BMCG 1013
DIFFERENTIAL EQUATIONS

LAPLACE TRANSFORM
(UNIT STEP FUNCTION)

Irma Wani binti Jamaludin

irma@utem.edu.my

Learning Outcomes

At the end of this topic, student should be able to:

1. Express a function in the form of unit step function.
2. Find Laplace transform of a unit step function

3.2 UNIT STEP FUNCTION

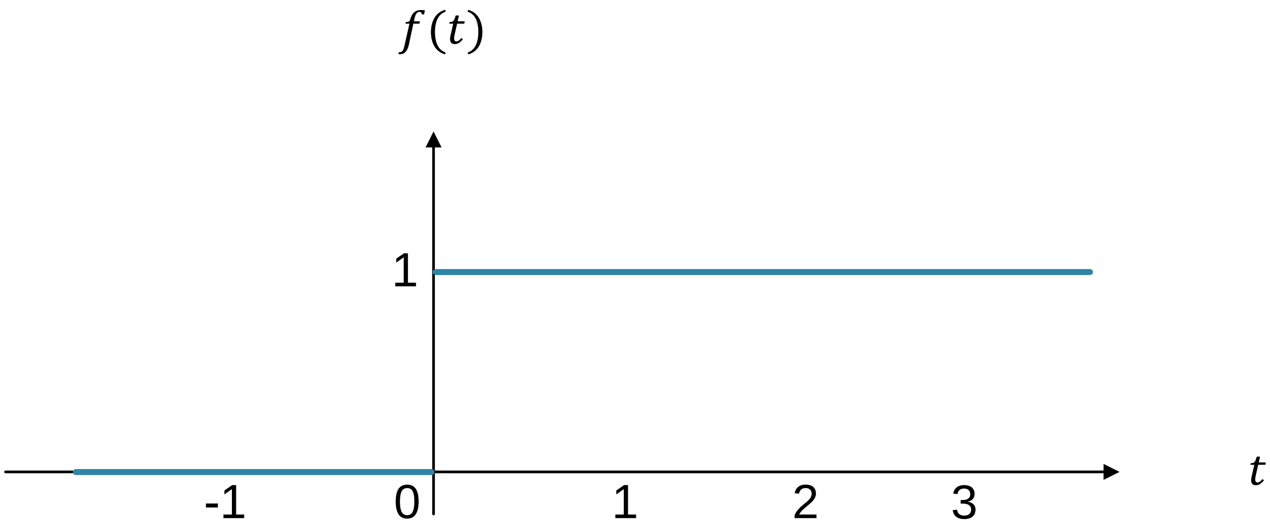
Definition and Notation

Unit step function $u(t)$ is defined by

$$u(t) = \begin{cases} 0, & t < 0 \\ 1, & t > 0 \end{cases}$$

That is, u is a function of time t, and u has value zero when its argument is negative; and 1 when its argument is positive.

3.2 UNIT STEP FUNCTION



Graph of unit step function

Shifted Unit Step Function

- In many circuits, waveforms are applied at specified intervals other than at t = 0. Such a function may be described using the Shifted Unit Step Function.

Definition of Shifted Unit Step Function:

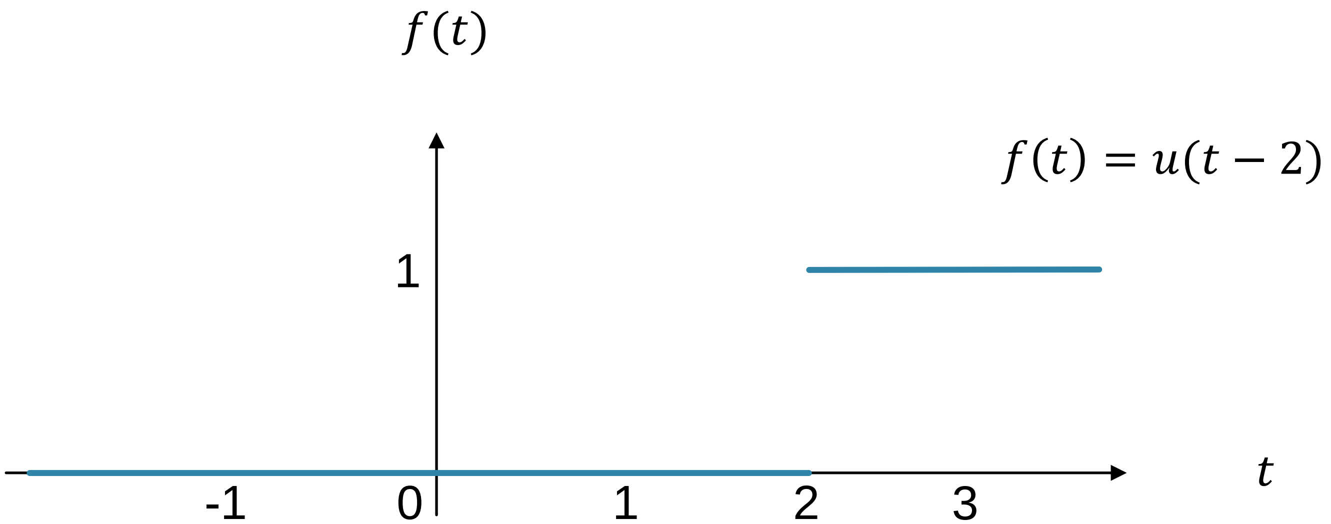
$$u(t-a) = \begin{cases} 0, & t < a \\ 1, & t > a \end{cases}$$

Example of Shifted Unit Step Function:

$$f(t) = u(t-2)$$

This means *f*(*t*) has value of 0 when *t* < 2 and 1 when *t* > 2.

The sketch of the waveform is as follows:



3.2.1 Writing Piecewise Continuous Function In Terms of Unit Step

- Suppose we are given piecewise continuous function as such below

$$f(t) = \begin{cases} f_1; & 0 < t \le a \\ f_2; & a < t \le b \\ f_3; & b < t \le c \\ f_4; & t > c \end{cases}$$

- We can write the function in terms of unit step function as:

$$f(t) = f_1 + [f_2 - f_1]u(t-a) + [f_3 - f_2]u(t-b) + [f_4 - f_3]u(t-c)$$

Example 3.10:

Write the following functions in terms of **unit step** function(s).

$$f(t)=\begin{cases}5t-3; & 0\le t<2\\ 7\quad ; & 2\le t<5\\ 0\quad ; & t\ge 5\end{cases}$$

Answer:

$$f(t)=f_1+[f_2-f_1]u(t-a)+[f_3-f_2]u(t-b)$$

$$=(5t-3)+[7-(5t-3)]u(t-2)+[0-7]u(t-5)$$

$$=(5t-3)+(10-5t)\,u(t-2)-7u(t-5)$$

Rectangular Pulse

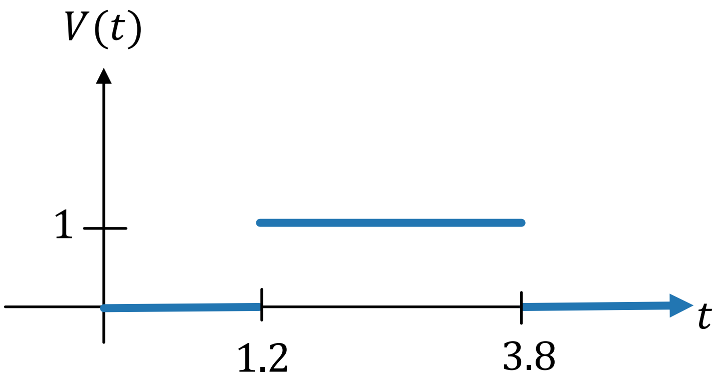
A common situation in a circuit is for a voltage to be applied at a particular time, $t = a$ and removed later, at $t = b$. This function is written using unit step function as

$$V(t) = u(t - a) - u(t - b)$$

and it has strength 1, duration $(b - a)$.

Example 3.11

The graph of $V(t) = u(t - 1.2) - u(t - 3.8)$



Here, the duration is 3.8 − 1.2 = 2.6.

Example 3.12

Given a piecewise function:

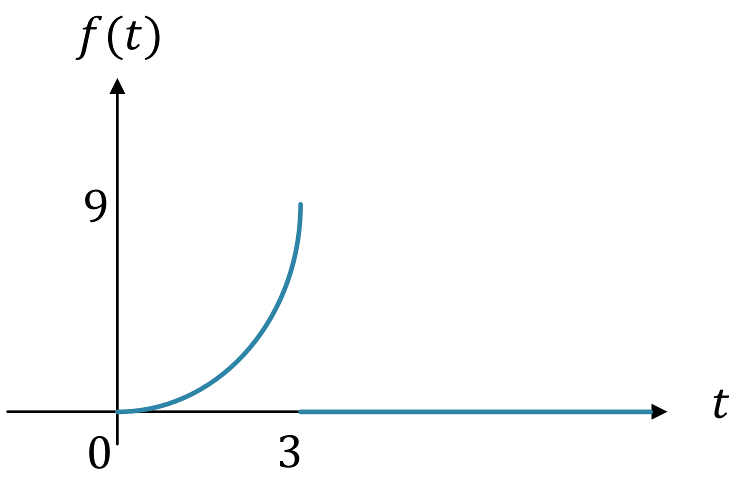
$$f(t) = \begin{cases} t^2; & t < 3 \\ 0; & t \geq 3 \end{cases}$$

(a) Sketch the graph of $f(t)$.

(b) Express $f(t)$ in terms of unit step function.

Solution:

$$f(t) = f_1 + [f_2 - f_1]u(t-a)$$
$$= t^2 + [0 - t^2]u(t-3)$$
$$= t^2 - t^2 u(t-3)$$



Example 3.13

Sketch the graph of the unit step function

$$f(t) = u(t-3) - u(t-6)$$

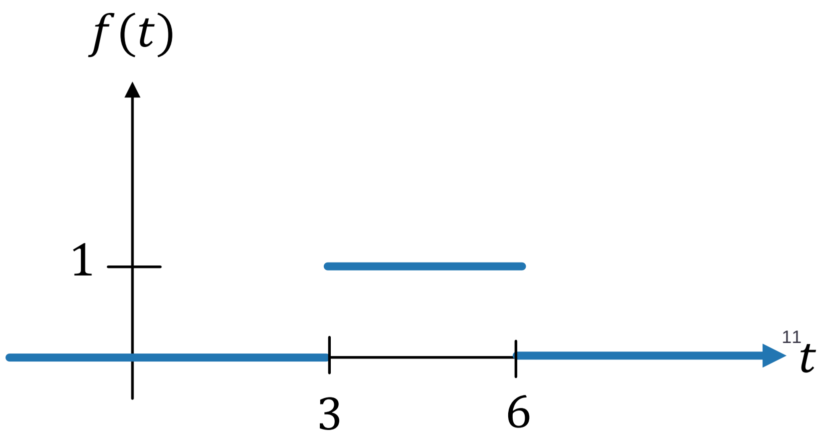
Solution:

Using the definition of shifted unit step function,

$$f(t) = u(t-3) - u(t-6)$$

$$= \begin{cases} 0, & t < 3 \\ 1, & t > 3 \end{cases} - \begin{cases} 0, & t < 6 \\ 1, & t > 6 \end{cases}$$

$$= \begin{cases} 0, & t < 3 \\ 1, & 3 < t < 6 \\ 0, & t > 6 \end{cases}$$



Exercise 3.5

Sketch the graph of the given piecewise function, and write the function in terms of unit step function.

1. $f(t) = \begin{cases} t; & 0 < t \leq \pi \\ 0; & \pi < t \leq 2\pi \\ 2; & t > 2\pi \end{cases}$

2. $f(t) = \begin{cases} -t^2; & 0 \leq t < 2 \\ 1 + t; & t \geq 2 \end{cases}$

Answer:

1. $f(t) = t - tu(t - \pi) + 2u(t - 2\pi)$

2. $f(t) = -t^2 + \left(1 + t + t^2\right)u(t - 2)$

3.2.2 LAPLACE TRANSFORM OF UNIT STEP FUNCTION

$$\mathcal{L}\{u(t-a)\} = \frac{e^{-as}}{s}, a > 0$$

Proof:

From the definition of unit step and Laplace Transform,

for $a > 0$, the Laplace transform of unit step function is

$$\mathcal{L}\{u(t-a)\} = \mathcal{L}\{f(t)\} = F(s) = \int_0^a e^{-st}(0)dt + \int_a^\infty e^{-st}(1)dt$$

$$= \left[-\frac{e^{-st}}{s}\right]_a^\infty = \frac{e^{-as}}{s}$$

Example 3.14

Find $\mathcal{L}\{f(t)\}$ if $f(t) = u(t-2) - u(t-6)$.

Solution:

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{u(t-2) - u(t-6)\}$$

Using the Linearity Property,

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{u(t-2)\} - \mathcal{L}\{u(t-6)\}$$

$$= \frac{e^{-2s}}{s} - \frac{e^{-6s}}{s}$$

$$= \frac{1}{s}\left(e^{-2s} - e^{-6s}\right)$$

Example 3.15

Find the Laplace transform of the following functions:

(a) $f(t) = u(t-1) - 2u(t-3)$

(b) $f(t) = 3u(t-2) - 5u(t-4) - 7u(t-6)$

Solution:

(a) $F(s) = \mathcal{L}\{u(t-1)\} - 2\mathcal{L}\{u(t-3)\}$

$$= \frac{e^{-s}}{s} - 2\frac{e^{-3s}}{s} = \frac{1}{s}\left(e^{-s} - 2e^{-3s}\right)$$

(b) $F(s) = 3\mathcal{L}\{u(t-2)\} - 5\mathcal{L}\{u(t-4)\} - 7\mathcal{L}\{u(t-6)\}$

$$= \frac{3e^{-2s}}{s} - 5\frac{e^{-4s}}{s} - 7\frac{e^{-6s}}{s}$$

$$= \frac{1}{s}\left(3e^{-2s} - 5e^{-4s} - 7e^{-6s}\right)$$

3.2.2.1 Second Shifting Property

If $a > 0$ and $\mathcal{L}\{f(t)\}$ then

$$\mathcal{L}\{g(t-a)u(t-a)\} = e^{-as}\mathcal{L}\{g(t)\} = e^{-as}G(s)$$

Example 3.16

Find the Laplace Transform of the following functions:

(a) $f(t) = (t-3)u(t-3)$

(b) $f(t) = (t-3)^2 u(t-3)$

$$\mathcal{L}\{g(t-a)u(t-a)\} = e^{-as}\mathcal{L}\{g(t)\} = e^{-as}G(s)$$

Solution (a):

Compare $f(t) = (t-3)u(t-3)$ with $g(t-a)u(t-a)$, we have

$$a = 3, \text{ and } g(t-a) = g(t-3) = t-3.$$

From $g(t-3) = t-3$, we can modify the function by letting $t-3 = t$ and becomes $g(t) = t$.

Thus, $G(s) = \mathcal{L}\{g(t)\} = \mathcal{L}\{t\} = \frac{1}{s^2}$.

Hence, $\mathcal{L}\{(t-3)u(t-3)\} = e^{-3s}\mathcal{L}\{t\} = e^{-3s}\left(\frac{1}{s^2}\right) = \frac{1}{s^2}e^{-3s}$

$$\mathcal{L}\{g(t-a)u(t-a)\} = e^{-as}\mathcal{L}\{g(t)\} = e^{-as}G(s)$$

Solution (b):

Compare $f(t) = (t-3)^2 u(t-3)$ with $g(t-a)u(t-a)$, we have

$$a = 3, \text{ and } g(t-a) = g(t-3) = (t-3)^2.$$

From $g(t-3) = (t-3)^2$, we can modify the function by letting $t-3 = t$ and becomes $g(t) = t^2$.

Thus, $G(s) = \mathcal{L}\{g(t)\} = \mathcal{L}\{t^2\} = \frac{2}{s^3}$.

Hence, $\mathcal{L}\{(t-3)^2 u(t-3)\} = e^{-3s}\mathcal{L}\{t^2\} = e^{-3s}\left(\frac{2}{s^3}\right) = \frac{2}{s^3}e^{-3s}$.

Example 3.17

Find the Laplace Transform of

$$f(t) = \sin(t-3)\,u(t-3).$$

$$\mathcal{L}\{g(t-a)u(t-a)\} = e^{-as}\mathcal{L}\{g(t)\} = e^{-as}G(s)$$

Solution:

Compare $f(t) = \sin(t-3)\,u(t-3)$ with $g(t-a)u(t-a)$, we have

$$a = 3, \text{ and } g(t-a) = g(t-3) = \sin(t-3).$$

From $g(t-3) = \sin(t-3)$, we can modify the function by letting $t-3 = t$ and becomes $g(t) = \sin t$.

Thus, $G(s) = \mathcal{L}\{g(t)\} = \mathcal{L}\{\sin t\} = \frac{1}{s^2+1}$.

Hence, $\mathcal{L}\{\sin(t-3)u(t-3)\} = e^{-3s}\mathcal{L}\{\sin t\} = e^{-3s}\left(\frac{1}{s^2+1}\right) = \frac{e^{-3s}}{s^2+1}$.

Exercise 3.6

Show that

(i) $\mathcal{L}\{e^{t-5}u(t-5)\} = \frac{e^{-5s}}{s-1}$

(ii) $\mathcal{L}\{\cos(t-4)u(t-4)\} = \frac{se^{-4s}}{s^2+1}$

Example 3.18

Find the Laplace Transform of

$$f(t) = tu(t-3)$$

$$\mathcal{L}\{g(t-a)u(t-a)\} = e^{-as}\mathcal{L}\{g(t)\} = e^{-as}G(s)$$

Solution:

Compare $f(t) = tu(t-3)$ with $g(t-a)u(t-a)$, we have

$$a = 3, \text{ and } g(t-a) = g(t-3) = t.$$

Since $t = (t-3) + 3$, then

$$g(t-3) = (t-3) + 3$$

$$g(t) = t + 3$$

Replace $(t-3)$ by t

Thus, $G(s) = \mathcal{L}\{g(t)\} = \mathcal{L}\{t+3\} = \frac{1}{s^2} + \frac{3}{s}$.

Hence, $\mathcal{L}\{tu(t-3)\} = e^{-3s}\left(\frac{1}{s^2} + \frac{3}{s}\right)$

Example 3.19

Find the Laplace Transform of

$$f(t) = (t + 5)u(t - 2)$$

$$\mathcal{L}\{g(t-a)u(t-a)\} = e^{-as}\mathcal{L}\{g(t)\} = e^{-as}G(s)$$

Solution:

Compare $f(t) = (t + 5)u(t - 2)$ with $g(t - a)u(t - a)$, we have

$$a = 2, \text{ and } g(t - a) = g(t - 2) = t + 5.$$

Since $t = (t - 2) + 2$, then

$$g(t - 2) = [(t - 2) + 2\,] + 5$$

$$g(t) = t + 7$$

Replace $(t - 2)$ by t

Thus, $\mathrm{G}(s) = \mathcal{L}\{g(t)\} = \mathcal{L}\{t + 7\} = \frac{1}{s^2} + \frac{7}{s}$.

Hence, $\mathcal{L}\{(t + 5)u(t - 2)\} = e^{-2s}\left(\frac{1}{s^2} + \frac{7}{s}\right)$

Example 3.20

Find the Laplace Transform of

$$f(t) = (t^2 + t - 2)u(t-1)$$

$$\mathcal{L}\{g(t-a)u(t-a)\} = e^{-as}\mathcal{L}\{g(t)\} = e^{-as}G(s)$$

Solution:

Compare $f(t) = \left(t^2 + t - 2\right)u(t-1)$ with $g(t-a)u(t-a)$, we have

$$a = 1, \text{ and } g(t-a) = g(t-1) = t^2 + t - 2$$

Since $t = (t-1) + 1$, then

$$g(t-1) = [(t-1) + 1\,]^2 + [(t-1) + 1\,] - 2$$

$$g(t) = (t+1)^2 + (t+1) - 2 = t^2 + 3t$$

Replace $(t-1)$ by t

Thus, $\mathrm{G}(s) = \mathcal{L}\{g(t)\} = \mathcal{L}\{t^2 + 3t\} = \frac{2}{s^3} + \frac{3}{s^2}$.

Hence, $\mathcal{L}\{(t^2 + t - 2)u(t-1)\} = e^{-s}\left(\frac{2}{s^3} + \frac{3}{s^2}\right)$

Example 3.21

Write $f(t)$ in terms of unit step function and finds its Laplace Transform.

$$f(t)=\begin{cases}1; & 0<t\leq 2\\ 0; & 2<t\leq 4\\ 2; & t>4\end{cases}.$$

<u>Solution:</u>

In terms of unit step function,

$$f(t)=1+(0-1)u(t-2)+(2-0)u(t-4)$$
$$f(t)=1-u(t-2)+2u(t-4)$$

Thus, the Laplace transform is

$$F(s)=\frac{1}{s}-\frac{e^{-2s}}{s}+\frac{2e^{-4s}}{s}=\frac{1}{s}\left(1-e^{-2s}+2e^{-4s}\right)$$

Example 3.22

Write $f(t)$ in terms of unit step function and finds its Laplace Transform.

$$f(t) = \begin{cases} e^{2t}; & 0 < t \leq 1 \\ 0; & 1 < t \leq 2 \\ 2; & t > 2 \end{cases}.$$

Solution:

In terms of unit step function,

$$f(t) = e^{2t} + \left(0 - e^{2t}\right)u(t-1) + (2-0)u(t-2)$$

$$f(t) = e^{2t} - e^{2t}u(t-1) + 2u(t-2)$$

Thus, the Laplace transform is

$$F(s) = \mathcal{L}\{e^{2t}\} - \mathcal{L}\{e^{2t}u(t-1)\} + 2\mathcal{L}\{u(t-2)\}$$

$$\mathcal{L}\{g(t-a)u(t-a)\} = e^{-as}\mathcal{L}\{g(t)\} = e^{-as}G(s)$$

For $\mathcal{L}\{e^{2t}u(t-1)\}$,

$$a = 1, \text{ and } g(t-a) = g(t-1) = e^{2t}.$$

Since $t = (t-1)+1$, then

$$g(t-1) = e^{2[(t-1)+1]}$$

$$g(t) = e^{2t+2} = e^{2}.e^{2t}$$

Replace $(t-1)$ by t

$$\mathrm{G(s)} = \mathcal{L}\{g(t)\} = \mathcal{L}\{e^{2}.e^{2t}\} = e^{2}\left(\frac{1}{s-2}\right)$$

Therefore, $\mathcal{L}\{e^{2t}u(t-1)\} = e^{-s}\left(e^{2}\left(\frac{1}{s-2}\right)\right) = e^{-s+2}\left(\frac{1}{s-2}\right)$.

Hence,

$$F(s) = \mathcal{L}\{e^{2t}\} - \mathcal{L}\{e^{2t}u(t-1)\} + 2\mathcal{L}\{u(t-2)\}$$

$$= \frac{1}{s-2} - e^{-s+2}\left(\frac{1}{s-2}\right) + \frac{2e^{-2s}}{s}.$$

Laplace Transform of Unit Step Functions

$\mathcal{L}\{f(t)\} = F(s)$	
$f(t)$	$F(s)$
$u(t-a)$	$\frac{1}{s}e^{-as}$
$f(t-a)u(t-a)$	$e^{-as}F(s)$

Exercise 3.7

1. Given

$$f(t) = \begin{cases} \cos t\,; & 0 \le t < \pi \\ \sin t\,; & t \ge \pi \end{cases}$$

   Write in terms of unit step function and determine the Laplace transform.

2. Find the Laplace transform of

$$f(t) = \begin{cases} 1 - t; & 0 \le t < 1 \\ 0; & t \ge 1 \end{cases}$$

Answer:

1. $f(t) = \cos t + (\sin t - \cos t)u(t-\pi)$; $F(s) = \frac{s}{s^2+1} + e^{-\pi s}\left(\frac{s-1}{s^2+1}\right)$

2. $f(t) = 1 - t + (t-1)u(t-1)$; $F(s) = \frac{1}{s} - \frac{1}{s^2} + \frac{1}{s^2}e^{-s}$

References

- Rahifa et. al, Differential Equations for Engineering Students, Penerbit UTeM 2019.

Thank You

Question & Answer