OPENCOURSEWARE

BMFG 1313
ENGINEERING MATHEMATICS 1
Nonlinear Equations

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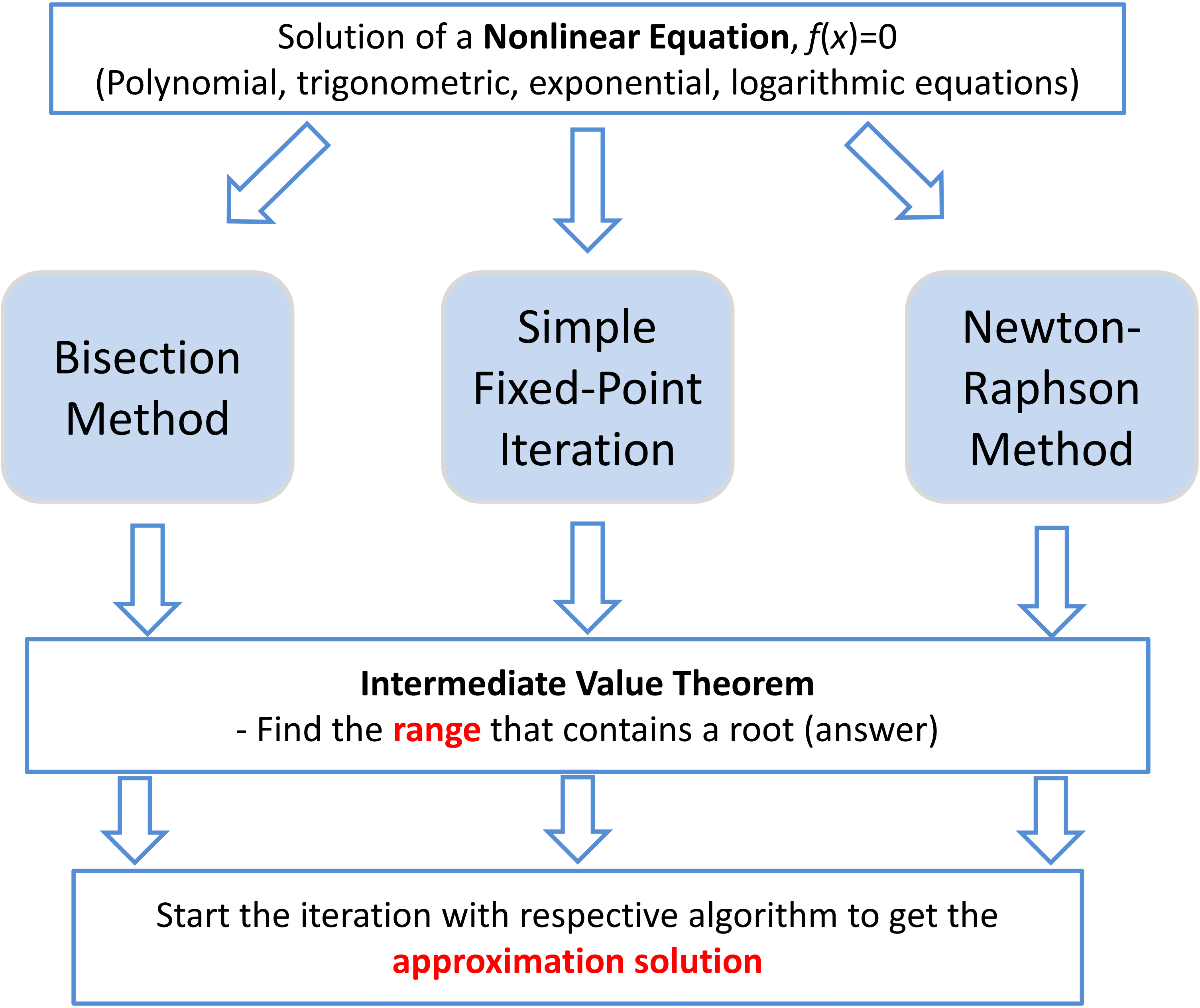
Lesson Outcome

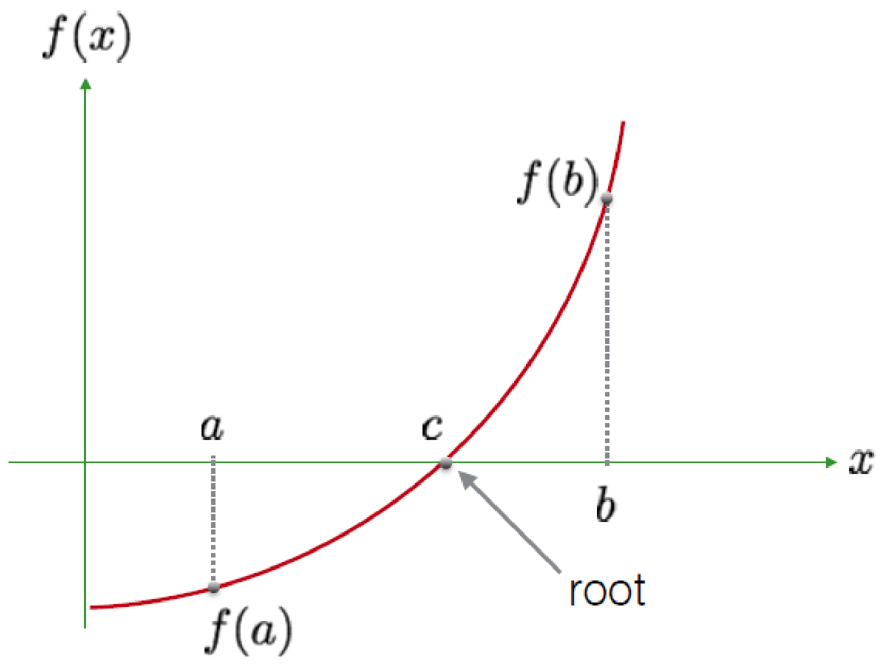
Upon completion of this lesson, the student should be able to:

1. Identify the range that contains root(s).
2. Compute roots for nonlinear equations by using Bisection method, Simple Fixed-Point iteration and Newton-Raphson method.


Solution of a **Nonlinear Equation**, $f(x)=0$
(Polynomial, trigonometric, exponential, logarithmic equations)
Bisection Method
Simple Fixed-Point Iteration
Newton-Raphson Method
Intermediate Value Theorem
- Find the **range** that contains a root (answer)
Start the iteration with respective algorithm to get the **approximation solution**

2.1 Intermediate Value Theorem

Let $f(x) = 0$ be a non-linear equation. If $f(x)$ is a continuous function and $\boldsymbol{f(a)f(b) < 0}$, then there exists at least a root in the interval (a, b).



When two points are connected by a curve:

- One point below x-axis
- One point above x-axis

Then there will be at least one root where the curve crosses the x-axis.

2.1 Intermediate Value Theorem

Example:

Given $f(x) = x^2 - 8x - 5$, use intermediate value theorem to find the interval that contains the negative root.

Solution:

$$f(0) = -5 < 0$$
$$f(-1) = 4 > 0$$
$$\therefore f(0)f(-1) < 0$$

Hence, the interval that contains the negative root is $(-1,0)$.

2.1 Intermediate Value Theorem

Exercise 2.1:

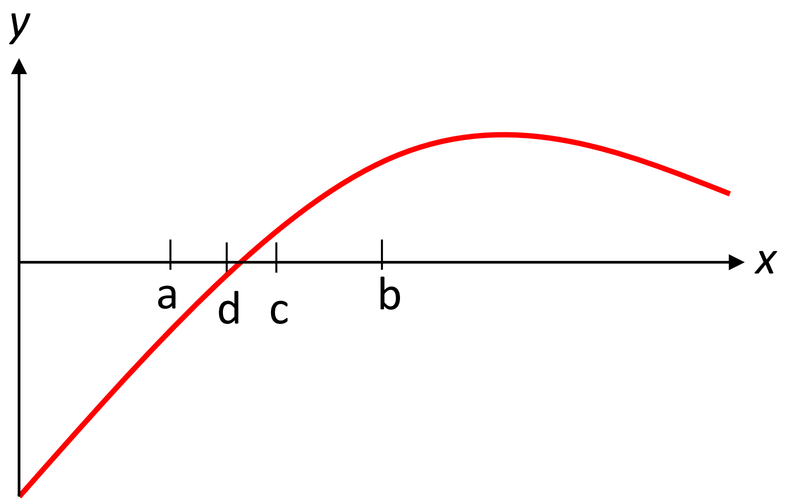
1) Use intermediate value theorem to find the interval that contains the root for $f(x) = x^3 + x + 3$.

2) Use intermediate value theorem to find the interval that contains the smallest positive root of $x = 2\sin x$.

[Ans: (-2,-1); (1,2)]

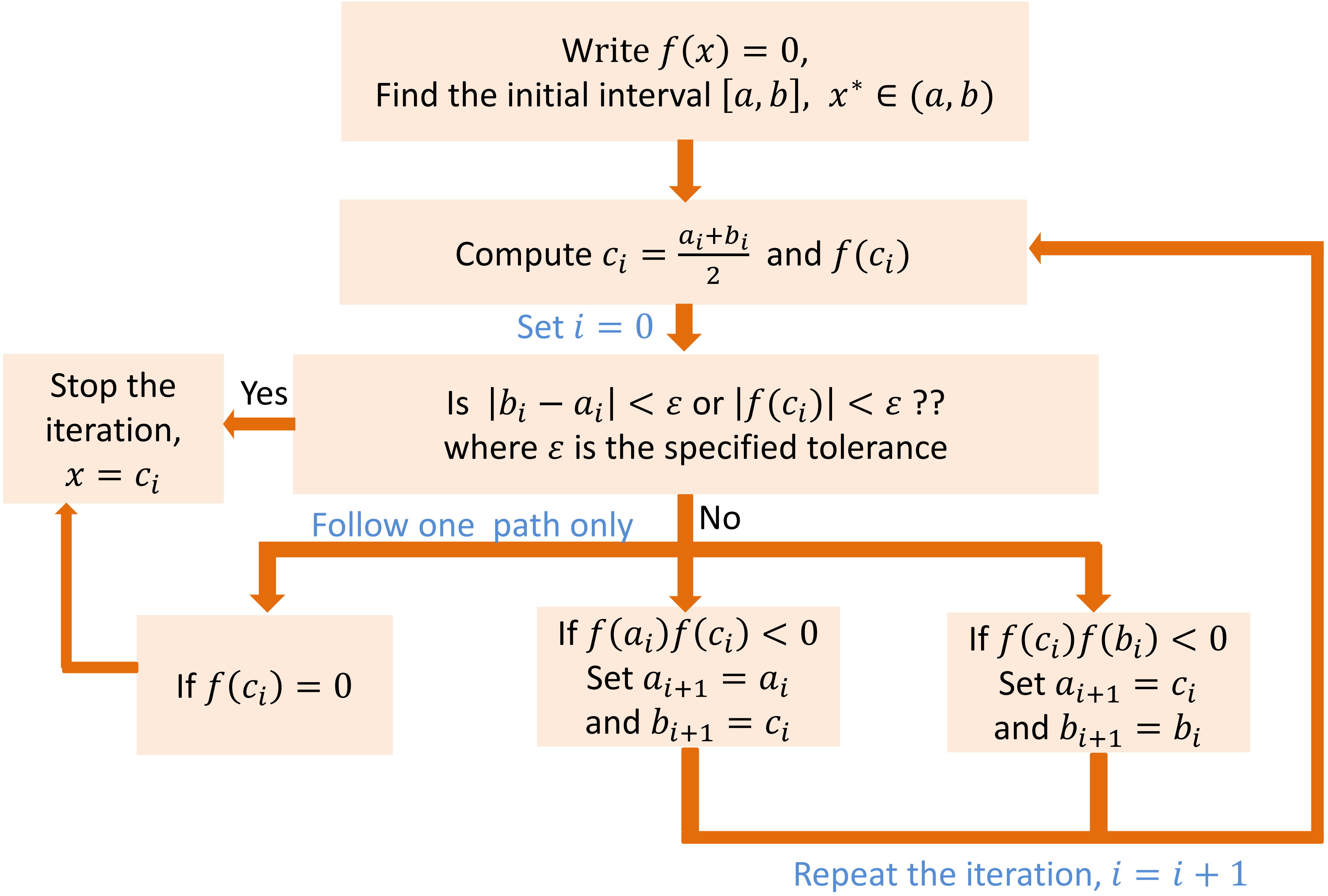
2.2 Bisection Method

The **bisection method** in mathematics is a root-finding method that **repeatedly bisects an interval** and then selects a subinterval that contains the root for further processing.



From (a,b), ↙ c is the midpoint of a and b
we choose (a,c), ↙ d is the midpoint of a and c
then we choose (d,c)
and so on…
until the range is small enough.

2.2.1 Bisection Method Algorithm



2.2.1 Bisection Method Algorithm

Example:

Find the root of $f(x) = x^2 - 3$ by using bisection method accurate to within $\varepsilon = 0.002$ and taking $(1,2)$ as starting interval.

(Answer correct to 4 decimal places)
Take that $|f(c_i)| < \varepsilon$ for your calculation.

2.2.1 Bisection Method Algorithm

Solution:

i	a_i	b_i	$f(a_i)$	$f(b_i)$	c_i	$f(c_i)$	$\lvert f(c_i)\rvert$
0	1.0000	2.0000	-2.0000	1.0000	1.5000	-0.7500	0.7500
1	1.5000	2.0000	-0.7500	1.0000	1.7500	0.0625	0.0625
2	1.5000	1.7500	-0.7500	0.0625	1.6250	-0.3594	0.3594
3	1.6250	1.7500	-0.3594	0.0625	1.6875	-0.1523	0.1523
4	1.6875	1.7500	-0.1523	0.0625	1.7188	-0.0459	0.0459
5	1.7188	1.7500	-0.0457	0.0625	1.7344	0.0081	0.0081
6	1.7188	1.7344	-0.0457	0.0081	1.7266	-0.0190	0.0190
7	1.7266	1.7344	-0.0188	0.0081	1.7305	-0.0055	0.0055
8	1.7305	1.7344	-0.0054	0.0081	**1.7325**	0.0013	0.0013

Root, $x = 1.7325$

2.2.1 Bisection Method Algorithm

Example:

Using the bisection method, find the root of

$$f(x) = x^6 - x - 1$$

accurate to within $\varepsilon = 0.003$.

(Answer correct to 4 decimal places)

Take that $|b_i - a_i| < \varepsilon$ for your calculation.

2.2.1 Bisection Method Algorithm

Solution:

i	a_i	b_i	$f(a_i)$	$f(b_i)$	c_i	$f(c_i)$	$\|b_i - a_i\|$
0	1.0000	2.0000	-1.0000	61.0000	1.5000	8.8906	1.0000
1	1.0000	1.5000	-1.0000	8.8906	1.2500	1.5647	0.5000
2	1.0000	1.2500	-1.0000	1.5647	1.1250	-0.0977	0.2500
3	1.1250	1.2500	-0.0977	1.5647	1.1875	0.6167	0.1250
4	1.1250	1.1875	-0.0977	0.6167	1.1562	0.2333	0.0625
5	1.1250	1.1562	-0.0977	0.2327	1.1406	0.0616	0.0312
6	1.1250	1.1406	-0.0977	0.0613	1.1328	-0.0196	0.0156
7	1.1328	1.1406	-0.0197	0.0613	1.1367	0.0206	0.0078
8	1.1328	1.1367	-0.0197	0.0204	1.1348	0.0004	0.0039
9	1.1328	1.1348	-0.0197	0.0008	**1.1338**	-0.0096	0.0020

The root is, ***x* = 1.1338**

2.2.1 Bisection Method Algorithm

Exercise 2.2:

Find the root of $f(x) = e^{x}(3.2\sin x - 0.5\cos x)$ on the interval $[3,4]$ by using bisection method accurate to within $\varepsilon = 0.05$.

(Answer correct to 4 decimal places)

Take that $|f(c_i)| < \varepsilon$ for your calculation

[Ans: 3.2969]

2.3 Simple Fixed-Point Iteration

The basic idea of the fixed-point iteration method is:

First, to reformulate an equation to an equivalent fixed-point problem:

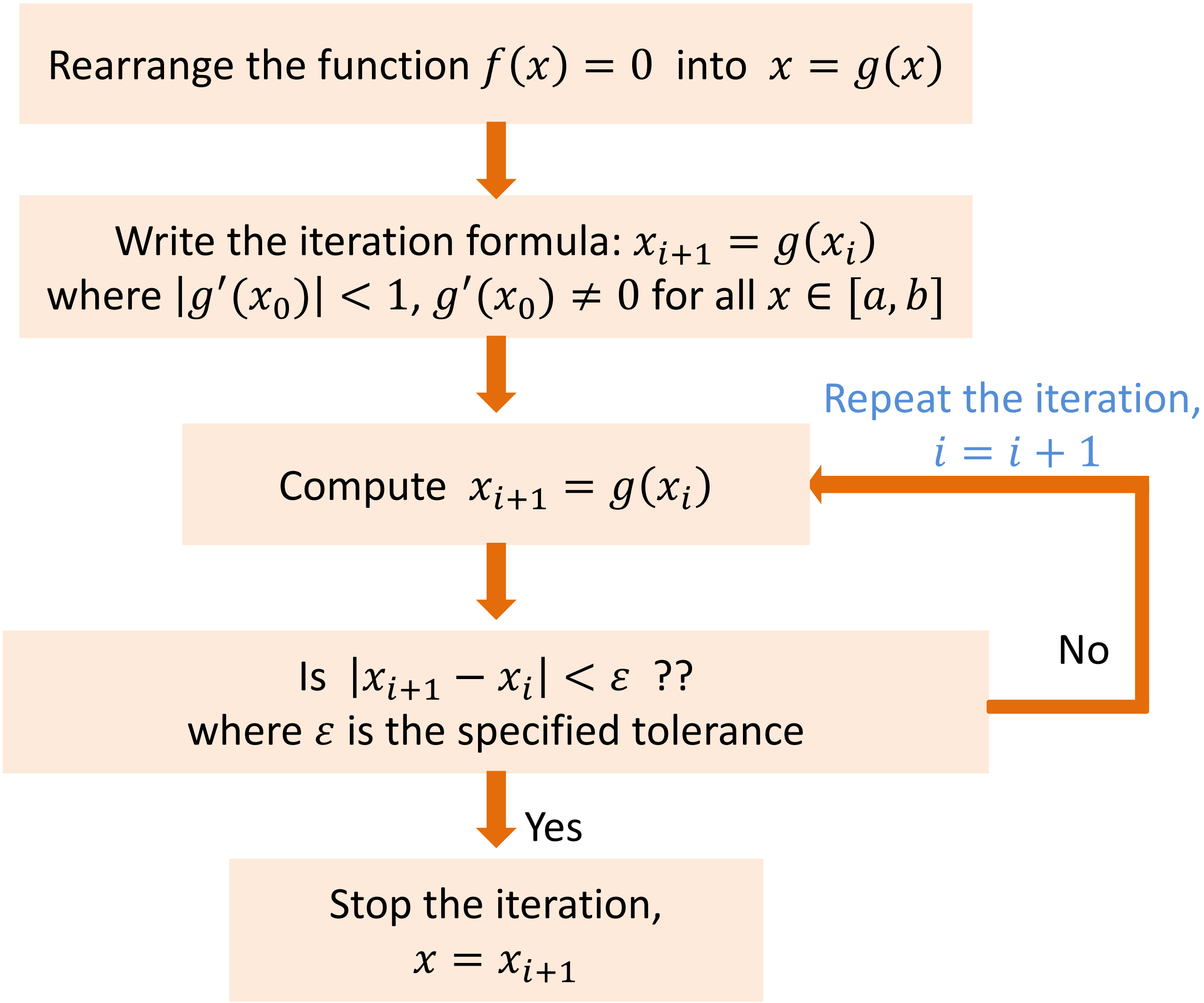
$$f(x) = 0 \Leftrightarrow x = g(x)$$

Secondly, choose an initial guess x_0 and use the iterative sequence $x_{n+1} = g(x_n)$ to compute an x in such a way $x = g(x)$.

*Note: A fixed point of a function $g(x)$ is a point k where $k = g(k)$. This point k is not the root of $g(x) = 0$ but the root of $f(x)$.

There are infinite ways to present an equivalent fixed-point problem for a given equation and it may lead to different roots found.

2.3 Simple Fixed-Point Iteration



Remarks:
The Fixed-point iteration may **converge to a root different from the expected one**, or it **may diverge**. Different rearrangement will converge at different rates.

2.3 Simple Fixed-Point Iteration

Example:

Given $f(x) = x^2 - 2x - 3$. Find the root of the function by using simple fixed-point method accurate to within $\varepsilon = 0.001$ and taking $x = 4$ as starting point.

(Answer correct to 4 decimal places)

2.3 Simple Fixed-Point Iteration

Solution:

a) $x = g(x) = \sqrt{2x+3}$; $g'(x) = \frac{1}{\sqrt{2x+3}}$ and $|g'(4)| = 0.3 < \mathbf{1}$

This form will converge and give a solution

i	x_i	$\lvert x_i - x_{i-1} \rvert$
0	4.0000	
1	3.3166	0.6834
2	3.1037	0.2129
3	3.0344	0.0693
4	3.0114	0.0230
5	3.0038	0.0076
6	3.0013	0.0025
7	3.0004	0.0009

The value converging to root of $\boldsymbol{x = 3.0004}$

2.3 Simple Fixed-Point Iteration

Solution:

b) $x = g(x) = \frac{3}{x-2}$; $g'(x) = -\frac{3}{(x-2)^2}$ and $|g'(4)| = 0.75 < \mathbf{1}$

This form will converge and give a solution

i	x_i	$\lvert x_i - x_{i-1} \rvert$
0	4.0000	
1	1.5000	2.5000
2	-6.0000	7.5000
3	-0.3750	5.6250
4	-1.2632	0.8882
5	-0.9193	0.3439
6	-1.0276	0.1083
7	-0.9909	0.0367
8	-1.0030	0.0121
9	-0.9990	0.0040

After 11 iterations, the value converging to root of $\boldsymbol{x = -1}$

2.3 Simple Fixed-Point Iteration

Solution:

c) $x = g(x) = \frac{x^2-3}{2}$; $g'(x) = x$ and $|g'(4)| = 4 \not< \mathbf{1}$

This form will diverge and give no solution

i	x_i	$\lvert x_i - x_{i-1} \rvert$
0	4.0000	
1	6.5000	2.5000
2	19.6250	13.1250
3	191.0703	171.4453
4	18252.4298	18061.3595
5	166575595.3	16557342.9

value diverges

$g(x) = \frac{x^2-3}{2}$ is not a suitable form for simple fixed-point iteration

2.3 Simple Fixed-Point Iteration

Example:

Find the root of

$$f(x) = 3xe^x - 1$$

by using simple fixed-point iteration accurate to within $\varepsilon = 0.0001$. Assume $x_0 = 1$.

(Answer correct to 4 decimal places)

2.3 Simple Fixed-Point Iteration

Solution:

There are two possible forms of $g(x)$:

$$x = g(x) = \frac{1}{3}e^{-x} \qquad \text{and} \qquad x = g(x) = \ln\left(\frac{1}{3x}\right)$$

$$g'(x) = -\frac{1}{3}e^{-x} \qquad\qquad g'(x) = -\frac{1}{x}$$

$$|g'(1)| = \mathbf{0.12} < \mathbf{1} \qquad\qquad |g'(1)| = \mathbf{1} \nless \mathbf{1}$$

Criteria is satisfied

Criteria is not satisfied

$$\therefore g(x) = \frac{1}{3}e^{-x}$$

2.3 Simple Fixed-Point Iteration

Solution:

i	x_i	$\lvert x_i - x_{i-1} \rvert$
0	1.0000	
1	0.1226	0.8774
2	0.2949	0.1722
3	0.2482	0.0467
4	0.2601	0.0119
5	0.2570	0.0031
6	0.2578	0.0008
7	0.2576	0.0002
8	**0.2576**	0.0000

Thus, the root that satisfies the stopping criteria is $x = 0.2576$.

2.3 Simple Fixed-Point Iteration

Exercise 2.3:

Locate the root of $f(x) = e^{-x} - x$ by using simple fixed-point iteration accurate to within $\varepsilon = 0.003$ where $x \in (0,1]$.

(Answer correct to 4 decimal places)

[Ans: 0.5664]

2.4 Newton-Raphson Method

From Taylor Series, expansion of $f(x)$ about a point $x = x_0$ is given by

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \cdots$$

Let x be a root of a function $f(x)$, i.e. $f(x) = 0$ and x_0 be an approximation to the root x. Since $f(x) = 0$ and only linear term is kept to approximate root x,

$$0 = f(x_0) + f'(x_0)(x - x_0)$$

Rearrangement of the equation gives

$$x = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Hence, Newton-Raphson apply this algorithm iteratively to generate sequence

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

2.4 Newton-Raphson Method

Write $f(x) = 0$,
Given x_0 reasonably close to the root

↓

Compute $f(x_0)$ and $f'(x_0)$
where $f(x_0) \neq 0$ and $f'(x_0) \neq 0$

↓ Set $i = 0$

Compute $x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$

↓

Is $|x_{i+1} - x_i| < \varepsilon$ or $|f(x_{i+1})| < \varepsilon$??
where ε is the specified tolerance

No → Repeat the iteration, $i = i + 1$ (back to Compute x_{i+1})

↓ Yes

Stop the iteration,
$x = x_{i+1}$

2.4 Newton-Raphson Method

Example:

Determine the root of the function

$$f(x) = e^x - \frac{2}{x}$$

by using Newton-Raphson method with $x_0 = 0.8$ accurate to within $\varepsilon = 0.0001.$

(Answer correct to 4 decimal places)

Take that $|f(x_i)| < \varepsilon$ for your calculation

2.4 Newton-Raphson Method

Solution:

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

Given $f(x) = e^x - \frac{2}{x}$

hence, $f'(x) = e^x + \frac{2}{x^2}$

Check:

$f(0.8) = -0.2745 \neq 0$

$f'(0.8) = 5.3505 \neq 0$

i	x_i	$f(x_i)$	$f'(x_i)$	$f(x_i)/f'(x_i)$	$\lvert f(x_i)\rvert$
0	0.8000	-0.2745	5.3505	-0.0513	0.2745
1	0.8513	-0.0067	5.1024	-0.0013	0.0067
2	0.8526	0	5.0970	0	0

Root, $x = 0.8526$

Reaching stopping criteria

2.4 Newton-Raphson Method

Example:

Use the Newton-Raphson method to estimate the root of

$$f(x) = 3x + \sin x - e^x$$

starting from $x_0 = 0$ accurate to within

$|x_i - x_{i-1}| \leq 0.0001.$

(Answer correct to 4 decimal places)

2.4 Newton-Raphson Method

Solution:

$$x_{i+1} = x_i - \frac{f(x_i)}{f'(x_i)}$$

Check:

$f(0) = -1 \neq 0$

$f'(0) = 3 \neq 0$

Given $f(x) = 3x + \sin x - e^x$

hence, $f'(x) = 3 + \cos x - e^x$

i	x_i	$f(x_i)$	$f'(x_i)$	$f(x_i)/f'(x_i)$	$\lvert x_i - x_{i-1} \rvert$
0	0	-1.0000	3.0000	-0.3333	
1	0.3333	-0.0685	2.5494	-0.0269	0.3333
2	0.3602	-0.0006	2.5022	-0.0002	0.0269
3	0.3604	-0.0001	2.5018	0	0.0002
4	0.3604	-0.0001	2.5018	0	0

Root, $x = 0.3604$

Reaching stopping criteria

2.4 Newton-Raphson Method

Exercise 2.4:

Use **Newton Raphson method** to find the solutions accurate to within 10^{-4} for the problem

$$f(x) = \ln(x-1) + \cos(x-1)$$

with initial guess $x_0 = 1.3$

(Answer correct to 4 decimal places)

Take that $|x_i - x_{i-1}| < \varepsilon$ for your calculation

[Ans: 1.3978]