ENGINEERING MATHEMATICS 1
BMFG 1313
MATRICES

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Lesson Outcome

Upon completion of this lesson, the student should be able to:

1. Apply basic operations of a matrix.
2. Compute determinant of a matrix.
3. Compute inverse matrix

1.1 Introduction

WHY WE NEED MATRICES?

In general, matrices are used as a **notation** that represents simplified form of a linear system problem

1.1 Introduction

A matrix with *m* rows and *n* columns has entries $a_{ij}, i = 1,2, \dots m, j = 1,2, \dots, n$ as follows:

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}_{m\times n}$$

Order of matrix

When

- $m = n$: Square matrix of order n
- $n = 1$: Column Vector, i.e. $\boldsymbol{B} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$
- $m = 1$: Row vector, i.e. $\boldsymbol{C} = [c_1 \quad c_2 \quad \cdots \quad c_n]$

1.1 Introduction

Symmetric matrix:

An n x n matrix $\boldsymbol{A}$ is a symmetric matrix if $\boldsymbol{A}^{\mathrm{T}} = \boldsymbol{A}$, i.e. $a_{ij} = a_{ji}$

$$\text{e.g.} \quad \boldsymbol{A} = \begin{bmatrix} 1 & -3 & 5 \\ -3 & 2 & 7 \\ 5 & 7 & 0 \end{bmatrix}$$

Skew-symmetric or antisymmetric matrix:

An n x n matrix $\boldsymbol{A}$ is known as antisymmetric matrix if $\boldsymbol{A}^{\mathrm{T}} = -\boldsymbol{A}$, i.e. $a_{ij} = -a_{ji}$

$$\text{e.g.} \quad \boldsymbol{A} = \begin{bmatrix} 0 & -4 & 6 \\ 4 & 0 & -7 \\ -6 & 7 & 0 \end{bmatrix}$$

1.1 Introduction

Diagonal matrix:

the entries other than main diagonal are all zeros

$$\text{e.g.} \quad \boldsymbol{B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix} \text{ and } \boldsymbol{C} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Identity matrix:

the entries are all zeros except $a_{ii} = 1, i = 1,2, \ldots n$

$$\text{e.g. } \boldsymbol{I}_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ and } \boldsymbol{I}_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

1.2 Basic Operation on Vectors and Matrices

a) Equality

Two matrices are equal if and only if all their elements are the same including their order.

$$\boldsymbol{A} = \boldsymbol{B}$$

b) Addition and Subtraction

$\boldsymbol{A} + \boldsymbol{B}$ and $\boldsymbol{A} - \boldsymbol{B}$ are defined only when $\boldsymbol{A}$ and $\boldsymbol{B}$ are the same order.

$\boldsymbol{A} + \boldsymbol{B}$ has elements $a_{ij} + b_{ij}$ and $\boldsymbol{A} - \boldsymbol{B}$ has elements $a_{ij} - b_{ij}$.

e.g.

$$\begin{bmatrix} 3 & -5 & 4 \\ -1 & 6 & 0 \end{bmatrix} + \begin{bmatrix} 2 & 3 & -4 \\ 0 & 5 & 1 \end{bmatrix} = \begin{bmatrix} 3+2 & -5+3 & 4+(-4) \\ -1+0 & 6+5 & 0+1 \end{bmatrix} = \begin{bmatrix} 5 & -2 & 0 \\ -1 & 11 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 5 & -1 \\ 4 & 6 \\ 2 & 3 \end{bmatrix} - \begin{bmatrix} 2 & -4 \\ 3 & 7 \\ -7 & 8 \end{bmatrix} = \begin{bmatrix} 5-2 & -1-(-4) \\ 4-3 & 6-7 \\ 2-(-7) & 3-8 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 1 & -1 \\ 9 & -5 \end{bmatrix}$$

1.2 Basic Operation on Vectors and Matrices

c) Multiplication by a scalar

Scalar *c* is multiplied to each of the elements of matrix.

e.g.

$$3\begin{bmatrix}-3 & 2 & 6\end{bmatrix} = \begin{bmatrix}3(-3) & 3(2) & 3(6)\end{bmatrix} = \begin{bmatrix}-9 & 6 & 18\end{bmatrix}$$

$$-2\begin{bmatrix}0 & 3 & -1\\ -4 & 2 & 6\\ 5 & -3 & 7\end{bmatrix} = \begin{bmatrix}-2(0) & -2(3) & -2(-1)\\ -2(-4) & -2(2) & -2(6)\\ -2(5) & -2(-3) & -2(7)\end{bmatrix} = \begin{bmatrix}0 & -6 & 2\\ 8 & -4 & -12\\ -10 & 6 & -14\end{bmatrix}$$

1.2 Basic Operation on Vectors and Matrices

d) Properties of the transpose matrix

i) $\left(\boldsymbol{A}^{\mathrm{T}}\right)^{\mathrm{T}} = \boldsymbol{A}$

ii) $(\boldsymbol{A} + \boldsymbol{B})^{\mathrm{T}} = \boldsymbol{A}^{\mathrm{T}} + \boldsymbol{B}^{\mathrm{T}}$

For example:

$$\left(\boldsymbol{A}^{\mathrm{T}} + \boldsymbol{A}\right)^{\mathrm{T}} = \left(\boldsymbol{A}^{\mathrm{T}}\right)^{\mathrm{T}} + \boldsymbol{A}^{\mathrm{T}} = \boldsymbol{A} + \boldsymbol{A}^{\mathrm{T}}$$

and this shows $\boldsymbol{A}^{\mathrm{T}} + \boldsymbol{A}$ must be a symmetric matrix.

1.2 Basic Operation on Vectors and Matrices

e) Basic Rules of Addition

If matrices $\boldsymbol{A}, \boldsymbol{B}$ and $\boldsymbol{C}$ have the same order:

$\boldsymbol{A} + \boldsymbol{B} = \boldsymbol{B} + \boldsymbol{A}$ (Commutative law)

$(\boldsymbol{A} + \boldsymbol{B}) + \boldsymbol{C} = \boldsymbol{A} + (\boldsymbol{B} + \boldsymbol{C})$ (Associative law)

$r(\boldsymbol{A} + \boldsymbol{B}) = r\boldsymbol{A} + r\boldsymbol{B}$ (Distributive law)

1.2 Basic Operation on Vectors and Matrices

Exercise 1.1:

Let $\boldsymbol{A} = \begin{bmatrix} 2 & -1 & 3 \\ 7 & 5 & 0 \\ -2 & 8 & 1 \end{bmatrix}$, $\boldsymbol{B} = \begin{bmatrix} 1 & 0 & 5 \\ -2 & 4 & 6 \\ 3 & 7 & -2 \end{bmatrix}$, $\boldsymbol{C} = \begin{bmatrix} 4 & 3 & 0 \\ -3 & 6 & -6 \end{bmatrix}$, $\boldsymbol{D} = \begin{bmatrix} 5 & 3 \\ 7 & -2 \\ 1 & 0 \end{bmatrix}$

Find

1) $\boldsymbol{A}^{\mathrm{T}} + 2\boldsymbol{B}$

2) $\boldsymbol{B} - 5\boldsymbol{C}$

3) $6\left(\boldsymbol{D}^{\mathrm{T}}\right) - 2\boldsymbol{C}$

[Ans: $\begin{bmatrix} 4 & 7 & 8 \\ -5 & 13 & 20 \\ 9 & 14 & -3 \end{bmatrix}$; undefined; $\begin{bmatrix} 22 & 36 & 6 \\ 24 & -24 & 12 \end{bmatrix}$]

1.3 Properties of Matrix Multiplication

Matrix multiplication:

Given matrix ***A*** with order $p \times q$ and matrix ***B*** with order $q \times r$,

product of ***AB*** = ***C*** has an order of $p \times r$.

e.g.

Given $\boldsymbol{A}_{2\times3}$ and $\boldsymbol{B}_{3\times5}$,

The order of matrix ***C*** = ***AB*** is 2×5,

but ***BA*** is undefined.

1.3 Properties of Matrix Multiplication

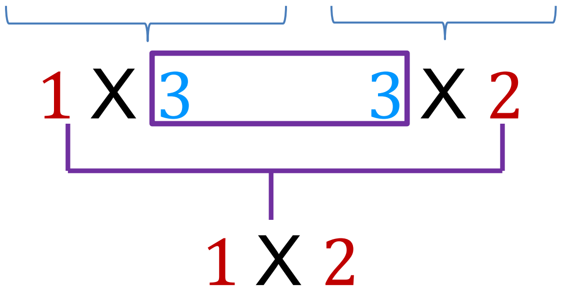
Matrix multiplication:

To multiply two matrices, take the row of the first matrix multiply to the column of the second matrix.

i.e. $\text{row}_1 \times \text{column}_2$ gives the value of a_{12}

For example:

$$\begin{bmatrix} 1 & 3 & 2 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 4 & -1 \\ 1 & 2 \end{bmatrix}$$



$$= \begin{bmatrix} 1(3) + 3(4) + 2(1) & 1(0) + 3(-1) + 2(2) \end{bmatrix} = \begin{bmatrix} 17 & 1 \end{bmatrix}$$

1.3 Properties of Matrix Multiplication

Matrix multiplication:

For example:

$\text{row}_2 \times \text{column}_3$ gives the value of $a_{23} = 4$

$$\begin{bmatrix} 2 & 3 & -1 \\ 0 & 5 & 2 \\ -1 & 6 & 4 \end{bmatrix}\begin{bmatrix} 3 & 0 & 8 \\ 4 & -1 & 2 \\ 1 & 2 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 2(3)+3(4)+(-1)(1) & 2(0)+3(-1)+(-1)(2) & 2(8)+3(2)+(-1)(-3) \\ 0(3)+5(4)+2(1) & 0(0)+5(-1)+2(2) & 0(8)+5(2)+2(-3) \\ -1(3)+6(4)+4(1) & -1(0)+6(-1)+4(2) & -1(8)+6(2)+4(-3) \end{bmatrix}$$

$$= \begin{bmatrix} 17 & -5 & 25 \\ 22 & -1 & 4 \\ 25 & 2 & -8 \end{bmatrix}$$

1.3 Properties of Matrix Multiplication

Some Properties:

Let $A \in M_{m\times n}$, let B and C have orders for which the indicated sums and products are defined.

- $\boldsymbol{A}(\boldsymbol{BC}) = (\boldsymbol{AB})\boldsymbol{C}$ (associative law of multiplication)
- $\boldsymbol{A}(\boldsymbol{B}+\boldsymbol{C}) = \boldsymbol{AB}+\boldsymbol{AC}$ (left distributive law)
- $(\boldsymbol{B}+\boldsymbol{C})\boldsymbol{A} = \boldsymbol{BA}+\boldsymbol{CA}$ (right distributive law)
- $r(\boldsymbol{AB}) = (r\boldsymbol{A})\boldsymbol{B} = \boldsymbol{A}(r\boldsymbol{B})$ for any scalar r
- $\boldsymbol{I}_m\boldsymbol{A} = \boldsymbol{A} = \boldsymbol{A}\boldsymbol{I}_n$ (identity for matrix multiplication)
- $(\boldsymbol{ABC})^{\mathrm{T}} = \boldsymbol{C}^{\mathrm{T}}\boldsymbol{B}^{\mathrm{T}}\boldsymbol{A}^{\mathrm{T}}$ (Transpose of a product)
- $\boldsymbol{A}^k = \boldsymbol{A} \dots \boldsymbol{A}$ for *k* times (Power of a matrix)

Warnings:

- $\boldsymbol{AB} \neq \boldsymbol{BA}$
- The cancellation laws do not hold for matrix multiplication.
  i.e., If $\boldsymbol{AB} = \boldsymbol{AC},\ \boldsymbol{B} \neq \boldsymbol{C}$ in general.
- If $\boldsymbol{AB} = \boldsymbol{0}_{m\times n}$, we cannot conclude either $\boldsymbol{A} = \boldsymbol{0}$ or $\boldsymbol{B} = \boldsymbol{0}$.

1.3 Properties of Matrix Multiplication

Exercise 1.2:

Given $\boldsymbol{A} = \begin{bmatrix} 2 & -1 & 3 \\ 7 & 5 & 0 \\ -2 & 8 & 1 \end{bmatrix}$, $\boldsymbol{B} = \begin{bmatrix} 1 & 0 & 5 \\ -2 & 4 & 6 \\ 3 & 7 & -2 \end{bmatrix}$, $\boldsymbol{C} = \begin{bmatrix} 2 & -1 & 0 \\ 1 & 2 & -6 \\ 3 & 0 & 5 \end{bmatrix}$,

Find

1) $3\boldsymbol{BC}$

2) $(\boldsymbol{AB})^{\mathrm{T}} + 2\boldsymbol{C}$

3) Verify Associative law of multiplication

[Ans: $\begin{bmatrix} 51 & -3 & 75 \\ 54 & 30 & 18 \\ 21 & 33 & -156 \end{bmatrix}$; $\begin{bmatrix} 17 & -5 & -15 \\ 19 & 24 & 27 \\ 4 & 65 & 46 \end{bmatrix}$]

1.4 Determinants

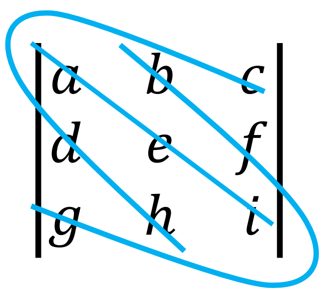
Computation of determinant, **Method 1**:

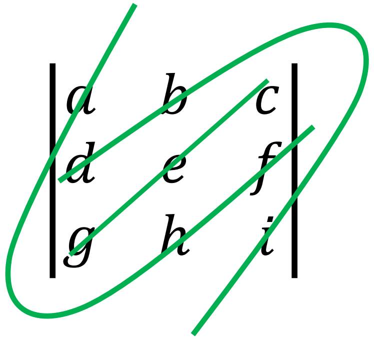
For a matrix with order 2×2,

i.e. $$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

For a matrix with order 3×3,

i.e. $$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = (aei + bfg + cdh) - (ceg + bdi + afh)$$





1.4 Determinants

Method 2:

Determinant of an $n \times n$ matrix $\boldsymbol{A}$ is denoted by $|\boldsymbol{A}|$ and it is computed by

Cofactors Expansion along a row:

$$|\boldsymbol{A}| = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \boldsymbol{M}_{ij}$$

i.e. $i = 2$ gives cofactor expansion along the second row

or Cofactors Expansion along a column:

$$|\boldsymbol{A}| = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \boldsymbol{M}_{ij}$$

i.e. $j = 3$ gives cofactor expansion along the third column

where a_{ij} is the entry of matrix $\boldsymbol{A}$ and

$\boldsymbol{M}_{ij}$ is known as minor.

1.4 Determinants

Minor of a matrix, $\boldsymbol{M}_{ij}$:

Minor, $\boldsymbol{M}_{ij}$, of an $n \times n$ matrix $\boldsymbol{A}$ is the determinant of $(n-1) \times (n-1)$ matrix formed from $\boldsymbol{A}$ by deleting the row and column that contains a_{ij}.

Example:

Given $\boldsymbol{A} = \begin{bmatrix} 2 & -1 & 3 \\ 6 & 5 & 8 \\ -4 & 7 & 1 \end{bmatrix}$, find the minor $\boldsymbol{M}_{32}$.

Solution:

Delete the row and column that contains $a_{32} = 7$:

$$\boldsymbol{A} = \begin{bmatrix} 2 & -1 & 3 \\ 6 & 5 & 8 \\ -4 & 7 & 1 \end{bmatrix}$$

$$\boldsymbol{M}_{32} = \begin{vmatrix} 2 & 3 \\ 6 & 8 \end{vmatrix} = 16 - 18 = -2$$

1.4 Determinants

The sign associated with the minor is given as follows:

$$A = \begin{bmatrix} + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{bmatrix}$$

Alternating signs start with "+" at a_{11}

A minor multiplied by the appropriate sign is known as cofactor, $\boldsymbol{A}_{ij}$.

So, $\boldsymbol{A}_{ij} = (-1)^{i+j}\boldsymbol{M}_{ij}$

e.g. Given $\boldsymbol{A} = \begin{bmatrix} 2 & -1 & 3 \\ 6 & 5 & 8 \\ -4 & 7 & 1 \end{bmatrix}$,

$$\boldsymbol{A}_{21} = (-1)\begin{vmatrix} -1 & 3 \\ 7 & 1 \end{vmatrix} = 22$$

$$\boldsymbol{A}_{13} = (+1)\begin{vmatrix} 6 & 5 \\ -4 & 7 \end{vmatrix} = 62$$

Example:

$$|\boldsymbol{A}| = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \boldsymbol{M}_{ij}$$

$$A = \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

Find the determinant of matrix $\boldsymbol{A} = \begin{bmatrix} 2 & -1 & 3 \\ 7 & 5 & 0 \\ -2 & 8 & 1 \end{bmatrix}$.

Solution:

Cofactor expansion across the first row ($i = 1$):

$$|\boldsymbol{A}| = (+1)(2)\begin{vmatrix} 5 & 0 \\ 8 & 1 \end{vmatrix} + (-1)(-1)\begin{vmatrix} 7 & 0 \\ -2 & 1 \end{vmatrix} + (+1)(3)\begin{vmatrix} 7 & 5 \\ -2 & 8 \end{vmatrix}$$
$$= 2(5) + 1(7) + 3(66)$$
$$= 215$$

Cofactor expansion across the second column ($j = 2$):

$$|\boldsymbol{A}| = -(-1)\begin{vmatrix} 7 & 0 \\ -2 & 1 \end{vmatrix} + 5\begin{vmatrix} 2 & 3 \\ -2 & 1 \end{vmatrix} - 8\begin{vmatrix} 2 & 3 \\ 7 & 0 \end{vmatrix}$$
$$= 1(7) + 5(8) - 8(-21)$$
$$= 215 \text{ (same answer)}$$

1.4 Determinants

Properties of determinants: (Method 3)

Theorem 1:

If $\boldsymbol{A}$ is a triangular matrix, then $|\boldsymbol{A}| = a_{11}a_{22}a_{33}\ldots a_{nn}$.

e.g.

$$\begin{vmatrix} 1 & 2 & 0 & 1 \\ 0 & 3 & 1 & 1 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 1 \end{vmatrix} = (1)(3)(2)(1) = 6$$

Upper triangular

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 1 & 3 & 2 & 0 \\ 2 & 1 & 1 & 4 \end{vmatrix} = (1)(-1)(2)(4) = -8$$

Lower triangular

1.4 Determinants

Properties of determinants:

Theorem 2:

Let A be a square matrix.

a) If a multiple of one row of $\boldsymbol{A}$ is added to another row to produce a matrix $\boldsymbol{B}$, then $|\boldsymbol{B}| = |\boldsymbol{A}|$.

e.g. $\begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix}_{-2r_1+r_2} = \begin{vmatrix} 1 & 2 \\ 0 & -1 \end{vmatrix}$

b) If two rows of $\boldsymbol{A}$ are interchanged to produce $\boldsymbol{B}$, then $|\boldsymbol{B}| = -|\boldsymbol{A}|$.

e.g. $\begin{vmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 2 & 3 & 0 \end{vmatrix}_{r_1 \leftrightarrow r_2} = -\begin{vmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 2 & 3 & 0 \end{vmatrix}$

c) If one row of $\boldsymbol{A}$ multiplied by k to produce $\boldsymbol{B}$, then $|\boldsymbol{B}| = k|\boldsymbol{A}|$.

e.g. $\begin{vmatrix} 5 & 2 \\ 3 & 6 \end{vmatrix} = 3\begin{vmatrix} 5 & 2 \\ 1 & 2 \end{vmatrix}$, $\begin{vmatrix} 2 & 12 \\ 4 & 3 \end{vmatrix} = 2\begin{vmatrix} 1 & 6 \\ 4 & 3 \end{vmatrix} = (2)(3)\begin{vmatrix} 1 & 2 \\ 4 & 1 \end{vmatrix}$

1.4 Determinants

Properties of determinants:

Theorem 3:

If A is an $n \times n$ matrix, $\left|\boldsymbol{A}^{\mathrm{T}}\right| = |\boldsymbol{A}|$.

$$\text{e.g. } \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = \begin{vmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{vmatrix}$$

Theorem 4:

If two rows (columns) of $\boldsymbol{A}$ are equal, then $|\boldsymbol{A}| = 0$.

$$\text{e.g. } \begin{vmatrix} 1 & 0 & 3 & 1 \\ 1 & 0 & 5 & 1 \\ 2 & 1 & 7 & 2 \\ 1 & 0 & 1 & 1 \end{vmatrix} = 0, \quad \begin{vmatrix} 2 & 5 & 8 & 3 \\ 1 & 0 & 1 & 0 \\ 5 & 8 & 4 & 6 \\ 1 & 0 & 1 & 0 \end{vmatrix} = 0.$$

1.4 Determinants

Properties of determinants:

Theorem 5:

If a row (column) of $\boldsymbol{A}$ consists entirely of zeroes, then $|\boldsymbol{A}| = 0$.

e.g. $\begin{vmatrix} 1 & 0 & 1 & 5 \\ 4 & 0 & 2 & 2 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 3 & 4 \end{vmatrix} = 0,\quad \begin{vmatrix} 1 & 2 & 5 \\ 4 & -2 & 7 \\ 0 & 0 & 0 \end{vmatrix} = 0.$

Theorem 6:

- If $\boldsymbol{A}$ and $\boldsymbol{B}$ are $n \times n$ matrices, $|\boldsymbol{AB}| = |\boldsymbol{A}||\boldsymbol{B}|$.
- If $\boldsymbol{A}$ is an $n \times n$ matrix, then $\boldsymbol{A}$ is invertible or nonsingular matrix iff $|\boldsymbol{A}| \neq 0$.
- $|\boldsymbol{A} + \boldsymbol{B}| \neq |\boldsymbol{A}| + |\boldsymbol{B}|$ in general.

1.4 Determinants

Exercise 1.3:

Evaluate the following determinants:

1) $\begin{vmatrix} 5 & 2 & 0 \\ 0 & -2 & 5 \\ 0 & 0 & 4 \end{vmatrix}$

2) $\begin{vmatrix} 4 & 2 & 2 & 0 \\ 2 & 0 & 0 & 0 \\ 3 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{vmatrix}$

3) $\begin{vmatrix} 1 & -3 & 1 & -2 \\ 2 & -5 & -1 & -2 \\ 0 & -4 & 5 & 1 \\ -3 & 10 & -6 & 8 \end{vmatrix}$ by using cofactor expansion across third column.

[Ans: - 40; 4; 0]

1.4 Determinants

Exercise 1.3:

Evaluate the following determinants:

4) Given $\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 7$, find

a) $\begin{vmatrix} a & b & c \\ 3d & 3e & 3f \\ g & h & i \end{vmatrix}$ b) $\begin{vmatrix} a+d & b+e & c+f \\ d & e & f \\ g & h & i \end{vmatrix}$

5) $\begin{vmatrix} 1 & 3 & 2 \\ -2 & 3 & -4 \\ 5 & 5 & 6 \end{vmatrix}$

[Ans: 21; 7; -36]

1.5 Inverse of a Matrix

Given a matrix $\boldsymbol{A}$,

if $\boldsymbol{BA} = \boldsymbol{AB} = \boldsymbol{I}$, it means that $\boldsymbol{B}$ is the inverse of $\boldsymbol{A}$ and hence,

$$\boldsymbol{B} = \boldsymbol{A}^{-1}$$

To compute an inverse from a matrix,

$$\boldsymbol{A}^{-1} = \frac{1}{|\boldsymbol{A}|} \text{adj } \boldsymbol{A}$$

where $|\boldsymbol{A}| \neq 0$ ($\boldsymbol{A}$ is a nonsingular matrix) and adj $\boldsymbol{A}$ is an adjoint matrix of $\boldsymbol{A}$ formed by transpose matrix which consists of cofactors of each of the elements in $\boldsymbol{A}$.

1.5 Inverse of a Matrix

Example:

Given $\boldsymbol{A} = \begin{bmatrix} 3 & -2 & 1 \\ 5 & 6 & 2 \\ 1 & 0 & -3 \end{bmatrix}$, find the inverse matrix of $\boldsymbol{A}$.

Solution:

Step 1: Find the determinant of matrix

$$\begin{vmatrix} 3 & -2 & 1 \\ 5 & 6 & 2 \\ 1 & 0 & -3 \end{vmatrix} = [3(6)(-3) + (-2)(2)(1) + 5(0)(1)]$$

$$-[1(6)(1) + 5(-2)(-3) + 3(2)(0)]$$

$$= -94$$

1.5 Inverse of a Matrix

Solution:

*Step 2: Find adj **A**, which is the transpose of cofactor matrix*

$$\text{adj } \mathbf{A} = \begin{bmatrix} +\begin{vmatrix} 6 & 2 \\ 0 & -3 \end{vmatrix} & -\begin{vmatrix} 5 & 2 \\ 1 & -3 \end{vmatrix} & +\begin{vmatrix} 5 & 6 \\ 1 & 0 \end{vmatrix} \\ -\begin{vmatrix} -2 & 1 \\ 0 & -3 \end{vmatrix} & +\begin{vmatrix} 3 & 1 \\ 1 & -3 \end{vmatrix} & -\begin{vmatrix} 3 & -2 \\ 1 & 0 \end{vmatrix} \\ +\begin{vmatrix} -2 & 1 \\ 6 & 2 \end{vmatrix} & -\begin{vmatrix} 3 & 1 \\ 5 & 2 \end{vmatrix} & +\begin{vmatrix} 3 & -2 \\ 5 & 6 \end{vmatrix} \end{bmatrix}^T$$

$$= \begin{bmatrix} +(-18) & -(-17) & +(-6) \\ -(6) & +(-10) & -(2) \\ +(-10) & -(1) & +(28) \end{bmatrix}^T = \begin{bmatrix} -18 & 17 & -6 \\ -6 & -10 & -2 \\ -10 & -1 & 28 \end{bmatrix}^T = \begin{bmatrix} -18 & -6 & -10 \\ 17 & -10 & -1 \\ -6 & -2 & 28 \end{bmatrix}$$

1.5 Inverse of a Matrix

Solution:

Step 3: Find inverse matrix $\boldsymbol{A}^{-1} = \frac{1}{|\boldsymbol{A}|}\text{adj }\boldsymbol{A}$

$$\boldsymbol{A}^{-1} = \frac{1}{-94}\begin{bmatrix} -18 & -6 & -10 \\ 17 & -10 & -1 \\ -6 & -2 & 28 \end{bmatrix} = \begin{bmatrix} \frac{9}{47} & \frac{3}{47} & \frac{5}{47} \\ -\frac{17}{94} & \frac{5}{47} & \frac{1}{94} \\ \frac{3}{47} & \frac{1}{47} & -\frac{14}{47} \end{bmatrix}$$

1.5 Inverse of a Matrix

Exercise 1.4:

Find the inverse of the following matrices:

1) $\boldsymbol{A} = \begin{bmatrix} 2 & 4 \\ 5 & 10 \end{bmatrix}$

2) $\boldsymbol{B} = \begin{bmatrix} -1 & 2 \\ 3 & 5 \end{bmatrix}$

3) $\boldsymbol{C} = \begin{bmatrix} 2 & 1 & 3 \\ 1 & -1 & 1 \\ 1 & 4 & -2 \end{bmatrix}$

4) $\boldsymbol{D} = \begin{bmatrix} 5 & 0 & 3 \\ 6 & 4 & -2 \\ 1 & 0 & -3 \end{bmatrix}$

[Ans: no inverse; $\begin{bmatrix} -5/11 & 2/11 \\ 3/11 & 1/11 \end{bmatrix}$; $\begin{bmatrix} -1/7 & 1 & 2/7 \\ 3/14 & -1/2 & 1/14 \\ 5/14 & -1/2 & -3/14 \end{bmatrix}$; $\begin{bmatrix} 1/6 & 0 & 1/6 \\ -2/9 & 1/4 & -7/18 \\ 1/18 & 0 & -5/18 \end{bmatrix}$]