are the Fourier series coefficients obtained by

$$a_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx, \qquad n = 1,2,3,\ldots$$

$$b_n = \frac{2}{n\pi c}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)dx, \qquad n = 1,2,3,\ldots \qquad (5.33)$$

Example 5.15:

Solve the following wave equation by using the method of separation of variables.

$$u_{tt} = 4u_{xx} \quad , \quad 0 < x < 2\pi, \quad t > 0.$$

$$u(0,t) = u(2\pi,t) = 0; \qquad u(x,0) = 3\sin x\,, \qquad u_t(x,0) = 1.$$

Next, by using the first three non-zero terms of the series, find the displacement of spring at the point $x = \pi$ at $t = 10$.

Solution:

Let $u(x,t) = X(x)T(t)$, the wave equation becomes

$$XT'' = 4X''T \tag{5.34}$$

Separate the variables, we have

$$\frac{X''}{X} = \frac{T''}{4T} = p \tag{5.35}$$

where constant p is known as separation constant.

From Eqn. (5.35),

$$\frac{X''}{X} = p \quad \text{and} \quad \frac{T''}{4T} = p \tag{5.36}$$

Hence X and T must satisfy

$$X'' - pX = 0 \ \text{ and } \ T'' - 4pT = 0 \tag{5.37}$$

Since $u(x,t) = X(x)T(t)$ and $T(t) \neq 0$, the boundary conditions become

$$\begin{aligned} u(0,t) &= X(0)T(t) = 0 \Rightarrow X(0) = 0 \\ u(2\pi,t) &= X(2\pi)T(t) = 0 \Rightarrow X(2\pi) = 0 \end{aligned} \tag{5.38}$$

Now, we need to consider three possible cases for p.

Case (1): $p = 0.$

From Eqn (5.37), (substitute $p = 0$ into $X'' - pX = 0$)

$$X'' = 0$$

$$X' = A$$

$$\therefore X(x) = Ax + B \qquad (5.39)$$

By substituting boundary conditions (5.38) into Eqn. (5.39),

$$X(0) = 0,\ \ X(0) = A(0) + B$$

$$\therefore B = 0$$

$$X(2\pi) = 0,\ \ X(2\pi) = A(2\pi) + B$$

$$0 = A(2\pi) + 0$$

$$A = 0$$

From Eqn. (5.39), we found that $X(x) = 0$ and this implies

$$u(x,t) = X(x)T(t) = 0.$$

This case gives a trivial solution. Hence, we omit this case.

Case (2): $p > 0$.

Let $p = \lambda^2$ and $\lambda > 0$. Substitute $p = \lambda^2$ into Eqn (5.37) leads to ODE

$$X'' - \lambda^2 X = 0$$

Characteristic equation:

$$m^2 - \lambda^2 = 0$$

$$m = \pm\lambda$$

$$\therefore X(x) = Ae^{\lambda x} + Be^{-\lambda x} \qquad (5.40)$$

$X(0) = 0,\ X(0) = Ae^{\lambda(0)} + Be^{-\lambda(0)}$

$$\therefore 0 = A + B$$

$$A = -B$$

$X(2\pi) = 0,\ 0 = Ae^{2\pi\lambda} + Be^{-2\pi\lambda}$

$$0 = -Be^{2\pi\lambda} + Be^{-2\pi\lambda}$$

$$0 = B(-e^{2\pi\lambda} + e^{-2\pi\lambda})$$

Since $-e^{2\pi\lambda} + e^{-2\pi\lambda} \neq 0$, $B = 0$ and then $A = 0$. From Eqn. (5.40),

$$X(x) = 0 \quad \Rightarrow \quad u(x,t) = X(x)T(t) = 0$$

This case gives a trivial solution. Hence, we omit this case.

Case (3): $p < 0$.

Let $p = -\lambda^2$ and $\lambda > 0$. From Eqn (5.37), it leads to ODE

$$X'' + \lambda^2 X = 0 \quad \text{and} \quad T'' + 4\lambda^2 T = 0$$

Characteristic equation for $X'' + \lambda^2 X = 0$:

$$m^2 + \lambda^2 = 0 \quad \Rightarrow \quad m = \pm\lambda i$$

$$\therefore X(x) = A\cos\lambda x + B\sin\lambda x \tag{5.41}$$

$X(0) = 0,\ X(0) = A\cos 0 + B\sin 0,\quad X(2\pi) = 0,\ X(2\pi) = A\cos 2\pi\lambda + B\sin 2\pi\lambda$

$$\therefore 0 = A \qquad\qquad 0 = B\sin 2\pi\lambda$$

$$B \neq 0,\ \therefore \sin 2\pi\lambda = 0$$

Given $\sin n\pi = 0, n = 1,2,3,\ldots$
So, $\sin 2\pi\lambda = 0$ implies $2\pi\lambda = n\pi$

$$2\pi\lambda = n\pi,\ \ n = 1,2,3,\ldots$$

$$\lambda_n = \frac{n}{2},\ \ n = 1,2,3,\ldots \tag{5.42}$$

$$\therefore X_n(x) = B_n \sin\lambda_n x$$

$$= B_n \sin\frac{nx}{2},\quad n = 1,2,3,\ldots \tag{5.43}$$

Characteristic equation for $T'' + 4\lambda^2 T = 0$:

$$m^2 + 4\lambda^2 = 0$$

$$m = \pm 2\lambda i$$

$$\therefore T(t) = C\cos 2\lambda t + D\sin 2\lambda t$$

Since $\lambda_n = \frac{n}{2}$ from Eqn. (5.42),

$$\therefore T_n(t) = C_n\cos nt + D_n\sin nt\,, \qquad n = 1,2,3,\ldots \tag{5.44}$$

Hence,

$$\begin{aligned} u_n(x,t) &= X_n(x)T_n(t) \\ &= \left(B_n\sin\frac{nx}{2}\right)(C_n\cos nt + D_n\sin nt) \\ &= (E_n\cos nt + F_n\sin nt)\sin\frac{nx}{2}, \qquad n = 1,2,3,\ldots \end{aligned} \tag{5.45}$$

Let $E_n = B_nC_n$ and $F_n = B_nD_n$

By superposition principle:

$$u(x,t) = \sum_{n=1}^{\infty}(E_n\cos nt + F_n\sin nt)\sin\frac{nx}{2} \tag{5.46}$$

Apply initial condition $u(x,0) = 3\sin x$ to the Eqn. (5.46)

$$3\sin x = \sum_{n=1}^{\infty} E_n \sin\frac{nx}{2} \tag{5.47}$$

By applying Fourier Sine series to Eqn. (5.47):

$$E_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$$

$$= \frac{2}{2\pi}\int_0^{2\pi} 3\sin x \sin\frac{nx}{2}\,dx$$

$$\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$$

$$= \frac{3}{2\pi}\int_0^{2\pi}\left[\cos\left(\frac{2-n}{2}\right)x - \cos\left(\frac{2+n}{2}\right)x\right]dx$$

$$= \frac{3}{2\pi}\left[\frac{2}{2-n}\sin\left(\frac{2-n}{2}\right)x - \frac{2}{2+n}\sin\left(\frac{2+n}{2}\right)x\right]_0^{2\pi}, \qquad n \neq 2$$

$$= \frac{3}{2\pi}\left[\left(\frac{2}{2-n}\sin(2-n)\pi - \frac{2}{2+n}\sin(2+n)\pi\right) - 0\right], \qquad n \neq 2 \tag{5.48}$$

$$= 0, \;\; n \neq 2$$

From Eqn. (5.48),

$$E_n = \frac{3}{2\pi}\left[\frac{2}{2-n}\sin(2-n)\pi - \frac{2}{2+n}\sin(2+n)\pi\right], \qquad n \neq 2$$

When $n = 2$,

$$E_2 = \frac{3}{\pi}\left[\lim_{n\to 2}\frac{\sin(2-n)\pi}{2-n} - \frac{\sin 4\pi}{4}\right]$$

$$= \frac{3}{\pi}\left[\lim_{n\to 2}\frac{-\pi\cos(2-n)\pi}{-1} - 0\right]$$

$$= \frac{3}{\pi}(\pi)$$

$$E_2 = 3$$

Apply L'Hopital's Rule

Alternative Way to find E_n for this question:

From Eqn. (5.47),

$$3\sin x = \sum_{n=1}^{\infty} E_n \sin\frac{nx}{2}$$

By expanding the summation,

$$3\sin x = E_1 \sin\frac{x}{2} + E_2 \sin x + E_3 \sin\frac{3x}{2} + E_4 \sin 2x + \cdots$$

By comparing coefficients of both sides,

$$E_2 = 3 \text{ and } E_1 = E_3 = E_4 = \cdots = 0$$

Hence, the only possible E_n for $u(x,t)$ is E_2 when $n = 2$.

Differentiate Eqn. (5.46) with respect to t,

$$u_t(x,t) = \sum_{n=1}^{\infty} (-nE_n \sin nt + nF_n \cos nt) \sin \frac{nx}{2} \qquad (5.49)$$

Apply initial velocity, $u_t(x,0) = 1$ into Eqn. (5.49):

$$1 = \sum_{n=1}^{\infty} nF_n \sin \frac{nx}{2}$$

By applying Fourier sine series:

$$nF_n = \frac{2}{L}\int_0^L f(x) \sin \frac{n\pi x}{L}\, dx = \frac{2}{2\pi}\int_0^{2\pi} \sin \frac{nx}{2}\, dx$$

$$= \frac{1}{\pi}\left[-\frac{2\cos\frac{nx}{2}}{n}\right]_0^{2\pi}$$

$$= \frac{2}{n\pi}[-\cos n\pi + 1] = \frac{2}{n\pi}[1-(-1)^n]$$

$$\therefore F_n = \frac{2}{n^2\pi}[1-(-1)^n], \qquad n = 1,2,3,\ldots \qquad (5.50)$$

Finally, substitute E_n and F_n into Eqn. (5.46),

$$u(x,t) = \sum_{n=1}^{\infty} (E_n \cos nt + F_n \sin nt) \sin \frac{nx}{2}$$

$$= \sum_{n=1}^{\infty} (E_n \cos nt) \sin \frac{nx}{2} + \sum_{n=1}^{\infty} (F_n \sin nt) \sin \frac{nx}{2}$$

$$= E_2 \cos 2t \sin x + \sum_{n=1}^{\infty} \left[\frac{2}{n^2 \pi} [1 - (-1)^n] \right] (\sin nt) \sin \frac{nx}{2}$$

$$= 3 \cos 2t \sin x + \frac{4}{\pi} \sum_{n\ odd}^{\infty} \frac{1}{n^2} (\sin nt) \sin \frac{nx}{2}$$

Next,

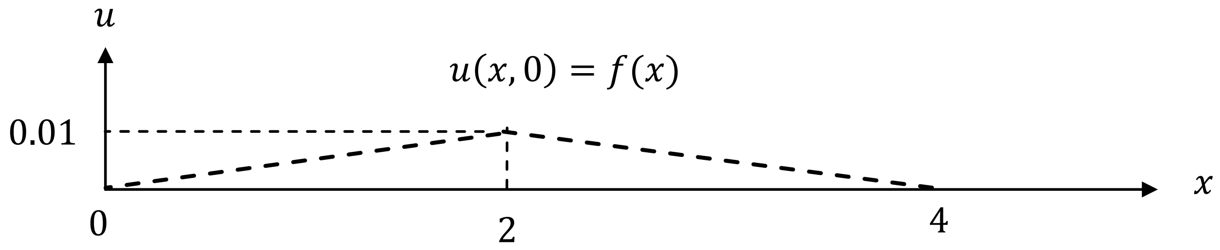
$$u(\pi, 10) = 3 \cos 20 \sin \pi + \frac{4}{\pi} \sum_{n\ odd}^{\infty} \frac{1}{n^2} (\sin 10n) \sin \frac{n\pi}{2}$$

$$= 0 + \frac{4}{\pi} \left(\frac{1}{1^2} \sin 10 \sin \frac{\pi}{2} + \frac{1}{3^2} \sin 30 \sin \frac{3\pi}{2} + \frac{1}{5^2} \sin 50 \sin \frac{5\pi}{2} \right)$$

$$\approx -0.5663$$

Example 5.16:

A string is stretched between two points and plucked at $t = 0$ as follows:



Length in meter

If the string is being released initially with zero velocity, apply the wave equation

$$\frac{\partial^2 u}{\partial t^2} = 100\frac{\partial^2 u}{\partial x^2}$$

to determine the subsequent motion of a point *P* with distance x from the origin at time t.

Solution:

From the figure, we need to find the $f(x)$ for the intervals of $0 \leq x \leq 2$ and $2 \leq x \leq 4$.

For the interval $0 \leq x \leq 2$, the line between $(0,0)$ and $(2,0.01)$ is

$$y - y_0 = \frac{y_1 - y_0}{x_1 - x_0}(x - x_0)$$

$$y - 0 = \frac{0.01 - 0}{2 - 0}(x - 0)$$

$$y = 0.005x$$

For the interval $2 \leq x \leq 4$, the line between $(2,0.01)$ and $(4,0)$ is

$$y - 0 = \frac{0 - 0.01}{4 - 2}(x - 4)$$

$$y = 0.02 - 0.005x$$

Hence,

$$f(x) = \begin{cases} 0.005x, & 0 \leq x \leq 2 \\ 0.02 - 0.005x, & 2 \leq x \leq 4 \end{cases}$$

The string is being released initially with zero velocity,

$$u_t(x,0) = 0$$

Hence, the model of the wave equation is as follow:

$$u_{tt} = 100u_{xx} \quad , \quad 0 < x < 4, \quad t > 0.$$
$$u(0,t) = u(4,t) = 0, \qquad u_t(x,0) = 0,$$
$$u(x,0) = f(x) = \begin{cases} 0.005x, & 0 \le x \le 2 \\ 0.02 - 0.005x, & 2 \le x \le 4 \end{cases}. \tag{5.51}$$

Apply the method of separation of variables:

Let $u(x,t) = X(x)T(t)$, the wave equation becomes

$$XT'' = 100X''T \tag{5.52}$$

Separate the variables, we have

$$\frac{X''}{X} = \frac{T''}{100T} = p$$

Hence X and T must satisfy

$$X'' - pX = 0 \text{ and } T'' - 100pT = 0 \qquad (5.53)$$

Since $u(x,t) = X(x)T(t)$ and $T(t) \neq 0$, the boundary conditions become

$$u(0,t) = X(0)T(t) = 0 \Rightarrow X(0) = 0$$
$$u(4,t) = X(4)T(t) = 0 \Rightarrow X(4) = 0$$

Now, we need to consider three possible cases for p.

The first two cases are similar to the previous example where they produce trivial solution. I leave it to you to work out the first two cases.

We will move to the third case directly.

Case (3): $p < 0$.

Let $p = -\lambda^2$ and $\lambda > 0$. Substitute $p = -\lambda^2$ into Eqn (5.53) leads to ODE

$$X'' + \lambda^2 X = 0 \qquad \text{and} \qquad T'' + 100\lambda^2 T = 0$$

Characteristic equation for $X'' + \lambda^2 X = 0$:

$$m^2 + \lambda^2 = 0 \quad \Rightarrow \quad m = \pm\lambda i$$

$$\therefore X(x) = A\cos\lambda x + B\sin\lambda x \tag{5.54}$$

$X(0) = 0,\ X(0) = A\cos 0 + B\sin 0, \quad X(4) = 0,\ X(4) = A\cos 4\lambda + B\sin 4\lambda$

$$\therefore 0 = A \qquad\qquad 0 = B\sin 4\lambda$$

$$B \neq 0,\ \therefore \sin 4\lambda = 0$$

Given $\sin n\pi = 0, n = 1,2,3,\ldots$
So, $\sin 4\lambda = 0$ implies $4\lambda = n\pi$

$$4\lambda = n\pi,\ n = 1,2,3,\ldots$$

$$\lambda_n = \frac{n\pi}{4},\ n = 1,2,3,\ldots \tag{5.55}$$

$$\therefore X_n(x) = B_n \sin\lambda_n x$$

$$= B_n \sin\frac{n\pi x}{4}, \quad n = 1,2,3,\ldots \tag{5.56}$$

Characteristic equation for $T'' + 100\lambda^2 T = 0$:

$$m^2 + 100\lambda^2 = 0$$

$$m = \pm 10\lambda i$$

$$\therefore T(t) = C\cos 10\lambda t + D\sin 10\lambda t$$

Since $\lambda_n = \frac{n\pi}{4}$ from Eqn. (5.55),

$$\therefore T_n(t) = C_n \cos\frac{5n\pi t}{2} + D_n \sin\frac{5n\pi t}{2}, \qquad n = 1,2,3,\ldots \tag{5.57}$$

Hence,

$$u_n(x,t) = X_n(x)T_n(t)$$

$$= \left(B_n \sin\frac{n\pi x}{4}\right)\left(C_n\cos\frac{5n\pi t}{2} + D_n\sin\frac{5n\pi t}{2}\right)$$

$$= \left(E_n\cos\frac{5n\pi t}{2} + F_n\sin\frac{5n\pi t}{2}\right)\sin\frac{n\pi x}{4} \tag{5.58}$$

Let $E_n = B_nC_n$ and $F_n = B_nD_n$

By superposition principle:

$$u(x,t)=\sum_{n=1}^{\infty}\left(E_n\cos\frac{5n\pi t}{2}+F_n\sin\frac{5n\pi t}{2}\right)\sin\frac{n\pi x}{4} \tag{5.59}$$

Apply initial condition $u(x,0)=f(x)=\begin{cases}0.005x, & 0\le x\le 2\\ 0.02-0.005x, & 2\le x\le 4\end{cases}$ to the Eqn. (5.59),

$$f(x)=\sum_{n=1}^{\infty}E_n\sin\frac{n\pi x}{4} \tag{5.60}$$

By applying Fourier Sine series to Eqn. (5.60):

$$E_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$$

$$= \frac{2}{4}\left[\int_0^2 0.005x\sin\frac{n\pi x}{4}\,dx + \int_2^4 (0.02-0.005x)\sin\frac{n\pi x}{4}\,dx\right]$$

Tabular Method:

	Diff.	Integrate
$+$	$0.005x$	$\sin\frac{n\pi x}{4}$
$-$	0.005	$-\frac{4}{n\pi}\cos\frac{n\pi x}{4}$
$+$	0	$-\frac{16}{n^2\pi^2}\sin\frac{n\pi x}{4}$

Tabular Method:

	Diff.	Integrate
$+$	$0.02-0.005x$	$\sin\frac{n\pi x}{4}$
$-$	-0.005	$-\frac{4}{n\pi}\cos\frac{n\pi x}{4}$
$+$	0	$-\frac{16}{n^2\pi^2}\sin\frac{n\pi x}{4}$

$$E_n = \frac{1}{2}\left[-\frac{0.02x}{n\pi}\cos\frac{n\pi x}{4} + \frac{0.08}{n^2\pi^2}\sin\frac{n\pi x}{4}\Big|_0^2 + \frac{4(0.005x-0.02)}{n\pi}\cos\frac{n\pi x}{4} - \frac{0.08}{n^2\pi^2}\sin\frac{n\pi x}{4}\Big|_2^4\right]$$

$$E_n = \frac{1}{2}\left[-\frac{0.02x}{n\pi}\cos\frac{n\pi x}{4} + \frac{0.08}{n^2\pi^2}\sin\frac{n\pi x}{4}\Big|_0^2 + \frac{4(0.005x - 0.02)}{n\pi}\cos\frac{n\pi x}{4} - \frac{0.08}{n^2\pi^2}\sin\frac{n\pi x}{4}\Big|_2^4\right]$$

$$= \frac{1}{2}\left[-\frac{0.04}{n\pi}\cos\frac{n\pi}{2} + \frac{0.08}{n^2\pi^2}\sin\frac{n\pi}{2} + \left(0 - \left(-\frac{0.04}{n\pi}\cos\frac{n\pi}{2} - \frac{0.08}{n^2\pi^2}\sin\frac{n\pi}{2}\right)\right)\right]$$

$$= \frac{1}{2}\left[\frac{0.16}{n^2\pi^2}\sin\frac{n\pi}{2}\right]$$

$$= \frac{0.08}{n^2\pi^2}\sin\frac{n\pi}{2}$$

Differentiate Eqn. (5.59) with respect to t,

$$u_t(x,t) = \sum_{n=1}^{\infty}\left(-\frac{5n\pi}{2}E_n \sin\frac{5n\pi t}{2} + \frac{5n\pi}{2}F_n \cos\frac{5n\pi t}{2}\right)\sin\frac{n\pi x}{4} \qquad (5.61)$$

Apply initial velocity, $u_t(x,0) = 0$ into Eqn. (5.61):

$$0 = \sum_{n=1}^{\infty}\frac{5n\pi}{2}F_n \sin\frac{n\pi x}{4}$$

By applying Fourier sine series:

$$\frac{5n\pi}{2}F_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$$

$$= \frac{2}{4}\int_0^4 (0)\sin\frac{n\pi x}{4}\,dx$$

$$= 0$$

$$\therefore F_n = 0$$

Finally, substitute E_n and F_n into Eqn. (5.59),

$$u(x,t) = \sum_{n=1}^{\infty} \left(E_n \cos\frac{5n\pi t}{2} + F_n \sin\frac{5n\pi t}{2} \right) \sin\frac{n\pi x}{4}$$

$$= \sum_{n=1}^{\infty} \left(\frac{0.08}{n^2\pi^2} \sin\frac{n\pi}{2} \right) \left(\cos\frac{5n\pi t}{2} \right) \sin\frac{n\pi x}{4}$$

Exercise 5.9:

1) Solve the following wave equation by using method of separation of variables:

$$u_{tt} = u_{xx} \quad , \quad 0 < x < \pi, \quad t > 0.$$

$$u(0,t) = u(\pi,t) = 0; \quad u(x,0) = 4\sin x - 3\sin 2x\,; \quad u_t(x,0) = 0.5.$$

[Ans: $u(x,t) = 4\cos t \sin x - 3\cos 2t \sin 2x + \frac{2}{\pi}\sum_{n\ odd}^{\infty}\frac{1}{n^2}(\sin nt)\sin nx$]

2) A stretched string of length 20cm is set oscillating by displacing its midpoint a distance 1cm from its rest position and releasing it with zero initial velocity. Solve the wave equation $u_{tt} = u_{xx}$ to determine the resulting motion, $u(x,t)$.

[Ans: $u(x,t) = \frac{8}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}(\sin\frac{n\pi}{2})\left(\cos\frac{n\pi t}{20}\right)\sin\frac{n\pi x}{20}$]

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Thank You

Questions & Answer?