BMCG 1013
DIFFERENTIAL EQUATIONS

LAPLACE TRANSFORM
(INVERSE LAPLACE TRANSFORM)

Irma Wani binti Jamaludin

irma@utem.edu.my

Learning Outcome

At the end of this chapter you should be able to:

1. Find the Inverse Laplace transform for a given function.
2. Use the linearity property of Inverse Laplace Transform Laplace Transform
3. Use first shifting property in order to determine the Inverse Laplace Transform for a given function
4. Use second shifting property in order to determine the Inverse Laplace Transform for a given function
5. Find the inverse Laplace transform for a given function by using Convolution theorem

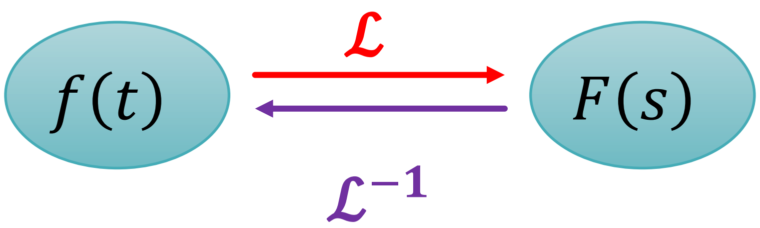
3.3 INVERSE LAPLACE TRANSFORM

Notation for Laplace Transform is written by

$$\mathcal{L}\{f(t)\} = F(s)$$

Thus, the inverse of Laplace transform is

$$\mathcal{L}^{-1}\{F(s)\} = f(t)$$



Example 3.23

Find the inverse Laplace transform of the following functions:

(a) $F(s) = \frac{5}{s-3}$

(b) $F(s) = \frac{1}{s^2}$

(c) $F(s) = \frac{1}{s^3}$

Solution:

(a) $f(t) = \mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\{\frac{5}{s-3}\} = 5e^{3t}$

(b) $f(t) = \mathcal{L}^{-1}\{\frac{1}{s^2}\} = t$

(c) $f(t) = \mathcal{L}^{-1}\{\frac{1}{s^3}\} = \frac{1}{2}\mathcal{L}^{-1}\{\frac{2}{s^3}\} = \frac{1}{2}t^2$

3.3.1 LINEARITY PROPERTY OF INVERSE LAPLACE TRANSFORM

If $\mathcal{L}^{-1}\{F(s)\} = f(t)$ and $\mathcal{L}^{-1}\{G(s)\} = g(t)$ with α and β are constants, then

$$\begin{aligned}\mathcal{L}^{-1}\{\alpha F(s) + \beta G(s)\} &= \alpha\mathcal{L}^{-1}\{F(s)\} + \beta\mathcal{L}^{-1}\{G(s)\} \\ &= \alpha f(t) + \beta g(t)\end{aligned}$$

Example 3.24

Find the inverse Laplace transform of the following functions:

(a) $F(s) = \frac{2}{s-3} + \frac{5s}{s^2-3}$

(b) $F(s) = \frac{2}{s^4} - \frac{1}{s^2+4}$

Solution:

(a) $F(s) = \frac{2}{s-3} + \frac{5s}{s^2-3}$

$f(t) = \mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\{\frac{2}{s-3} + \frac{5s}{s^2-3}\}$

$= 2\mathcal{L}^{-1}\{\frac{1}{s-3}\} + 5\mathcal{L}^{-1}\{\frac{s}{s^2-3}\} = 2e^{3t} + 5\cosh\sqrt{3}t$

(b) $F(s) = \frac{2}{s^4} - \frac{1}{s^2+4}$

$f(t) = \mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\{\frac{2}{s^4} - \frac{1}{s^2+4}\}$

$= 2\left(\frac{1}{3!}\right)\mathcal{L}^{-1}\{\frac{3!}{s^{3+1}}\} - \frac{1}{2}\mathcal{L}^{-1}\{\frac{2}{s^2+2^2}\}$

$= \frac{1}{3}t^3 - \frac{1}{2}\sin 2t$

Exercise 3.8

Find the inverse Laplace transform for the following functions:

(a) $Y(s) = \frac{s}{s^2+9} + \frac{2}{s^2+16}$

(b) $Y(s) = \frac{6-s}{s^2+25}$

(c) $Y(s) = \frac{2}{s^2} - \frac{s+2}{s^2-9}$

Answer:

(a) $y(t) = \cos 3t + \frac{1}{2}\sin 4t$; (b) $y(t) = \frac{6}{5}\sin 5t - \cos 5t$; (c) $y(t) = 2t - \cosh 3t - \frac{2}{3}\sinh 3t$

3.3.2 FIRST SHIFTING PROPERTY FOR INVERSE LAPLACE TRANSFORM

If $\mathcal{L}^{-1}\{G(s)\} = g(t)$ with a as a constant, then

$$\mathcal{L}^{-1}\{G(s-a)\} = e^{at}\mathcal{L}^{-1}\{G(s)\} = e^{at}g(t)$$

or we can write as

$$\mathcal{L}^{-1}\{G(s-a)\} = e^{at}g(t)\}$$

Example 3.25

Find the inverse Laplace transform of

$$F(s) = \frac{8}{(s-2)^3}$$

Solution:

Compare with $\mathcal{L}^{-1}\{G(s-a)\} = e^{at}g(t)$, we have $a = 2$,

$$F(s) = G(s-2) = \frac{8}{(s-2)^3}$$

$$G(s) = \frac{8}{(s)^3} \to g(t) = \mathcal{L}^{-1}\left\{\frac{8}{s^3}\right\} = 4t^2$$

Therefore,

$$\mathcal{L}^{-1}\left\{\frac{8}{(s-2)^3}\right\} = e^{at}g(t) = e^{2t}\left(4t^2\right) = 4t^2e^{2t}$$

Example 3.26

Find the inverse Laplace transform of

$$F(s) = \frac{8}{(s-4)^2+1}$$

Solution:

Compare with $\mathcal{L}^{-1}\{G(s-a)\} = e^{at}g(t)$, we have $a = 4$,

$$F(s) = G(s-4) = \frac{8}{(s-4)^2+1}$$

$$G(s) = \frac{8}{s^2+1} \to g(t) = 8\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = 8\sin t$$

Thus,

$$\mathcal{L}^{-1}\left\{\frac{8}{(s-4)^2+1}\right\} = e^{at}g(t) = e^{4t}(8\sin t) = 8e^{2t}\sin t$$

Example 3.27

Find the inverse Laplace transform of

$$F(s) = \frac{s}{(s-1)^2 + 4}$$

<u>Solution:</u>

Compare with $\mathcal{L}^{-1}\{G(s-a)\} = e^{at}g(t)$, we have $a = 1$,

$$F(s) = G(s-1) = \frac{s}{(s-1)^2 + 4} = \frac{\boldsymbol{s-1} + 1}{(s-1)^2 + 4}$$

$$G(s) = \frac{s+1}{s^2+4} = \frac{s}{s^2+4} + \frac{1}{s^2+4}$$

$$g(t) = \mathcal{L}^{-1}\left\{\frac{s}{s^2+2^2}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^2+2^2}\right\} = \cos 2t + \frac{1}{2}\sin 2t$$

Thus,

$$\mathcal{L}^{-1}\left\{\frac{s}{(s-1)^2+4}\right\} = e^{t}g(t) = e^{4t}\left(\cos 2t + \frac{1}{2}\sin 2t\right)$$

Exercise 3.9

Find the inverse Laplace transform below:

(a) $\mathcal{L}^{-1}\left\{\frac{s-1}{(s+1)^3}\right\}$

(b) $\mathcal{L}^{-1}\left\{\frac{s+2}{s^2+4s+5}\right\}$

(c) $\mathcal{L}^{-1}\left\{\frac{2s-3}{(s+5)^2+36}\right\}$

Answer:

(a) $e^{-t}(t-t^2)$ (b) $e^{-2t}\cos t$ (c) $e^{-5t}\left(2\cos 6t-\frac{13}{6}\sin 6t\right)$

3.3.3 SECOND TRANSLATION PROPERTY FOR INVERSE LAPLACE TRANSFORM

Translation on the *t*-axis

If $\mathcal{L}^{-1}\{G(s)\} = g(t)$ and let a be a constant, then

$$\mathcal{L}^{-1}\{e^{-as}G(s)\} = g(t-a)u(t-a)$$

where $u(t-a)$ is a unit step function.

Example 3.28

Find the inverse Laplace of the given expression:

(a) $F(s) = \dfrac{2e^{-2s}}{s^3}$

(b) $F(s) = \dfrac{2e^{-s}}{s^2+9}$

Solution:

(a) $F(s) = \frac{2e^{-2s}}{s^3}$

Compare with $e^{-as}G(s)$ from $\mathcal{L}^{-1}\{e^{-as}G(s)\} = g(t-a)u(t-a)$, we have

$$a = 2,\ G(s) = \frac{2}{s^3}.$$

Then, $g(t) = \mathcal{L}^{-1}\left\{\frac{2}{s^3}\right\} = t^2 \ \rightarrow g(t-2) = (t-2)^2$

Therefore, $f(t) = \mathcal{L}^{-1}\left\{\frac{2e^{-2s}}{s^3}\right\} = (t-2)^2 u(t-2)$.

Solution:

(b) $F(s) = \frac{2e^{-s}}{s^2+9}$

Compare with $e^{-as}G(s)$ from $\mathcal{L}^{-1}\{e^{-as}G(s)\} = g(t-a)u(t-a)$, we have

$$a = 1,\ G(s) = \frac{2}{s^2+9}.$$

Then, $g(t) = \mathcal{L}^{-1}\left\{\frac{2}{s^2+3^2}\right\} = \frac{2}{3}\sin 3t \rightarrow g(t-1) = \frac{2}{3}\sin 3(t-1)$

Therefore, $f(t) = \mathcal{L}^{-1}\left\{\frac{2e^{-s}}{s^2+9}\right\} = \frac{2}{3}\sin 3(t-1)\,u(t-1)$.

Exercise 3.10

Find the inverse Laplace transform of the following expressions:

(a) $\frac{5se^{-3s}}{s^2+4}$

(b) $\frac{2e^{-2s}}{(s-1)^2+4}$

Answer:

(a) $5\cos 2(t-3)\,u(t-3)$ (b) $e^{(t-2)}\ \sin 2(t-2)\,u(t-2).$

3.3.4 PARTIAL FRACTIONS DECOMPOSITION

The basic knowledge of partial fractions is very important as it can be used to solve the inverse Laplace transform.

Steps for writing in terms of Partial Fraction

1 : Factorize denominator of the fraction.

2 : Express the fraction in corresponding form.

- ✓Linear factor
- ✓Quadratic factor
- ✓Repeated factors

3 : Determine the value of the constant on each of partial fraction by

Method 1 : Comparing the coefficient of x^n.

Method 2 : Substitution of any values x.

Example 3.29

Find the inverse Laplace transforms of the functions below by using partial fraction.

a) $F(s) = \frac{1}{s(s-2)}$

b) $F(s) = \frac{2}{s(s^2-4)}$

c) $F(s) = \frac{s-4}{s^3-s^2-2s}$

Solution (a):

Step 1: Rewrite the function in terms of partial fraction.

$$F(s) = \frac{1}{s(s-2)} = \frac{A}{s} + \frac{B}{s-2}$$

$$A(s-2) + Bs = 1$$

Step 2: Choose suitable values of s:

Let $s = 0,\ A(0-2) + B(0) = 1 \ \rightarrow A = -\frac{1}{2}$

Let $s = 2,\ A(2-2) + B(2) = 1 \ \rightarrow B = \frac{1}{2}$

$$F(s) = \frac{-\frac{1}{2}}{s} + \frac{\frac{1}{2}}{s-2} = \frac{1}{2}\left(-\frac{1}{s} + \frac{1}{s-2}\right)$$

Step 3: Find the inverse Laplace transform

$$f(t) = \mathcal{L}^{-1}\{F\ (s)\} = \frac{1}{2}(-1 + e^{2t})$$

Solution (b):

Step 1: Rewrite the function in terms of partial fraction:

$$F(s)=\frac{2}{s(s^2-4)}=\frac{A}{s}+\frac{Bs+C}{s^2-4}$$

$$A(s^2-4)+(Bs+C)s=2$$

Step 2: Choose suitable values of s:

Let $s=0,\ A(0-4)+(0+C)(0)=2\ \rightarrow A=-\frac{1}{2}$

Let $s=2,\ A(4-4)+(2B+C)(2)=2\ \rightarrow 4B+2C=2$

Let $s=-2,\ A(4-4)+(-2B+C)(-2)=2\ \rightarrow 4B-2C=2$

$$\rightarrow B=\frac{1}{2}, C=0$$

$$F(s)=\frac{-\frac{1}{2}}{s}+\frac{\frac{1}{2}s}{s^2-4}=\frac{1}{2}\left(-\frac{1}{s}+\frac{s}{s^2-4}\right)$$

Step 3: Find the inverse Laplace transform

$$f(t)=\mathcal{L}^{-1}\{F\ (s)\}=\frac{1}{2}(-1+\cosh 2t)$$

Solution (c):

$$F(s)=\frac{s-4}{s^3-s^2-2s}=\frac{s-4}{s(s^2-s-2)}=\frac{s-4}{s(s+1)(s-2)}$$

Step 1: Rewrite the function in terms of partial fraction.

$$F(s)=\frac{s-4}{s(s+1)(s-2)}=\frac{A}{s}+\frac{B}{s+1}+\frac{C}{s-2}$$

$$A(s+1)(s-2)+Bs(s-2)+Cs(s+1)=s-4$$

Step 2: Choose suitable values of s:

Let $s=0,\ A(0+1)(0-2)+B(0)(0-2)+C(0)(0+1)=0-4\ \rightarrow A=2$

Let $s=-1,\ A(-1+1)(-1-2)+B(-1)(-1-2)+C(-1)(-1+1)=-1-4$

$$\rightarrow B=-5/3$$

Let $s=2,\ A(2+1)(2-2)+B(2)(2-2)+C(2)(2+1)=2-4$

$$\rightarrow C=-1/3$$

$$F(s)=\frac{2}{s}+\frac{-5/3}{s+1}+\frac{-1/3}{s-2}$$

Solution (c):

$$F(s) = \frac{s-4}{s^3 - s^2 - 2s}$$

$$= \frac{2}{s} + \frac{-5/3}{s+1} + \frac{-1/3}{s-2}$$

Step 3: Find the inverse Laplace transform

$$f(t) = \mathcal{L}^{-1}\{F\,(s)\} = 2 - \frac{5}{3}e^{-t} - \frac{1}{3}e^{2t}$$

Exercise 3.11

Find the inverse Laplace transform of the following expressions:

(a) $\frac{1}{s(s-4)}$

(b) $\frac{2}{s^2+s-6}$

(c) $\frac{2}{s^2(s+1)}$

Answer:

(a) $\frac{1}{4}(e^{4t}-1)$ (b) $\frac{2}{5}\left(e^{2t}-e^{-3t}\right)$ (c) $2t-2+2e^{-t}$

3.3.5 CONVOLUTION THEOREM

- This theorem will be used to find the inverse Laplace Transform for the function in the form of $F(s)G(s)$.
- The convolution of $f(t)$ and $g(t)$, denoted by $f * g$ is defined by

$$f(t) * g(t) = \int_0^t f(u)g(t-u)\,du$$

3.3.5 CONVOLUTION THEOREM

Theorem:

If $\mathcal{L}^{-1}\{F(s)\} = f(t)$ and $\mathcal{L}^{-1}\{G(s)\} = g(t)$, then

$$\mathcal{L}^{-1}\{F(s)G(s)\} = f(t) * g(t) = \int_0^t f(u)g(t-u)\,du$$

Example 3.30

Find the inverse Laplace Transform for $\frac{1}{s(s-2)}$ by using the convolution theorem.

Solution 1:

From the expression, $\frac{1}{s(s-2)} = \frac{1}{s}\frac{1}{s-2}$.

Thus, we set

$F(s) = \frac{1}{s}$ and $G(s) = \frac{1}{s-2}$ where the inverse Laplace transforms are given as:

$$f(t) = 1 \text{ and } g(t) = e^{2t}$$

$$\rightarrow f(u) = 1 \qquad \text{and} \qquad g(t-u) = e^{2(t-u)} = e^{2t}.e^{-2u}$$

Using the theorem $\boldsymbol{\mathcal{L}^{-1}\{F(s)G(s)\} = f(t) * g(t) = \int_0^t f(u)g(t-u)\,du}$,

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s-2)}\right\} = \int_0^t (1)(e^{2t}.\,e^{-2u})\,du$$

$$= e^{2t}\left[-\frac{1}{2}e^{-2u}\right]_0^t$$

$$= e^{2t}\left[-\frac{1}{2}e^{-2t} + \frac{1}{2}e^{0}\right]$$

$$= -\frac{1}{2}e^{0} + \frac{1}{2}e^{2t} = \frac{1}{2}\left(e^{2t} - 1\right)$$

Solution 2:

From the expression, $\frac{1}{s(s-2)} = \frac{1}{s}\frac{1}{s-2}$.

Thus, we set

$$F(s) = \frac{1}{s-2} \qquad G(s) = \frac{1}{s}$$

$$f(t) = e^{2t} \qquad g(t) = 1$$

$$f(u) = e^{2u} \qquad g(t-u) = 1$$

Using the theorem $\mathcal{L}^{-1}\{F(s)G(s)\} = f(t) * g(t) = \int_0^t f(u)g(t-u)\,du$,

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s-2)}\right\} = \int_0^t e^{2u}(1)\,du = \left[\frac{1}{2}e^{2u}\right]_0^t$$

$$= \left[\frac{1}{2}e^{2t} - \frac{1}{2}e^{0}\right] = \frac{1}{2}\left(e^{2t} - 1\right)$$

Example 3.31

Find the inverse Laplace Transform for $\frac{1}{s(s^2+4)}$ by using the convolution theorem.

Solution:

From the expression, $\frac{1}{s(s^2+4)} = \frac{1}{s}\left(\frac{1}{s^2+4}\right)$.

Thus, we set $G(s) = \frac{1}{s}$ and $F(s) = \frac{1}{s^2+4}$ where the inverse Laplace transforms are given as:

$$f(t) = \frac{1}{2}\sin 2t \text{ and } g(t) = 1 \rightarrow f(u) = \frac{1}{2}\sin 2u \text{ and } g(t-u) = 1$$

Using the theorem $\boldsymbol{\mathcal{L}^{-1}\{F(s)G(s)\} = f(t) * g(t) = \int_0^t f(u)g(t-u)\,du}$,

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+4)}\right\} = \int_0^t (\frac{1}{2}\sin 2u)(1)\,du$$

$$= \left[-\frac{1}{4}\cos 2u\right]_0^t$$

$$= \left[-\frac{1}{4}\cos 2t + \frac{1}{4}\cos 0\right] = \frac{1}{4} - \frac{1}{4}\cos 2t$$

Exercise 3.12

By using convolution theorem, find the inverse Laplace transform of the following expressions:

(a) $\dfrac{1}{s(s-4)}$

(b) $\dfrac{2}{s^2(s+1)}$

(c) $\dfrac{2}{s^2(s+1)^2}$

Answer:

(a) $\frac{1}{4}(e^{4t}-1)$ (b) $2t-2+2e^{-t}$ (c) $2te^{-t}+4e^{-t}+2t-4$

Reference

- Rahifa et. al, Differential Equations for Engineering Students, Penerbit UTeM 2019.

Thank You

Question & Answer