ENGINEERING MATHEMATICS 1
BMFG 1313
COMPLEX NUMBER
- Loci in the Complex Plane
- Function of a Complex Variable

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Learning Outcomes

Upon completion of this lesson, the student should be able to:

1. Use complex numbers to represent a locus of points in the Argand diagram.
2. Apply transformations from the z-plane to the w-plane.

4.3 Loci in the Complex Plane

A locus (plural loci) is a set of points where their locations satisfy one or more specified properties.

Example:

1. A circle is the locus of a set of points in a plane where they all have a fixed distance (its radius) from a fixed point (its centre).

2. A straight line is also can be described by the locus.

The properties may be defined in sentences or algebraically.

4.3.1 Straight lines

A straight line can be presented in the form of complex numbers in many ways.

We will illustrate straight lines with few examples.

Example:

Describe and sketch the locus of z given that

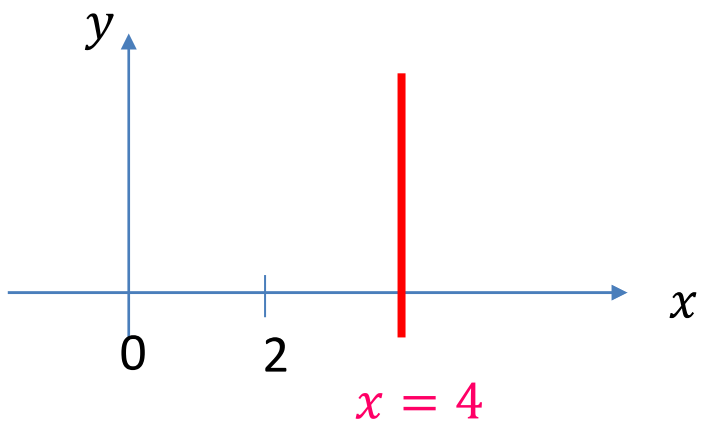
(a) $\text{Re}(z) = 4$

(b) $\text{Re}(z) = -3$

4.3.1 Straight lines

Solution:

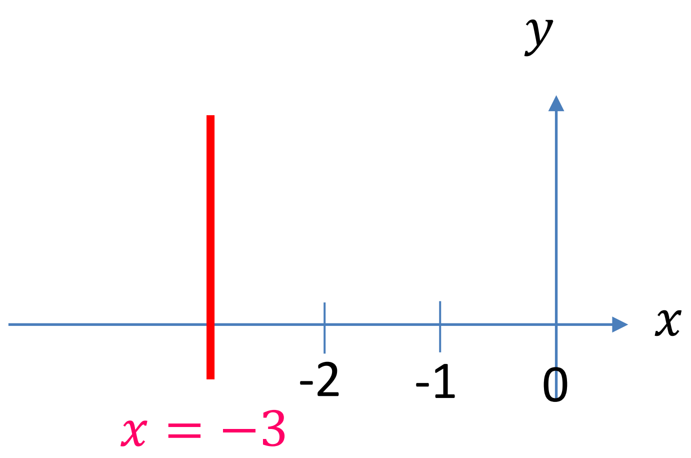
(a) For $\mathrm{Re}(z) = 4$, we will have $z = 4 + \mathrm{j}b$ for any real b. Thus the locus is the vertical straight line and the equation is given by $x = 4$.



4.3.1 Straight lines

Solution:

(b) For $\text{Re}(z) = -3$, we will have $z = -3 + jb$ for any real b. Thus the locus is the vertical straight line on the left side and the equation is given by $x = -3$.



4.3.1 Straight lines

Example:

Describe the locus of z given by

$$\left|\frac{z-\mathrm{j}2}{z+1}\right|=1$$

4.3.1 Straight lines

Solution:

From $\left|\frac{z-\text{j}2}{z+1}\right| = 1$, we have $|z - \text{j}2| = |z + 1|$.

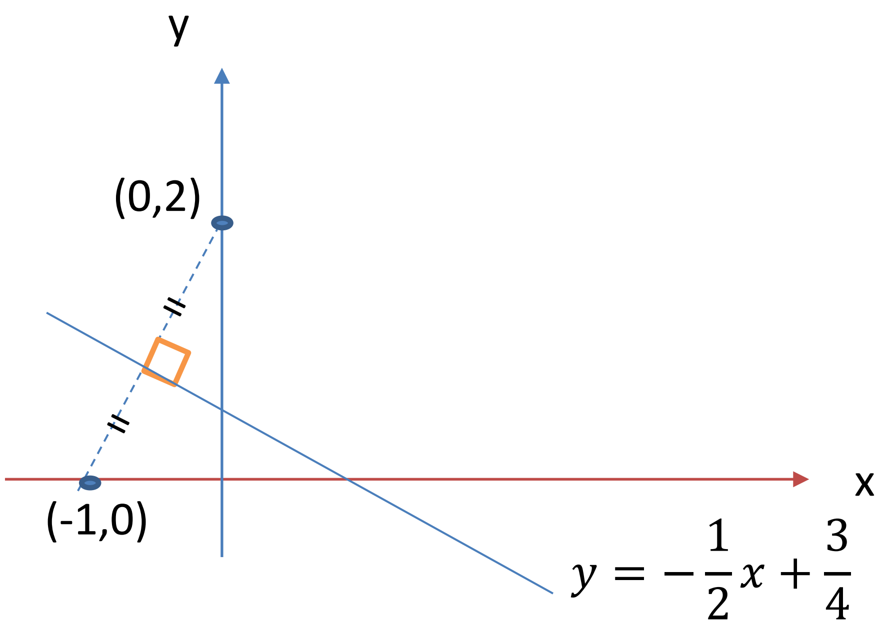
We know that $z = x + \text{j}y$, thus

$$|x + \text{j}y - \text{j}2| = |x + \text{j}y + 1|$$

Using the definition of modulus, we can rewrite as

$$\sqrt{x^2 + (y-2)^2} = \sqrt{(x+1)^2 + y^2}$$
$$x^2 + y^2 - 4y + 4 = x^2 + 2x + 1 + y^2$$
$$4y + 2x - 3 = 0$$
$$y = -\frac{1}{2}x + \frac{3}{4}$$

This equation describes a straight line with negative slope.



The locus of z is the perpendicular bisector of the line segment joining the points (0,2) and (-1,0)

4.3.1 Straight lines

Example:

If $|z| = |z - \mathrm{j}4|$,

(a) sketch the locus of $Q(x, y)$ which represented by *z* on an Argand diagram

(b) Find the Cartesian equation of this locus by using an algebraic method.

4.3.1 Straight lines

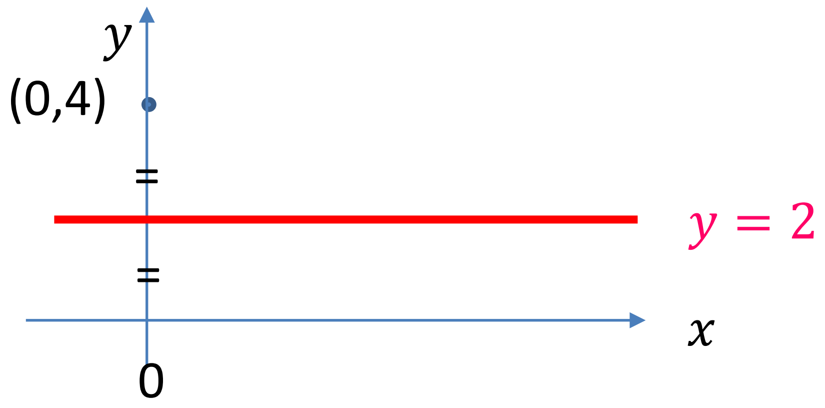
Solution:

(a) $|z|$ represents the distance from the origin (0,0) to Q.

$|z - \mathrm{j}4|$ represents the distance from the point (0,4) to Q.

As $|z| = |z - \mathrm{j}4|$, then Q is the locus of points which have the same distance between the points (0,0) and (0,4).

Thus the locus of Q is the perpendicular bisector of the line joining the points (0,0) and (0,4). The equation is given by $y = 2$.



4.3.1 Straight lines

Solution:

(b) $|z| = |z - \text{j}4|$

Let $z = x + \text{j}y$. Thus we have

$$|x + \text{j}y| = |x + \text{j}y - \text{j}4|$$

$$\sqrt{x^2 + y^2} = \sqrt{x^2 + (y - 4)^2}$$

$$x^2 + y^2 = x^2 + y^2 - 8y + 16$$

$$8y = 16$$

Thus, the Cartesian equation of the locus Q is $y = 2$.

4.3.1 Straight lines

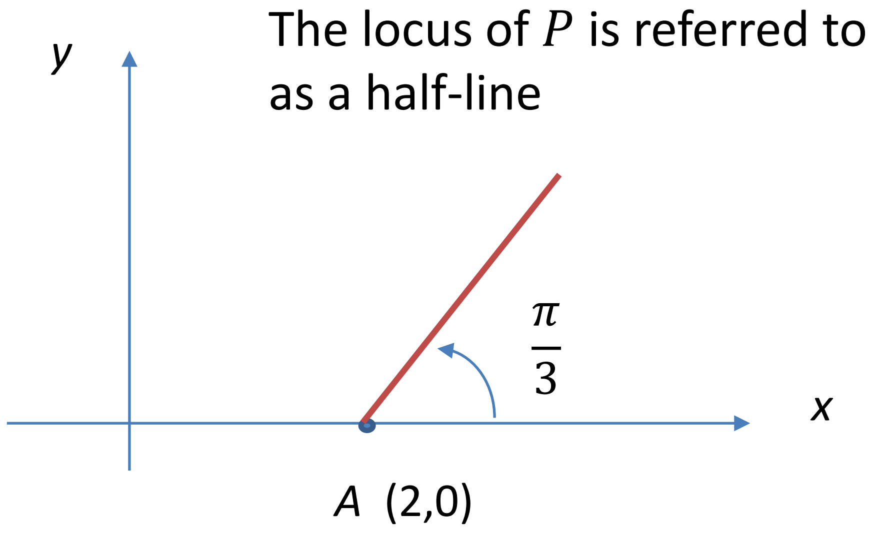
Example:

Given $\arg(z-2)=\frac{\pi}{3}$. Sketch the locus of P represented by z on an Argand diagram. Then, find the Cartesian equation of the locus using the algebraic method.

4.3.1 Straight lines

Solution:

From $\arg(z-2)=\frac{\pi}{3}$, we have $z=2$ and this is a point at (2,0) in Argand diagram. Thus The locus of P is referred to as a half-line positive slope from $(2,0)$ making an angle of $\frac{\pi}{3}$ in an anti-clockwise sense.



4.3.1 Straight lines

To find the Cartesian equation,

$$\arg(z-2)=\frac{\pi}{3}$$

$$\arg(x+\mathrm{j}y-2)=\frac{\pi}{3}$$

$$\frac{y}{x-2}=\tan\left(\frac{\pi}{3}\right)=\sqrt{3}$$

$$y=\sqrt{3}(x-2)$$

Thus, the Cartesian equation of the locus P is

$$y=\sqrt{3}x-2\sqrt{3}$$

Exercise 4.3

1. Describe the locus of z given by

   (a) $\mathrm{Re}(z) = -1$

   (b) $|z| = |z - \mathrm{j}6|$

2. Sketch the locus of z and give the Cartesian equation of the locus of z for:

   (a) $\frac{|z+3|}{|z-5|} = 1$

   (b) $|z - \mathrm{j}3| = |z + 2|$

   (c) $|z - 3| = |z + \mathrm{j}|$

Exercise 4.3

3. If $\arg(z + 3 + j2) = \frac{3\pi}{4}$, sketch the locus on an Argand diagram. Find the Cartesian equation of this locus.

[Ans: Straight line vertically at $x = -1$, Straight line horizontally at $y = 3$;
$x = 1, y = -\frac{2}{3}x + \frac{5}{6}, y = -3x + 4$;
half-line from $(-3, -2)$ making an angle of $\frac{3\pi}{4}$ in an anti-clockwise sense from a line in the same direction as the positive *x*-axis, $y = -x - 5$]

4.3.2 Circles

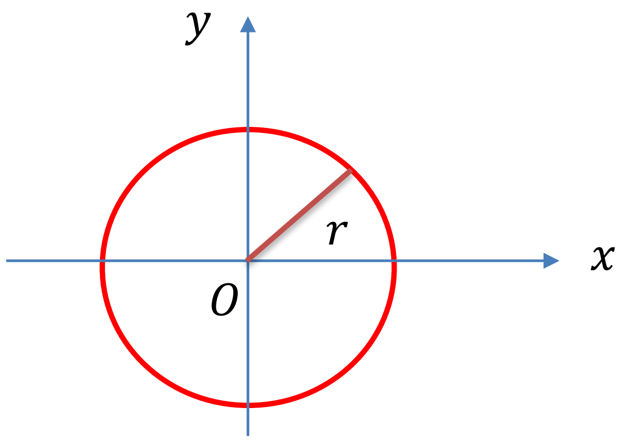
The locus of points which satisfy $|z| = r$ is given as a circle where the centre is at the origin with radius r. This can be proven as below:

Suppose $z = x + \mathrm{j}y$. Thus $|x + \mathrm{j}y| = \sqrt{x^2 + y^2} = r$.

Squaring both sides we will have

$$x^2 + y^2 = r^2.$$

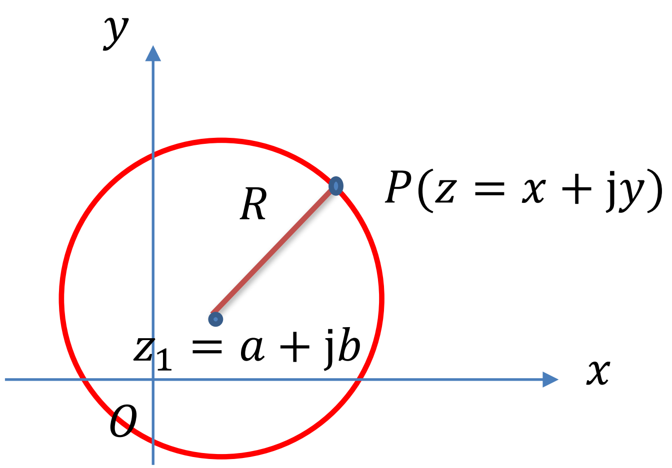
This is the equation of a circle which the centre is at the origin with radius r.



4.3.2 Circles

A circle on the Argand diagram reflects that $|z - z_1|$ is the distance between the point $z = x + \mathrm{j}y$ and the point $z_1 = a + \mathrm{j}b$. Hence a circle of radius R, centred at (a, b) is written as :

$$|z - z_1| = R$$



4.3.2 Circles

Example:

Given $|z-5-j2|=2$.

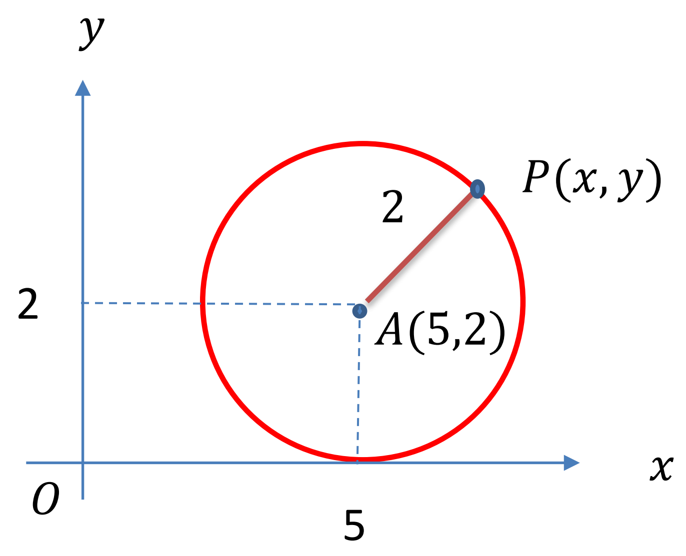
(a) Sketch the locus of $P(x,y)$ which is represented by z on an Argand diagram.

(b) Find the Cartesian equation of this locus by using an algebraic method.

4.3.2 Circles

Solution:

(a) From $|z - 5 - j2| = 2$, we can rewrite as $|z - (5+j2)| = 2$ and this represents the distance between the fixed point $A(5,2)$ and the variable point $P(x, y)$ where the distance is always equal to 2.



4.3.2 Circles

Solution:

(b) By substituting $z = x + \mathrm{j}y$ we have

$$|z - 5 - \mathrm{j}2| = 2$$

$$|x + \mathrm{j}y - 5 - \mathrm{j}2| = 2$$

Applying the definition of modulus, we then have

$$\sqrt{(x-5)^2 + (y-2)^2} = 2$$

Squaring both sides to get

$$(x-5)^2 + (y-2)^2 = 4$$

Hence, the equation is given as

$$x^2 - 10x + 25 + y^2 - 4y + 4 = 4$$

$$x^2 + y^2 - 10x - 4y + 25 = 0$$

4.3.2 Circles

Example:

Find the Cartesian equation of the circle

$$|z-(1+j2)|=2$$

Solution:

By substituting $z = x + jy$ we have $|x + jy - 1 - j2| = 2$.
Applying the definition of modulus, we then have

$$\sqrt{(x-1)^2+(y-2)^2}=2$$

Squaring both sides of the equation and hence the equation is given as

$$x^2-2x+1+y^2-4y+4=4$$
$$x^2+y^2-2x-4y+1=0$$

Exercise 4.4:

Give a geometrical interpretation of the locus of points z represented by:

1. $|z - \mathrm{j}| = 2$
2. $|z - \mathrm{j}4| = 2$
3. $|z - 2 - \mathrm{j}3| = 5$
4. $|2 - \mathrm{j}5 - z| = 3$

[Ans: Circle centered at (0,1) with radius 2; Circle centered at (0,4) with radius 2;
Circle centered at (2,3) with radius 5; Circle centered at (2,-5) with radius 3]

4.4 Functions of Complex Number

A complex-valued function *f* of the complex variable *z* is a kind of mapping of each complex number *z* in a set *D* to one and only one complex number *w*.

We write $w = f(z)$ which means w *is* the image of *z* under the function *f*. Thus, the set *D* is known as the domain of *f* while the set of all images is known as the range of *f*.

As discussed before, *z* can be expressed by $z = x + \mathrm{j}y$. Hence we write $f(z) = w = u + \mathrm{j}v$, where u and v are the real and imaginary parts of w, respectively. Therefore we will have

$$w = f(z) = f(x, y) = f(x + \mathrm{j}y) = u + \mathrm{j}v$$

4.4 Functions of Complex Number

Since u and v depend on x and y, respectively, we can write them as:

$$u = u(x, y) \text{ and } v = v(x, y).$$

Combining these ideas, we will have a complex-valued function f in the form

$$f(z) = f(x + \mathrm{j}y) = u(x, y) + \mathrm{j}v(x, y)$$

4.4 Functions of Complex Number

Example:

Write $f(z) = z^2$ in the form $f(z) = u(x, y) + \mathrm{j}v(x, y)$.

Solution:

$$\begin{aligned} f(z) = z^2 &= (x + \mathrm{j}y)^2 \\ &= x^2 + 2\mathrm{j}xy - y^2 \\ &= (x^2 - y^2) + \mathrm{j}2xy \end{aligned}$$

4.4 Functions of Complex Number

Example:

Express u and v in terms of x and y where

$$w = u + \mathrm{j}v, z = x + \mathrm{j}y, w = f(z) \text{ and } f(z) = \frac{z - \mathrm{j}2}{z+1}, z \neq -1.$$

4.4 Functions of Complex Number

Solution:

$$f(z) = \frac{z - j2}{z + 1} = \frac{x + jy - j2}{x + jy + 1}$$

$$= \frac{x + j(y-2)}{(x+1) + jy} \times \frac{(x+1) - jy}{(x+1) - jy}$$

$$= \frac{x(x+1) - jxy + j(y-2)(x+1) + y(y-2)}{(x+1)^2 + y^2}$$

Thus, we have

$$u = \frac{x(x+1) + y(y-2)}{(x+1)^2 + y^2} \quad \text{and} \quad v = \frac{y - 2x - 2}{(x+1)^2 + y^2}$$

4.4 Functions of Complex Number

Example:

Express the function

$$f(z) = \bar{z}\,\text{Re}\,(z) + z^2 + \text{Im}\,(z)$$

in the form $f(z) = u(x, y) + \text{j}v(x, y)$.

4.4 Functions of Complex Number

Solution:

Using the properties of complex numbers, the function becomes

$$
\begin{aligned}
f(z) &= \bar{z}\,\mathrm{Re}\,(z) + z^2 + \mathrm{Im}\,(z) \\
&= (x - \mathrm{j}y)x + (x+\mathrm{j}y)^2 + y \\
&= x^2 - \mathrm{j}xy + x^2 + \mathrm{j}2xy - y^2 + y \\
&= (2x^2 - y^2 + y) + \mathrm{j}xy
\end{aligned}
$$

Exercise 4.5:

1. Express u and v in terms of x and y where
   $w = u + \mathrm{j}v,\ z = x + \mathrm{j}y,\ w = f(z)$ and $f(z) = \bar{z}^2$.

2. Express $f(z) = \frac{z+2-\mathrm{j}}{z-1+\mathrm{j}}$ in the form of $u + \mathrm{j}v$.

3. Express $f(z) = \bar{z}^2 + (2 - \mathrm{j}3)z$ in the form of $u + \mathrm{j}v$.

[Ans: $u = x^2 - y^2, v = -2xy$;

$f(z) = \frac{x^2+y^2+x-3}{x^2+y^2-2x+2y+2} + \mathrm{j}\frac{-2x-3y-1}{x^2+y^2-2x+2y+2}$;

$f(z) = (x^2 - y^2 + 2x + 3y) + \mathrm{j}(2y - 3x - 2xy)$]

References

Glyn James, Modern Engineering Mathematics Fourth Edition, Pearson Prentice Hall, 2008.

John H. Mathews & Russell W. Howell, Complex Analysis for Mathematics and Engineering, Jones and Bartlett Publishers, 2001.