BMCG 1013
DIFFERENTIAL EQUATIONS

APPLICATION OF LAPLACE TRANSFORM

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Lesson outcomes

Upon completion of this week lesson, students should be able to:

i. Use Laplace transform to solve the initial value problem

ii. Use Laplace transform to solve the transfer function in a control system

3.4 Application of Laplace transform

- Solving an initial value problem
  - Transform of derivatives
  - Solve the initial value problem
- Solving a transfer function

3.4.1 Solving an initial value problem

$y^{(n)}(t)$, $n=1,2,3,\ldots$	$s^n Y(s) - s^{n-1}y(0) - s^{n-2}y'(0)\ldots - sy^{(n)}(0) - y^{(n-1)}(0)$

* Refer to the table of Laplace transform table

Let $Y(s) = L\{f(t)\}$

Substitute $n=1$

First derivative, $n=1$

$$L\{y'(t)\} = sY(s) - y(0)$$

$y^{(n)}(t)$, $n=1,2,3,....$	$s^nY(s)-s^{n-1}y(0)-s^{n-2}y'(0)...-sy^{(n)}(0)-y^{(n-1)}(0)$

* Refer to the table of Laplace transform table

Let $Y(s) = L\{f(t)\}$

Substitute $n = 2$

Second derivative, $n = 2$

$$L\{y''(t)\} = s^2Y(s) - sy(0) - y'(0)$$

Example 3.32:

Determine the $Y(s)$ for the following initial value problem:

$$\frac{d^2y}{dt^2}+3\frac{dy}{dt}+2y=0, \quad y(0)=0, y'(0)=3$$

Solution:

$$L\left\{\frac{d^2y}{dx^2}+3\frac{dy}{dx}+2y\right\}=L\{0\}$$

Initial conditions:
$y(0)=0,\ y'(0)=3$

$$s^2Y(s)-sy(0)-y'(0)+3\big(sY(s)-y(0)\big)+2Y(s)=0$$

$$s^2Y(s)-s(0)-3+3\big(sY(s)-0\big)+2Y(s)=0$$

$$\left(s^2+3s+2\right)Y(s)=3$$

$$Y(s)=\frac{3}{s^2+3s+2}$$

Example 3.33:

Solve following IVP by using the Laplace transform:

$$\frac{d^2y}{dt^2}+3\frac{dy}{dt}+2y=0, \quad y(0)=0,\ y'(0)=3$$

By referring to the previous example, we know

$$Y(s) = \frac{3}{s^2 + 3s + 2}$$

Then

$$y(t) = L^{-1}\{f(t)\} = L^{-1}\left\{\frac{3}{s^2 + 3s + 2}\right\}$$

$$= L^{-1}\left\{\frac{3}{(s+1)(s+2)}\right\}$$

Let $Y(s) = F(s)G(s) = \dfrac{3}{(s+1)(s+2)}$

Now choose

$$F(s) = \frac{3}{s+2},\ G(s) = \frac{1}{s+1}$$

Then

$$f(t) = L^{-1}\{F(s)\} = L^{-1}\left\{\frac{3}{s+2}\right\} = 3e^{-2t}$$

$$g(t) = L^{-1}\{G(s)\} = L^{-1}\left\{\frac{1}{s+1}\right\} = e^{-t}$$

So

$$f(\tau) = 3e^{-2\tau}, \; g(t-\tau) = e^{-(t-\tau)}$$

Determine $y(t)$ by using the convolution theorem:

$$y(t) = L^{-1}\{Y(s)\} = L^{-1}\{F(s)G(s)\}$$

$$= \int_0^t f(\tau)g(t-\tau)d\tau$$

$$= \int_0^t 3e^{-2\tau} \cdot e^{-(t-\tau)} \, d\tau$$

$$= \int_0^t 3e^{-t} \cdot e^{-\tau} \, d\tau$$

$$= 3e^{-t}\left(-e^{-\tau}\right)_0^t$$

$$= 3e^{-t}\left(-e^{-t} - (-e^{-0})\right)$$

$$= 3\left(e^{-t} - e^{-2t}\right)$$

*You may double check your answer by solving the homogeneous equation with characteristic equation as well.

Example 3.34:

Solve following IVP by using the Laplace transform:

$$\frac{d^2y}{dt^2}+2\frac{dy}{dt}+y=t, \quad y(0)=0,\ y'(0)=0$$

Solution:

Initial conditions:

$y(0) = 0,\ y'(0) = 0$

$$L\left\{\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y\right\} = L\{t\}$$

$$s^2Y(s) - sy(0) - y'(0) + 2\big(sY(s) - y(0)\big) + Y(s) = \frac{1}{s^2}$$

$$s^2Y(s) - s(0) - 0 + 2\big(sY(s) - 0\big) + Y(s) = \frac{1}{s^2}$$

$$\left(s^2 + 2s + 1\right)Y(s) = \frac{1}{s^2}$$

$$Y(s) = \frac{1}{s^2\left(s^2 + 2s + 1\right)} = \frac{1}{s^2(s+1)^2}$$

Then

$$y(t) = L^{-1}\{f(t)\} = L^{-1}\left\{\frac{1}{s^2(s+1)^2}\right\}$$

Let

$$Y(s) = F(s)G(s) = \frac{1}{s^2(s+1)^2}$$

Choose

$$F(s) = \frac{1}{(s+1)^2},\ G(s) = \frac{1}{s^2}$$

Then

$$f(t) = L^{-1}\{F(s)\} = L^{-1}\left\{\frac{1}{(s+1)^2}\right\} = te^{-t}$$

$$g(t) = L^{-1}\{G(s)\} = L^{-1}\left\{\frac{1}{s^2}\right\} = t$$

So

$$f(\tau) = te^{-\tau},\ g(t-\tau) = t-\tau$$

Determine the $y(t)$ by using the convolution theorem:

$$y(t) = L^{-1}\{Y(s)\} = L^{-1}\{F(s)G(s)\}$$

$$= \int_0^t f(\tau)g(t-\tau)d\tau$$

$$= \int_0^t \tau e^{-\tau} \cdot (t-\tau)\, d\tau$$

$$= \int_0^t t\tau e^{-\tau}\, d\tau - \int_0^t \tau^2 e^{-\tau}\, d\tau$$

$$= \left(-t\tau e^{-\tau} - te^{-\tau}\right)_0^t - \left(-\tau^2 e^{-\tau} - 2\tau e^{-\tau} - 2e^{-\tau}\right)_0^t$$

$$= \left(-t^2 e^{-t} - te^{-t} - (-te^0)\right) + \left(t^2 e^{-t} + 2te^{-t} + 2e^{-t} - 2\right)$$

$$= te^{-t} + 2e^{-t} + t - 2$$

Exercise 3.13

Find the solution for

$$\frac{d^2y}{dt^2}+4\frac{dy}{dt}+4y=t^2,\quad y(0)=0,\ y'(0)=0$$

by using the Laplace transform.

Answer:

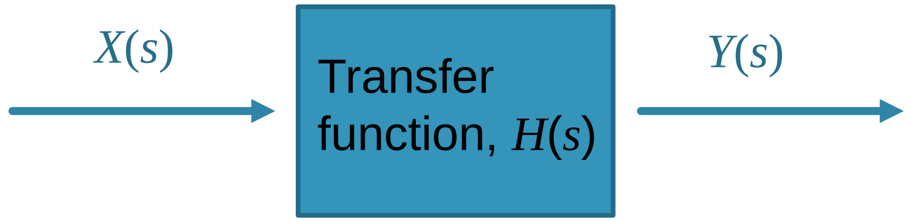
$$y(t)=-e^{-2t}\left(\frac{t}{4}+\frac{3}{8}\right)+\frac{t^2}{4}-\frac{t}{2}+\frac{3}{8}$$

3.4.2 Solving a transfer function

A transfer function of a control system models the output signal for all possible input values .

The input, $X(s)$ and the output $Y(s)$ can be related by a transfer function

$$H(s) = \frac{Y(s)}{X(s)}$$



In the time domain, the input and the output of a control system are denoted as $x(t)$ *and* $y(t)$ respectively

The inverse Laplace transform

$$h(t) = L^{-1}\{H(s)\}$$

is called the impulse response.

In Laplace transform, all initial conditions are assumed to be **zero**.

Example 3.35:

Given the transfer function of a system is given by

$$H(s) = \frac{s}{s^2 + 4}$$

with the input signal $X(s) = \frac{1}{s}$. Find the output response $y(t)$ for the system.

Solution:

Given the transfer function is

$$H(s) = \frac{s}{s^2+4} = \frac{Y(s)}{\left(\frac{1}{s}\right)} \longleftarrow H(s) = \frac{Y(s)}{X(s)}$$

So

$$Y(s) = \frac{s}{s^2+4} \cdot \frac{1}{s}$$

Now use convolution theorem to find $y(t)$, we choose

$$F(s) = \frac{s}{s^2+4}, \; G(s) = \frac{1}{s}$$

So

$$f(t) = L^{-1}\{F(s)\} = L^{-1}\left\{\frac{s}{s^2+4}\right\} = \cos 2t$$

$$g(t) = L^{-1}\{G(s)\} = L^{-1}\left\{\frac{1}{s}\right\} = 1$$

Then

$$f(\tau) = \cos 2\tau,\ g(t-\tau) = 1$$

Determine the $y(t)$ by using the convolution theorem:

$$y(t) = \int_0^t \cos 2\tau \cdot 1 \, d\tau \quad \longleftarrow \quad \int_0^t f(\tau) g(t-\tau) d\tau$$

$$= \left[\frac{\sin 2\tau}{2} \right]_0^t$$

$$= \frac{1}{2}\left(\sin 2t - \sin 0\right) = \frac{1}{2} \sin 2t$$

Example 3.36:

Given the input $x(t)$ and the output $y(t)$ of an electronic system can be related by

$$y'' + 4y = 2x(t).$$

Determine the transfer function $H(s)$ and the impulse response $h(t)$ for the system.

Solution:

$$L\{y'' + 4y\} = 2L\{x(t)\}$$

$$s^2Y(s) - sy(0) - y'(0) + 4Y(s) = 2X(s)$$

$$s^2Y(s) - s(0) - 0 + 4Y(s) = 2X(s)$$

$$(s^2 + 4)Y(s) = 2X(s)$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{2}{s^2 + 4}$$

Initial conditions are assumed to be zero in Laplace transform

Transfer function

The impulse response $h(t)$ for the system

$$h(t) = L^{-1}\{H(s)\}$$

$$= L^{-1}\left\{\frac{2}{s^2+4}\right\}$$

$$= \sin 2t$$

Exercise 3.14

The input $x(t)$ and output $y(t)$ of an electronic system can be represented by

$$y'' + 4y' + 5y = x(t)$$

Find the transfer function $H(s)$ and the impulse response $h(t)$ for the system.

Answer: $H(s) = \dfrac{1}{(s^2 + 4s + 5)}, \; h(t) = e^{-2t} \sin t$

☞ Are you able to

i. use Laplace transform to solve the initial value problem

ii. use Laplace transform to solve the transfer function in the control system

References

Edwards C. H., Penny D.E. & Calvis D. (2016). Differential Equations and Boundary Value Problems, 5thEdition. Pearson Education Inc..

Zill, D. G. (2017). Differential Equations with Boundary-Value Problems. Cengage Learning Inc

Thank You

Questions & Answer?