BMCG 1013
DIFFERENTIAL EQUATIONS

LAPLACE TRANSFORM (PROPERTIES OF THE LAPLACE TRANSFORM)

Irma Wani binti Jamaludin

irma@utem.edu.my

Learning Outcomes

At the end of this topic, you will be able to:

1. Find the Laplace transform for a given function.
2. Use the linearity property of Laplace Transform.
3. Use shifting property in order to determine the Laplace Transform for a given function.

3.0 INTRODUCTION

- **Laplace transform** is an integral transform which transforms a function of a real variable, t domain(often time) to a function of a complex variable, s domain (frequency).

- It has many applications in science and engineering.

- The Laplace transform is particularly useful in solving linear ordinary differential equations such as those arising in the analysis of electronic circuits.

3.0 INTRODUCTION

Definition 3.1

The Laplace transform of a function is defined by

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\, dt$$

Example 3.1

Find the Laplace Transform of $f(t) = a$ where a is a constant.

Solution:

$$F(s) = \int_0^\infty e^{-st} f(t)\,dt = \int_0^\infty e^{-st}.a\,dt$$

$$= \left[\frac{ae^{-st}}{-s}\right]_0^\infty = -\frac{a}{s}[e^{-\infty} - e^0] = -\frac{a}{s}[0-1] = \frac{a}{s}$$

Thus, we have

$$\mathcal{L}\{a\} = \frac{a}{s}, s > 0$$

Example 3.2

Find $F(s)$ if $f(t) = e^{at}$ where a is a constant.

Solution:

$$F(s) = \int_0^{\infty} e^{-st} f(t)\, dt = \int_0^{\infty} e^{-st} . e^{at}\, dt$$

$$= \int_0^{\infty} e^{-(s-a)t}\; dt = \left[\frac{e^{-(s-a)t}}{-(s-a)}\right]_0^{\infty}$$

$$= -\frac{1}{s-a}\left[e^{-\infty} - e^{0}\right] = -\frac{1}{s-a}[0-1] = \frac{1}{s-a}$$

Thus, we have $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}, s > a$

Exercise 3.1

Using the definition of Laplace Transform, find $F(s)$ if:

(a) $f(t) = t$

(b) $f(t) = \begin{cases} 1; & 0 \leq t < 1 \\ 0; & t \geq 1 \end{cases}$

Answer: (a) $\frac{1}{s^2}$ *(b)* $\frac{1-e^{-s}}{s}$

Table 3.1 Elementary Laplace Transform

$f(t)$	$F(s) = \mathcal{L}\{f(t)\}$
a; where a is a constant	$\frac{a}{s}$
t^n; where $n = 1,2,3,\ldots$	$\frac{n!}{s^{n+1}}$
e^{at}	$\frac{1}{s-a}$
$\sin at$	$\frac{a}{s^2+a^2}$
$\cos at$	$\frac{s}{s^2+a^2}$
$\sinh at$	$\frac{a}{s^2-a^2}$
$\cosh at$	$\frac{s}{s^2-a^2}$

3.1 PROPERTIES OF THE LAPLACE TRANSFORM

3.1.1 Linearity property of Laplace Transform.

Theorem 3.1

Let $f(t), f_1(t)$ and $f_2(t)$ be functions whose Laplace Transforms exist for $s > a$ and c be a constant. Then for $s > a$,

$$\mathcal{L}\{f_1(t) \pm f_2(t)\} = \mathcal{L}\{f_1(t)\} \pm \mathcal{L}\{f_2(t)\}$$

and

$$\mathcal{L}\{cf(t)\} = c\mathcal{L}\{f(t)\}$$

Example 3.3

Given a function $f(t) = 2\sin 3t - e^{4t}$. Find the Laplace transform of $f(t)$.

Solution:

$$F(s) = \mathcal{L}\{2\sin 3t - e^{4t}\}$$

$$= \mathcal{L}\{2\sin 3t\} - \mathcal{L}\{e^{4t}\}$$

$$= 2\left(\frac{3}{s^2+3^2}\right) - \frac{1}{s-4}$$

$$= \frac{6}{s^2+9} - \frac{1}{s-4}$$

Example 3.4

Find the Laplace transform of the following functions:

(a) $f(t) = 3e^{-2t} + t^3$

(b) $g(t) = 2\cos 5t - \sin 5t$

(c) $h(t) = \left(t - t^2\right)^2$

Solution (a):

$$F(s) = \mathcal{L}\{3e^{-2t} + t^3\}$$

$$= \mathcal{L}\{3e^{-2t}\} + \mathcal{L}\{t^3\}$$

$$= 3\left(\frac{1}{s-(-2)}\right) + \frac{3!}{s^{3+1}}$$

$$= \frac{3}{s+2} + \frac{6}{s^4}$$

Solution (b):

$$G(s) = \mathcal{L}\{\,2\cos 5t - \sin 5t\}$$

$$= \mathcal{L}\{2\cos 5t\} - \mathcal{L}\{\sin 5t\}$$

$$= 2\left(\frac{s}{s^2+5^2}\right) - \frac{5}{s^2+5^2}$$

$$= \frac{2s-5}{s^2+25}$$

Solution (c):

$$H(s) = \mathcal{L}\{(t - t^2)^2\}$$

$$= \mathcal{L}\{t^2 - 2t^3 + t^4\}$$

$$= \mathcal{L}\{t^2\} - 2\mathcal{L}\{t^3\} + \mathcal{L}\{t^4\}$$

$$= \frac{2}{s^3} - 2\frac{3!}{s^4} + \frac{4!}{s^5}$$

$$= \frac{2}{s^3} - \frac{12}{s^4} + \frac{24}{s^5}$$

Exercise 3.2

Determine the Laplace transform of the following functions:

(a) $f(t) = t - t^2 + t^3$

(b) $g(t) = (\sin 2t)^2$

(c) $h(t) = (e^t - e^{-t})^2$

Answer:

(a) $\frac{1}{s^2} - \frac{2}{s^3} + \frac{6}{s^4}$

(b) $\frac{8}{s^3+16s}$

(c) $-\frac{2}{s} + \frac{1}{s+2} + \frac{1}{s-2}$

3.1 PROPERTIES OF THE LAPLACE TRANSFORM

3.1.2 First Shifting Property

Theorem 3.2

The Laplace Transform of $f(t) = e^{at}g(t)$ is given by

$$\mathcal{L}\{e^{at}g(t)\} = G(s-a)$$

where $G(s) = \mathcal{L}\{g(t)\}$.

Example 3.5

Given a function $f(t) = te^{3t}$. Find the Laplace transform of $f(t)$.

Solution:

Using the first shifting property,

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{te^{3t}\} = \mathcal{L}\{e^{at}g(t)\} = G(s-a)$$

with $a = 3$ and $g(t) = t$.

From $g(t) = t$, we will have $G(s) = \mathcal{L}\{g(t)\} = \mathcal{L}\{t\} = \frac{1}{s^2}$.

Thus,

$$\mathcal{L}\{te^{3t}\} = G(s-3) = \frac{1}{(s-3)^2}.$$

Example 3.6

Find $\mathcal{L}\{e^{2t}\sin 4t\}$.

Solution:

$$\mathcal{L}\{e^{2t}\sin 4t\} = \mathcal{L}\{e^{at}g(t)\} = G(s-a)$$

with $a = 2$ and $g(t) = \sin 4t$.

From $g(t) = \sin 4t$, we will have $G(s) = \mathcal{L}\{\sin 4t\} = \frac{4}{s^2+4^2}$.

Thus,

$$\mathcal{L}\{e^{2t}\sin 4t\} = G(s-2) = \frac{4}{(s-2)^2+16} = \frac{4}{s^2-4s+20}$$

Exercise 3.3

Using the first shifting property, determine the Laplace transform of the following functions:

(a) $f(t) = t^2 e^{-4t}$

(b) $f(t) = e^t \cos \pi t$

(c) $f(t) = -2e^{-t} \sinh t$

Answer:

(a) $\frac{2}{(s+4)^3}$ (b) $\frac{s-1}{(s-1)^2+\pi^2}$ (c) $-\frac{2}{(s+1)^2-1}$

3.1 PROPERTIES OF THE LAPLACE TRANSFORM

3.1.3 Differentiation of a Transform

Theorem 3.3

If $\mathcal{L}\{f(t)\} = F(s)$, then for $n = 1,2,3, \ldots$

$$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}[F(s)]$$

Example 3.7

Given that $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$. Using the formula

$$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}[F(s)],$$

find:

(a) $\mathcal{L}\{te^{at}\}$

(b) $\mathcal{L}\{t^2 e^{at}\}$

$$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}[F(s)]$$

Solution:

(a) $n = 1;\ f(t) = e^{at} \rightarrow F(s) = \frac{1}{s-a};$

$$\mathcal{L}\{te^{at}\} = (-1)\frac{d}{ds}\left(\frac{1}{s-a}\right) = -\frac{d}{ds}(s-a)^{-1}$$

$$= (s-a)^{-2}$$

$$= \frac{1}{(s-a)^2}.$$

$$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}[F(s)]$$

Solution:

(b) $n = 2; f(t) = e^{at} \to F(s) = \frac{1}{s-a};$

$$\mathcal{L}\{t^2 e^{at}\} = (-1)^2 \frac{d^2}{ds^2}\left(\frac{1}{s-a}\right) = \frac{d^2}{ds^2}(s-a)^{-1}$$

$$= -\frac{d}{ds}(s-a)^{-2}$$

$$= 2(s-a)^{-3} = \frac{2}{(s-a)^3}.$$

#Notes: We also can solve (a) and (b) using the first shifting property.

Example 3.8

Find the Laplace transform of the following functions:

(a) $f(t) = t^2 e^{3t}$

(b) $f(t) = t\cos 2t$

Solution:

(a) By using the solution from Example 3.7 (b) and with $a = 3$, we will have

$$\mathcal{L}\{t^2 e^{3t}\} = \frac{2}{(s-3)^3}.$$

$$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}[F(s)]$$

Solution:

(b) $\mathcal{L}\{t\cos 2t\} = -\frac{d}{ds}\left(\frac{s}{s^2+4}\right)$

$$= -\frac{d}{ds}\left[s\left(s^2+4\right)^{-1}\right]$$

$$= -\left\{s\left[-1\left(s^2+4\right)^{-2}.2s\right] + \left(s^2+4\right)^{-1}\right\}$$

$$= \frac{2s^2}{(s^2+4)^2} - \frac{1}{s^2+4}$$

$$= \frac{2s^2-s^2-4}{(s^2+4)^2} = \frac{s^2-4}{(s^2+4)^2}$$

Example 3.9

Find $\mathcal{L}\{te^t \cos 2t\}$.

Solution: (Method 1)

Applying first shifting properties,

$\mathcal{L}\{te^t \cos 2t\} = \mathcal{L}\{e^{at} g(t)\} = G(s-a)$

with $a = 1$ and $g(t) = t\cos 2t$.

From $g(t) = t\cos 2t$, we will have

Quotient rule

$$G(s) = \mathcal{L}\{t\cos 2t\} = -\frac{d}{ds}\left(\frac{s}{s^2+4}\right) = -\frac{d}{ds}\left[s\left(s^2+4\right)^{-1}\right] = \frac{s^2-4}{(s^2+4)^2}.$$

Thus, $\mathcal{L}\{f(t)\} = G(s-1) = \dfrac{(s-1)^2-4}{\left((s-1)^2+4\right)^2}$.

Solution: (Method 2)

Apply the differentiation of transform

$$\mathcal{L}\{te^t \cos 2t\} = -\frac{d}{ds}[F(s)]$$

where $f(t) = e^t \cos 2t$, and hence, $F(s) = \frac{s-1}{(s-1)^2+4}$.

$$\frac{d}{ds}\left[\frac{s-1}{(s-1)^2+4}\right] = \frac{[(s-1)^2+4](1)-(s-1)2(s-1)}{\left((s-1)^2+4\right)^2}$$

$$= \frac{[(s-1)^2+4]-2(s-1)^2}{\left((s-1)^2+4\right)^2} = \frac{-(s-1)^2+4}{\left((s-1)^2+4\right)^2}$$

Thus, $\mathcal{L}\{te^t \cos 2t\} = -\frac{d}{ds}[F(s)] = \frac{(s-1)^2-4}{\left((s-1)^2+4\right)^2}$.

Exercise 3.4

Find the Laplace transform of the following functions:

(a) $f(t) = t\cosh 4t$

(b) $y(t) = te^{-t}\sin 3t$

Answer:

(a) $\frac{s^2+16}{(s^2-16)^2}$

(b) $\frac{6(s+1)}{\left((s+1)^2+9\right)^2}$

References

- Rahifa et. al, Differential Equations for Engineering Students, Penerbit UTeM 2019.

Thank You

Question & Answer