ENGINEERING MATHEMATICS 1
BMFG 1313
MATRICES (SOLUTION OF LINEAR SYSTEMS)

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Lesson Outcome

Upon completion of this lesson, the student should be able to:

1. Reduce a matrix into row echelon form using row reduction method.
2. Solve a linear system by using Gauss Elimination.
3. Decompose a matrix into a product of an upper and lower triangular matrices using LU Decomposition to solve a linear system.
4. Solve a linear system by using Gauss Seidel.

1.6 Introduction to a Linear System

System of linear equations is a collection of linear equations that involves a set of variables. It represents system mostly in engineering, physics, chemistry, computer science and economics.

For example,

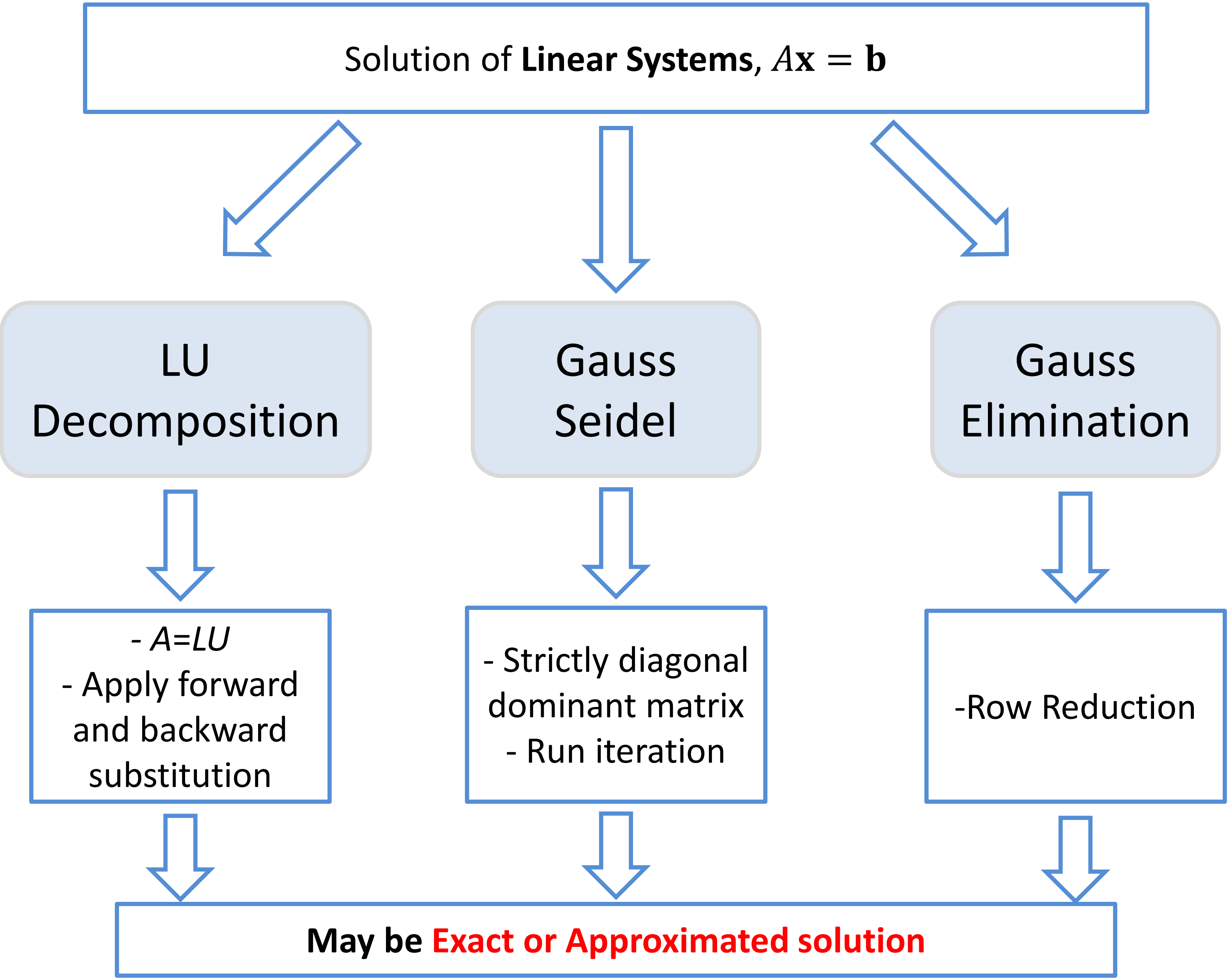
$$\begin{aligned} 3x + 2y - 4z &= 3 \\ 2x - 4y + z &= -5 \\ x + 3y + 5z &= 12 \end{aligned}$$

is a linear system with variables x, y and z. It is represented in the form of $A\mathbf{x} = \mathbf{b}$ as follows:

$$\begin{bmatrix} 3 & 2 & -4 \\ 2 & -4 & 1 \\ 1 & 3 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -5 \\ 12 \end{bmatrix}$$

The **solution** to the linear equation system is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}.$$


Solution of **Linear Systems**, $A\mathbf{x} = \mathbf{b}$
LU Decomposition
Gauss Seidel
Gauss Elimination
- *A=LU*
- Apply forward and backward substitution
- Strictly diagonal dominant matrix
- Run iteration
-Row Reduction
May be Exact or Approximated solution

1.7 Elementary Row Operation

Aim to reduce a matrix into an upper triangular matrix, which is also known as row echelon form.

i.e.

$$\begin{bmatrix} X & X & X \\ 0 & X & X \\ 0 & 0 & X \end{bmatrix}$$

X can be any different values

Pivot/Leading

1.7 Elementary Row Operation

1) Interchange any two rows

- Interchange two rows i and j: $r_i \leftrightarrow r_j$

E.g.

$$\begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix} \xrightarrow{r_1 \leftrightarrow r_2} \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}$$

2) Multiply any row with a scalar

- Multiply a scalar k to row r_i: kr_i

E.g.

$$\begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix} \xrightarrow{\frac{1}{2}r_1} \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}$$

1.7 Elementary Row Operation

3) Row replacement operation:

- Replace row j by $mr_i + r_j$:

$$mr_i + r_j \rightarrow r_j$$

$m = -\frac{3}{1} = -3$

$m = -\frac{-2}{1} = 2$

$m = -\frac{1}{5}$

$$\begin{bmatrix} 1 & 0 & 3 \\ 3 & 5 & -2 \\ -2 & 1 & -3 \end{bmatrix} \xrightarrow[2r_1+r_3]{-3r_1+r_2} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 5 & -11 \\ 0 & 1 & 3 \end{bmatrix} \xrightarrow{-\frac{1}{5}r_2+r_3} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 5 & -11 \\ 0 & 0 & 26/5 \end{bmatrix}$$

The first pivot entry is 1, two operations are needed to reduce row 2 and row 3

The second pivot entry is 5, one operation is needed to reduce row 3

The third pivot entry is 26/5, upper triangular is obtained and the ERO is done

1.7 Elementary Row Operation

Example:

Reduce the matrix into row echelon form by using only row replacement operation.

$$A = \begin{bmatrix} -1 & 2 & -5 \\ 2 & -1 & 6 \\ 1 & 1 & 3 \end{bmatrix}$$

Solution:

$$\begin{bmatrix} -1 & 2 & -5 \\ 2 & -1 & 6 \\ 1 & 1 & 3 \end{bmatrix} \xrightarrow[r_1+r_3]{2r_1+r_2} \begin{bmatrix} -1 & 2 & -5 \\ 0 & 3 & -4 \\ 0 & 3 & -2 \end{bmatrix} \xrightarrow{-r_2+r_3} \begin{bmatrix} -1 & 2 & -5 \\ 0 & 3 & -4 \\ 0 & 0 & 2 \end{bmatrix}$$

1.7 Elementary Row Operation

Example:

Reduce the matrix into row echelon form by using only row replacement operation.

$$A = \begin{bmatrix} 2 & 2 & 3 \\ -2 & 1 & 5 \\ 1 & 3 & 4 \end{bmatrix}$$

Solution:

$$\begin{bmatrix} 2 & 2 & 3 \\ -2 & 1 & 5 \\ 1 & 3 & 4 \end{bmatrix} \xrightarrow[-\frac{1}{2}r_1+r_3]{r_1+r_2} \begin{bmatrix} 2 & 2 & 3 \\ 0 & 3 & 8 \\ 0 & 2 & 5/2 \end{bmatrix} \xrightarrow{-\frac{2}{3}r_2+r_3} \begin{bmatrix} 2 & 2 & 3 \\ 0 & 3 & 8 \\ 0 & 0 & -17/6 \end{bmatrix}$$

Exercise 1.5:

Reduce the following matrices into row echelon form.

1) $\begin{bmatrix} 1 & 4 & -2 \\ -1 & 6 & 3 \\ 5 & 2 & 0 \end{bmatrix}$

2) $\begin{bmatrix} 3 & 8 & 5 \\ -2 & 4 & 3 \\ 0 & 1 & 2 \end{bmatrix}$

[Ans: $\begin{bmatrix} 1 & 4 & -2 \\ 0 & 10 & 1 \\ 0 & 0 & 59/5 \end{bmatrix}$; $\begin{bmatrix} 3 & 8 & 5 \\ 0 & 28/3 & 19/3 \\ 0 & 0 & 37/28 \end{bmatrix}$]

1.8 Gauss Elimination Algorithm

Basic idea:

Solve the equivalent system which is in a simpler form compared to the original linear system.

Flow:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned}$$

Linear System

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Matrix form

$$\left[\begin{array}{ccc|c} a_{11} & a_{12} & a_{13} & b_1 \\ a_{21} & a_{22} & a_{23} & b_2 \\ a_{31} & a_{32} & a_{33} & b_3 \end{array}\right]$$

Augmented Matrix

Row Reduction

Backward Substitution

Example:

Solve the following linear system by using Gauss Elimination.

$$2x_1 + 3x_2 - x_3 = 2$$
$$4x_1 + 4x_2 - x_3 = -1$$
$$-2x_1 - 3x_2 + 4x_3 = 1$$

Solution:

Step 1: Transform the linear system into matrix form:

$$\begin{bmatrix} 2 & 3 & -1 \\ 4 & 4 & -1 \\ -2 & -3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

Solution:

Step 2: Augmented Matrix:

$$\left[\begin{array}{ccc|c} 2 & 3 & -1 & 2 \\ 4 & 4 & -1 & -1 \\ -2 & -3 & 4 & 1 \end{array}\right]$$

Step 3: Apply row reduction:

$$\left[\begin{array}{ccc|c} 2 & 3 & -1 & 2 \\ 4 & 4 & -1 & -1 \\ -2 & -3 & 4 & 1 \end{array}\right] \xrightarrow[r_1+r_3]{-2r_1+r_2} \left[\begin{array}{ccc|c} 2 & 3 & -1 & 2 \\ 0 & -2 & 1 & -5 \\ 0 & 0 & 3 & 3 \end{array}\right]$$

Solution:

Step 4: Backward substitution:

$$\begin{matrix} x_1 & x_2 & x_3 & \\ \end{matrix}$$

$$\left[\begin{array}{ccc|c} 2 & 3 & -1 & 2 \\ 0 & -2 & 1 & -5 \\ 0 & 0 & 3 & 3 \end{array}\right]$$

Start from last row,

$$3x_3 = 3, \qquad x_3 = 1$$

$$-2x_2 + x_3 = -5, \qquad -2x_2 + (1) = -5, \qquad x_2 = 3$$

$$2x_1 + 3x_2 - x_3 = 2, \qquad 2x_1 + 3(3) - (1) = 2, \qquad x_1 = -3$$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -3 \\ 3 \\ 1 \end{bmatrix}$$

Example:

Solve the following linear system by using Gauss Elimination.

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 9 \\ 2x_1 - x_2 + x_3 &= 8 \\ 3x_1 \qquad - x_3 &= 3 \end{aligned}$$

Solution:

Augmented Matrix:

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 2 & -1 & 1 & 8 \\ 3 & 0 & -1 & 3 \end{array}\right]$$

Solution (cont.):

Row reduction:

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 2 & -1 & 1 & 8 \\ 3 & 0 & -1 & 3 \end{array}\right] \xrightarrow[-3r_1+r_3]{-2r_1+r_2} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 0 & -5 & -5 & -10 \\ 0 & -6 & -10 & -24 \end{array}\right]$$

$$\xrightarrow{-\frac{6}{5}r_2+r_3} \left[\begin{array}{ccc|c} 1 & 2 & 3 & 9 \\ 0 & -5 & -5 & -10 \\ 0 & 0 & -4 & -12 \end{array}\right]$$

Backward substitution:

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$$

Exercise 1.6:

Solve the following linear system by using Gauss Elimination.

$$3x_1 - 2x_2 + x_3 = -1$$
$$x_1 + 4x_2 - 2x_3 = 16$$
$$2x_1 - 5x_2 + 2x_3 = -13$$

[Ans: $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$]

1.9 LU Decomposition

Given a linear system,
$A\mathbf{x} = \mathbf{b}$

↓

Decompose $A = LU$

↓

$(LU)\mathbf{x} = \mathbf{b}$

↓

$L(U\mathbf{x}) = \mathbf{b}$

↓

Let $U\mathbf{x} = \mathbf{y}, \quad \therefore L\mathbf{y} = \mathbf{b}$

↓

solve $L\mathbf{y} = \mathbf{b}$ for $\mathbf{y}$
(forward substitution)

↓

solve $U\mathbf{x} = \mathbf{y}$ for $\mathbf{x}$
(backward substitution)

1.9 LU Decomposition

An $n \times n$ **nonsingular matrix** where all its leading principal minors are non-zero, can be decomposed into a lower triangular matrix (L) and an upper triangular matrix (U), for example, a 3×3 matrix as follows:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix}$$

$$\boldsymbol{A} = \boldsymbol{L} \qquad \boldsymbol{U}$$

Leading Principal Minors of matrix *A* are:

$$D_1 = a_{11}, D_2 = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}, D_3 = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}, \dots, D_n = |A|$$

1.9.1 Steps of LU Decomposition

Step 1: Construct L and U

$$\boldsymbol{L\,U = A}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Multiply *L* and *U*:

$$\begin{bmatrix} \boxed{u_{11}} & \boxed{u_{12}} & \boxed{u_{13}} \\ l_{21}u_{11} & l_{21}u_{12} + u_{22} & l_{21}u_{13} + u_{23} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + l_{32}u_{23} + u_{33} \end{bmatrix} = \begin{bmatrix} \boxed{a_{11}} & \boxed{a_{12}} & \boxed{a_{13}} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Compare each of the entries to obtain the values for *u* and *l*.

i.e. $u_{11} = a_{11}$

1.9.1 Steps of LU Decomposition

Example 1:

Decompose $A = \begin{bmatrix} 1 & 4 & 2 \\ -2 & -1 & 1 \\ 4 & 2 & -3 \end{bmatrix}$ into LU.

$$LU = A$$

$$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix}\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} = \begin{bmatrix} u_{11} & u_{12} & u_{13} \\ l_{21}u_{11} & l_{21}u_{12} + u_{22} & l_{21}u_{13} + u_{23} \\ l_{31}u_{11} & l_{31}u_{12} + l_{32}u_{22} & l_{31}u_{13} + l_{32}u_{23} + u_{33} \end{bmatrix} = \begin{bmatrix} 1 & 4 & 2 \\ -2 & -1 & 1 \\ 4 & 2 & -3 \end{bmatrix}$$

Compare the entries from both sides:

$u_{11} = 1$	$u_{12} = 4$	$u_{13} = 2$
$l_{21}u_{11} = -2$ $l_{21}(1) = -2$ $\therefore l_{21} = -2$	$l_{21}u_{12} + u_{22} = -1$ $(-2)(4) + u_{22} = -1$ $\therefore u_{22} = 7$	$l_{21}u_{13} + u_{23} = 1$ $(-2)(2) + u_{23} = 1$ $\therefore u_{23} = 5$
$l_{31}u_{11} = 4$ $l_{31}(1) = 4$ $\therefore l_{31} = 4$	$l_{31}u_{12} + l_{32}u_{22} = 2$ $(4)(4) + l_{32}(7) = 2$ $\therefore l_{32} = -2$	$l_{31}u_{13} + l_{32}u_{23} + u_{33} = -3$ $(4)(2) + (-2)(5) + u_{33} = -3$ $\therefore u_{33} = -1$

Hence,

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 4 & -2 & 1 \end{bmatrix}$$

$$U = \begin{bmatrix} 1 & 4 & 2 \\ 0 & 7 & 5 \\ 0 & 0 & -1 \end{bmatrix}$$

1.9.1 Steps of LU Decomposition

*Step 1: Construct L and U (**Alternative Way**)*

Row reduction (Use only ERO Step 3: Row Replacement Operation):

Example 2:

Matrix A satisfies the condition where all its leading principal minors are non-zero.

$$A = \begin{bmatrix} 1 & 4 & 2 \\ -2 & -1 & 1 \\ 4 & 2 & -3 \end{bmatrix} \xrightarrow[-4r_1+r_3]{2r_1+r_2} \begin{bmatrix} 1 & 4 & 2 \\ 0 & 7 & 5 \\ 0 & -14 & -11 \end{bmatrix} \xrightarrow{2r_2+r_3} \begin{bmatrix} 1 & 4 & 2 \\ 0 & 7 & 5 \\ 0 & 0 & -1 \end{bmatrix} = U$$

$\div 1$ $\quad\div 7$ $\quad\div -1$

*To obtain matrix L, identify the columns that contains the pivots, divide all the entries underneath the pivot, including pivot itself, by their respective pivot values.

$$\therefore L = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 4 & -2 & 1 \end{bmatrix}$$

1.9.1 Steps of LU Decomposition

*Step 1: Construct L and U (**Alternative Way**)*

Example 3:

Matrix A satisfies the condition where all its leading principal minors are non-zero.

$$A = \begin{bmatrix} 1 & -2 & -2 & -3 \\ 3 & -9 & 0 & -9 \\ -1 & 2 & 4 & 7 \\ -3 & -6 & 26 & 2 \end{bmatrix} \xrightarrow{\substack{-3r_1+r_2 \\ r_1+r_3 \\ 3r_1+r_4}} \begin{bmatrix} 1 & -2 & -2 & -3 \\ 0 & -3 & 6 & 0 \\ 0 & 0 & 2 & 4 \\ 0 & -12 & 20 & -7 \end{bmatrix}$$

$\div 1$ $\quad\quad$ $\div -3$

$$\xrightarrow{-4r_2+r_4} \begin{bmatrix} 1 & -2 & -2 & -3 \\ 0 & -3 & 6 & 0 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & -4 & -7 \end{bmatrix} \xrightarrow{2r_3+r_4} \begin{bmatrix} 1 & -2 & -2 & -3 \\ 0 & -3 & 6 & 0 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix} = U$$

$\div 2$ $\quad\quad$ $\div 1$

$$\therefore L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 3 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -3 & 4 & -2 & 1 \end{bmatrix}$$

1.9.1 Steps of LU Decomposition

Example 4:

Matrix A does not satisfy the condition to have a pure LU decomposition since $D_2 = 0$. Hence, rows interchanging is applied as follows:

Note:
Rows interchange $r_k \leftrightarrow r_{k+1}$ is applied when $D_k = 0$.

$$A = \begin{bmatrix} 1 & -2 & -2 \\ -1 & 2 & 4 \\ 4 & 2 & 1 \end{bmatrix} \xrightarrow{r_2 \leftrightarrow r_3} \begin{bmatrix} 1 & -2 & -2 \\ 4 & 2 & 1 \\ -1 & 2 & 4 \end{bmatrix}$$

Note:
Hence, $\mathbf{b} = \begin{bmatrix} b_1 \\ b_3 \\ b_2 \end{bmatrix}$ according to this example.

$$A = \begin{bmatrix} 1 & -2 & -2 \\ 4 & 2 & 1 \\ -1 & 2 & 4 \end{bmatrix} \xrightarrow[r_1+r_3]{-4r_1+r_2} \begin{bmatrix} 1 & -2 & -2 \\ 0 & 10 & 9 \\ 0 & 0 & 2 \end{bmatrix} = U$$

$\div 1$ $\quad \div 10 \quad \div 2$

$$\therefore L = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

1.9.1 Steps of LU Decomposition

Step 2: Solve $L\mathbf{y} = \mathbf{b}$ for $\mathbf{y}$ by using forward substitution

$$A\mathbf{x} = \mathbf{b}$$

When $A = LU$,

$$L(U\mathbf{x}) = \mathbf{b}$$

Let $U\mathbf{x} = \mathbf{y}$,

$$L\mathbf{y} = \mathbf{b}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Forward substitution

$$\begin{aligned} y_1 &= b_1 \\ l_{21}y_1 + y_2 &= b_2 \\ l_{31}y_1 + l_{32}y_2 + y_3 &= b_3 \end{aligned}$$

1.9.1 Steps of LU Decomposition

Step 3: Solve $U\mathbf{x} = \mathbf{y}$ for $\mathbf{x}$ by using backward substitution

$$U\mathbf{x} = \mathbf{y}$$

$$\begin{bmatrix} u_{11} & u_{12} & u_{13} \\ 0 & u_{22} & u_{23} \\ 0 & 0 & u_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

Backward substitution

$$u_{33}x_3 = y_3$$

$$u_{22}x_2 + u_{23}x_3 = y_2$$

$$u_{11}x_1 + u_{12}x_2 + u_{13}x_3 = y_1$$

Example:

Solve the following linear system by using LU decomposition.

$$2x_1 + 3x_2 - x_3 = 2$$
$$4x_1 + 4x_2 - x_3 = -1$$
$$-2x_1 - 3x_2 + 4x_3 = 1$$

Solution:

Transform the linear system into matrix form:

$$\begin{bmatrix} 2 & 3 & -1 \\ 4 & 4 & -1 \\ -2 & -3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

Solution (cont.):

Step 1: Construct L and U

$$A = \begin{bmatrix} 2 & 3 & -1 \\ 4 & 4 & -1 \\ -2 & -3 & 4 \end{bmatrix} \xrightarrow[r_1+r_3]{-2r_1+r_2} \begin{bmatrix} 2 & 3 & -1 \\ 0 & -2 & 1 \\ 0 & 0 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 3 & -1 \\ 0 & -2 & 1 \\ 0 & 0 & 3 \end{bmatrix} = U$$

$\div 2$ $\quad\quad$ $\div -2$ $\quad\quad$ $\div 3$

$$\therefore \quad L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \qquad U = \begin{bmatrix} 2 & 3 & -1 \\ 0 & -2 & 1 \\ 0 & 0 & 3 \end{bmatrix}$$

Solution (cont.):

Step 2: Solve $L\mathbf{y} = \mathbf{b}$ for $\mathbf{y}$ by using forward substitution

$$A\mathbf{x} = \mathbf{b},$$

$$\text{when } A = LU, \quad (LU)\mathbf{x} = \mathbf{b} \quad \Rightarrow \quad L(U\mathbf{x}) = \mathbf{b}.$$

Let $U\mathbf{x} = \mathbf{y}$, we have

$$L\mathbf{y} = \mathbf{b}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

By forward substitution:

$$y_1 = 2, \quad 2y_1 + y_2 = -1, \quad -y_1 + y_3 = 1,$$

$$2(2) + y_2 = -1, \quad -(2) + y_3 = 1,$$

$$y_2 = -5 \qquad y_3 = 3$$

$$\therefore \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -5 \\ 3 \end{bmatrix}$$

Solution (cont.):

Step 3: Solve $U\mathbf{x} = \mathbf{y}$ for $\mathbf{x}$ by using backward substitution

$$U\mathbf{x} = \mathbf{y}$$

$$\begin{bmatrix} 2 & 3 & -1 \\ 0 & -2 & 1 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -5 \\ 3 \end{bmatrix}$$

By Backward substitution:

$$3x_3 = 3, \quad -2x_2 + x_3 = -5, \quad 2x_1 + 3x_2 - x_3 = 2$$

$$x_3 = 1 \quad -2x_2 + (1) = -5, \quad 2x_1 + 3(3) - (1) = 2$$

$$x_2 = 3 \qquad x_1 = -3$$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -3 \\ 3 \\ 1 \end{bmatrix}$$

Example:

Solve the following linear system by using LU decomposition.

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 9 \\ 2x_1 - x_2 + x_3 &= 8 \\ 3x_1 \qquad - x_3 &= 3 \end{aligned}$$

Solution:

Transform the linear system into matrix form:

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 3 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 9 \\ 8 \\ 3 \end{bmatrix}$$

Solution (Cont.):

Step 1: Construct L and U

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 3 & 0 & -1 \end{bmatrix} \xrightarrow[-3r_1+r_3]{-2r_1+r_2} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -5 & -5 \\ 0 & -6 & -10 \end{bmatrix} \xrightarrow{-\frac{6}{5}r_2+r_3} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -5 & -5 \\ 0 & 0 & -4 \end{bmatrix} = U$$

$\div 1$ $\quad \div -5$ $\quad \div -4$

$$\therefore \quad L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 6/5 & 1 \end{bmatrix} \qquad U = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -5 & -5 \\ 0 & 0 & -4 \end{bmatrix}$$

Solution (Cont.):

Step 2: Solve $L\mathbf{y} = \mathbf{b}$ for $\mathbf{y}$ by using forward substitution

$$A\mathbf{x} = \mathbf{b},$$

$$\text{when } A = LU, \quad (LU)\mathbf{x} = \mathbf{b} \quad \Rightarrow \quad L(U\mathbf{x}) = \mathbf{b}.$$

Let $U\mathbf{x} = \mathbf{y}$,

$$L\mathbf{y} = \mathbf{b}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 6/5 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 9 \\ 8 \\ 3 \end{bmatrix}$$

By forward substitution:

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 9 \\ -10 \\ -12 \end{bmatrix}$$

Solution (Cont.):

Step 3: Solve $U\mathbf{x} = \mathbf{y}$ for $\mathbf{x}$ by using backward substitution

$$U\mathbf{x} = \mathbf{y}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -5 & -5 \\ 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 9 \\ -10 \\ -12 \end{bmatrix}$$

By Backward substitution:

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix}$$

Exercise 1.7:

Solve the following linear system by using LU Decomposition.

$$1.012x_1 - 2.132x_2 + 3.104x_3 = 1.984$$
$$-2.132x_1 + 4.096x_2 - 7.013x_3 = -5.049$$
$$3.104x_1 - 7.013x_2 + 0.014x_3 = -3.895$$

Work out your solution in 3 decimal places.

[Ans: $L = \begin{bmatrix} 1 & 0 & 0 \\ -2.107 & 1 & 0 \\ 3.067 & 1.197 & 1 \end{bmatrix}, U = \begin{bmatrix} 1.012 & -2.132 & 3.104 \\ 0 & -0.396 & -0.473 \\ 0 & 0 & -8.940 \end{bmatrix}, \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$]

1.10 Gauss Seidel

Strictly Diagonal Dominant Matrix

- The **magnitude of diagonal entry** in a row is larger than the sum of the magnitudes of all the other entries in that row:

$$|a_{kk}| > \sum_{j\neq k}^{N} |a_{kj}| \text{ for } k = 1,2,\dots,N)$$

e.g.

$$\begin{bmatrix} 6 & 2 & 3 \\ -1 & -9 & 4 \\ 3 & 0 & -7 \end{bmatrix}$$

$|6| > |2| + |3|$

$|-9| > |-1| + |4|$

$|-7| > |3| + |0|$

1.10 Gauss Seidel

Basic Idea:

Solve unknown variable of a linear system iteratively by using previously computed results as soon as they are available.

Flow:

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1\\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2\\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3
\end{aligned}
$$

Linear System

Strictly diagonal dominant matrix arrangement

Form the iterative sequence

$$\left(x_1^{(k+1)}, x_2^{(k+1)}, x_3^{(k+1)}\right)$$

Initial guess: $(x_1^{(0)}, x_2^{(0)}, x_3^{(0)})$

Stop iteration when

$$\left\|\mathbf{x}^{(k)} - \mathbf{x}^{(k-1)}\right\|_\infty = \max_{1\le i\le n}\left\{\left|x_i^{(k)} - x_i^{(k-1)}\right|\right\} < \varepsilon$$

(known as infinity norm)

1.10.1 Gauss Seidel Iterative Sequence

From $A\mathbf{x} = \mathbf{b}$ (A is a strictly diagonal dominant matrix),

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

$$x_1^{(k+1)} = \frac{1}{a_{11}}\left(b_1 - a_{12}x_2^{(k)} - a_{13}x_3^{(k)}\right)$$

$$x_2^{(k+1)} = \frac{1}{a_{22}}\left(b_2 - a_{21}x_1^{(k+1)} - a_{23}x_3^{(k)}\right)$$

$$x_3^{(k+1)} = \frac{1}{a_{33}}\left(b_3 - a_{31}x_1^{(k+1)} - a_{32}x_2^{(k+1)}\right)$$

1.10.1 Gauss Seidel Iterative Sequence

Example:

Solve the following linear system by using Gauss Seidel.

$$12x_1 + 3x_2 - x_3 = 15$$
$$2x_1 - x_2 + 10x_3 = 30$$
$$x_1 + 8x_2 + x_3 = 20$$

Start the initial guess with $\mathbf{x}^{(0)} = \mathbf{0}$ and stop the iteration when $\left\|\mathbf{x}^{(k)} - \mathbf{x}^{(k-1)}\right\|_\infty < 0.001$

Solution:

Check for strictly diagonal dominant:

$$\begin{aligned} 12x_1 + 3x_2 - x_3 &= 15 \\ 2x_1 - x_2 + 10x_3 &= 30 \\ x_1 + 8x_2 + x_3 &= 20 \end{aligned} \quad \xrightarrow{r_2 \leftrightarrow r_3} \quad \begin{aligned} 12x_1 + 3x_2 - x_3 &= 15 \\ x_1 + 8x_2 + x_3 &= 20 \\ 2x_1 - x_2 + 10x_3 &= 30 \end{aligned}$$

Iterative Sequence:

$$x_1^{(k+1)} = \frac{1}{12}\left(15 - 3x_2^{(k)} + x_3^{(k)}\right)$$

$$x_2^{(k+1)} = \frac{1}{8}\left(20 - x_1^{(k+1)} - x_3^{(k)}\right)$$

$$x_3^{(k+1)} = \frac{1}{10}\left(30 - 2x_1^{(k+1)} + x_2^{(k+1)}\right)$$

Solution:

$$\max_{1\le i\le n}\left\{x_i^{(k)} - x_i^{(k-1)}\right\}$$

k	$x_1^{(k)}$	$x_2^{(k)}$	$x_3^{(k)}$	$\left\lVert \mathbf{x}^{(k)} - \mathbf{x}^{(k-1)} \right\rVert_\infty$
0	0	0	0	
1	1.2500	2.3438	2.9844	2.9844
2	0.9128	2.0129	3.0187	0.3372
3	0.9983	1.9979	3.0001	0.0855
4	1.0005	1.9999	2.9999	0.0022
5	1.0000	2.0000	3.0000	0.0005

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

1.10.1 Gauss Seidel Iterative Sequence

Example:

Solve the following linear system by using Gauss Seidel.

$$0.3x_1 - 0.2x_2 + 10x_3 = 71.4$$
$$3x_1 - 0.1x_2 - 0.2x_3 = 7.85$$
$$0.1x_1 + 7x_2 - 0.3x_3 = -19.3$$

Start the initial guess with $\mathbf{x}^{(0)} = \mathbf{0}$ and stop the iteration when $\left\|\mathbf{x}^{(k)} - \mathbf{x}^{(k-1)}\right\|_\infty < 0.0005$

Solution:

Check for strictly diagonal dominant:

$$3x_1 - 0.1x_2 - 0.2x_3 = 7.85$$
$$0.1x_1 + 7x_2 - 0.3x_3 = -19.3$$
$$0.3x_1 - 0.2x_2 + 10x_3 = 71.4$$

Iterative Sequence:

$$x_1^{(k+1)} = \frac{1}{3}\left(7.85 + 0.1x_2^{(k)} + 0.2x_3^{(k)}\right)$$
$$x_2^{(k+1)} = \frac{1}{7}\left(-19.3 - 0.1x_1^{(k+1)} + 0.3x_3^{(k)}\right)$$
$$x_3^{(k+1)} = \frac{1}{10}\left(71.4 - 0.3x_1^{(k+1)} + 0.2x_2^{(k+1)}\right)$$

Solution:

$\max_{1\le i\le n} \left\{x_i^{(k)} - x_i^{(k-1)}\right\}$

k	$x_1^{(k)}$	$x_2^{(k)}$	$x_3^{(k)}$	$\left\lVert \mathbf{x}^{(k)} - \mathbf{x}^{(k-1)} \right\rVert_\infty$
0	0	0	0	
1	2.6167	-2.7945	7.0056	7.0056
2	2.9906	-2.4996	7.0003	0.3739
3	3.0000	-2.5000	7.0000	0.0094
4	3.0000	-2.5000	7.0000	0

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ -2.5 \\ 7 \end{bmatrix}$$

Exercise 1.8:

Solve the following linear system by using Gauss Seidel.

$$-x_1 + x_2 + 7x_3 = -6$$
$$4x_1 - x_2 - x_3 = 3$$
$$-2x_1 + 6x_2 + x_3 = 9$$

Start the initial guess with $\mathbf{x}^{(0)} = \mathbf{0}$ and stop the iteration when $\left\|\mathbf{x}^{(k)} - \mathbf{x}^{(k-1)}\right\|_\infty < 0.001$

[Ans: $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$]