BMCG 1013
DIFFERENTIAL EQUATIONS

SOLVING NONHOMOGENEOUS EQUATIONS WITH CONSTANT COEFFICIENTS

Khoo Chin Foon
chinfoon@utem.edu.my

Lesson outcomes

Upon completion of this week lesson, students should be able to:

i. describe the basic concept for methods of undetermined coefficients and variation of parameters

ii. find the solution for second order linear nonhomogeneous equations

CHAPTER 2

Second Order Linear Differential Equations

- Solving homogeneous equations with constant coefficients

  - real and distinct roots
  - real and repeated roots
  - complex conjugate roots

- Solving nonhomogeneous equations with

  - undetermined coefficients method
  - variation of parameters method

2.2 Solving Non-homogeneous Equations

The general form of the nonhomogeneous equation is

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=f(x)$$

or

$$ay''+by'+cy=f(x)$$

where a, b, c are some constants and $f(x)\neq 0.$

A second order linear differential equation is **nonhomogeneous** if $f(x)$ in the equation above is a **non-zero function**.

The general solution of the nonhomogeeous equation is given as

$$y(x) = y_c(x) + y_p(x)$$

where $y_c(x)$ is the **general solution** of the corresponding homogeneous equation (it is also called the complementary solution), while $y_p(x)$ is a **particular solution** (or known as a particular integral).

There are two common methods can be used for determining the particular solution, i.e. the method of

- undetermined coefficients and
- variation of parameters.

2.2.1 Solving Non-homogeneous Equations with Undetermined Coefficients method

The method of undetermined coefficients is an approach to find a particular solution $y_p(x)$ for certain nonhomogeneous ordinary differential equations. However, this method only works if $f(x)$ is in three basic forms or their combinations.

The method of undetermined coefficients is quite simple. First, we need to look at $f(x)$ and make a guess to the form of $y_p(x)$ by leaving the coefficient(s) undetermined. Then put the guess into the differential equation and check whether values of the coefficients can be found. If the values for the coefficients can be determined, then our guess is correct, else if the values for the coefficients cannot be found, then our guess is incorrect.

$f(x)$	$y_p(x)$
$p_n(x) = A_n x^n + A_{n-1}x^{n-1} + \ldots + A_1 x + A_0$	$C_n x^n + C_{n-1}x^{n-1} + \ldots + C_1 x + C_0$
$e^{\alpha x}$	$C e^{\alpha x}$
$\cos\beta x$ or $\sin\beta x$	$C_1\cos\beta x + C_2\sin\beta x$
$p_n(x)\, e^{\alpha x}$	$(C_n x^n + C_{n-1}x^{n-1} + \ldots + C_1 x + C_0)\, e^{\alpha x}$
$p_n(x)\begin{cases}\cos\beta x\\ \sin\beta x\end{cases}$	$(C_n x^n + C_{n-1}x^{n-1} + \ldots + C_1 x + C_0)\cos\beta x + (D_n x^n + D_{n-1}x^{n-1} + \ldots + D_1 x + D_0)\sin\beta x$
$e^{\alpha x}\begin{cases}\cos\beta x\\ \sin\beta x\end{cases}$	$e^{\alpha x}\,(C_1\cos\beta x + C_2\sin\beta x)$
$p_n(x).e^{\alpha x}.\begin{cases}\cos\beta x\\ \sin\beta x\end{cases}$	$(C_n x^n + C_{n-1}x^{n-1} + \ldots + C_1 x + C_0)\, e^{\alpha x}\cos\beta x + (D_n x^n + D_{n-1}x^{n-1} + \ldots + D_1 x + D_0)\, e^{\alpha x}\sin\beta x$

In general, there are two cases to be considered:

(i) no function in $y_p(x)$ has the same form as in $y_c(x)$

(ii) there is function in $y_p(x)$ same with the term in $y_c(x)$

For case (ii), if $y_p(x)$ has the same term with $y_c(x)$, then we **multiply $y_p(x)$ with x^n**, where n is the least positive integer, to eliminate the term.

Example 2.7

Find the general solution of

$$\frac{d^2y}{dx^2} + 4y = x^2 + 2 \qquad (1)$$

Solution

The characteristic equation of the equation is

$$m^2 + 4 = 0$$

$$m^2 = -4 = 4i^2$$

$$m = \pm 2i$$

So

$$y_c(x) = A\cos 2x + B\sin 2x.$$

The particular integral is guessed as

$$y_p(x) = C_2 x^2 + C_1 x + C_0 \qquad (2)$$

Next, find the first and second derivatives of $y_p(x)$:

$$y_p'(x) = 2C_2 x + C_1$$

$$y_p''(x) = 2C_2 \qquad (3)$$

Substitute (2) and (3) into the left hand side (LHS) of the equation (1), hence

$$\begin{aligned} &2C_2 + 4\left(C_2x^2 + C_1x + C_0\right) \\ &= 2C_2 + 4C_2x^2 + 4C_1x + 4C_0 \\ &= 4C_2x^2 + 4C_1x + \left(2C_2 + 4C_0\right) \end{aligned} \qquad (4)$$

Now compare the coefficients of x^n in equation (4) with the right hand side (RHS) of the equation (1):

$$4C_2x^2 + 4C_1x + (2C_2 + 4C_0) = x^2 + 2$$

Then we have

coefficient for x^2:

$$4C_2 = 1 \quad \Rightarrow \quad C_2 = \frac{1}{4}.$$

coefficient for x^1 :

$$4C_1 = 0 \quad \Rightarrow \quad C_1 = 0$$

coefficient for x^0 :

$$2C_2 + 4C_0 = 2$$

$$2\left(\frac{1}{4}\right) + 4C_0 = 2 \Rightarrow \quad C_0 = \frac{1}{4}\left(2 - \frac{2}{4}\right) = \frac{3}{8}$$

$$\therefore y_p = \frac{1}{4}x^2 + \frac{3}{8}$$

$$y(x) = y_c + y_p$$

$$= A\cos 2x + B\sin 2x + \frac{1}{4}x^2 + \frac{3}{8}$$

Example 2.8

Find the general solution of

$$\frac{d^2 y}{dx^2} + \frac{dy}{dx} - 6y = e^{-3x} \qquad (5)$$

Solution

The characteristic equation of the equation is

$$m^2 + m - 6 = 0$$

$$(m-2)(m+3) = 0$$

$$m = 2, \;\; m = -3$$

So

$$y_c(x) = Ae^{2x} + Be^{-3x}.$$

The particular integral is guessed as

$$y_p(x) = Ce^{-3x}$$

The **term e^{-3x} in $y_p(x)$ is same with a term in $y_c(x)$**, hence we need a new particular integral, namely

$$y_p(x) = Cxe^{-3x} \qquad (6)$$

Find the first and second derivatives of $y_p(x)$:

$$y_p'(x) = -3Cxe^{-3x} + Ce^{-3x}$$

$$y_p''(x) = 9Cxe^{-3x} - 3Ce^{-3x} - 3Ce^{-3x} = 9Cxe^{-3x} - 6Ce^{-3x} \qquad (7)$$

Substitute (6) and (7) into the left hand side (LHS) of the equation (5), hence

$$9Cxe^{-3x} - 6Ce^{-3x} - 3Cxe^{-3x} + Ce^{-3x} - 6Cxe^{-3x} = -5Ce^{-3x} \tag{8}$$

Now compare the coefficients of e^{-3x} in equation (8) with the right hand side (RHS) of the equation (5):

$$e^{-3x} = -5Ce^{-3x}$$

$$C = -\frac{1}{5}$$

$$y_p(x) = -\frac{1}{5}xe^{-x}.$$

$$y(x) = y_c(x) + y_p(x)$$

$$= Ae^{2x} + Be^{-3x} - \frac{1}{5}xe^{-3x}.$$

Exercise 2.4

Find the solution of

$$\frac{d^2y}{dx^2} + y = e^{3x}.$$

Answer:

$$y(x) = A\cos x + B\sin x + \frac{1}{10}e^{3x}$$

Exercise 2.5

Find the solution of

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = \sin x.$$

Answer:

$$y(x) = Ae^{2x} + Be^{x} + \frac{1}{10}e^{3x}$$

Example 2.9

Find the particular solution for

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = x^2 - x + 2\sin x \qquad (9)$$

where $y(0) = 0,\ y'(0) = 1.$

Solution

The characteristic equation of the equation is

$$m^2 - 4m + 4 = 0$$

$$(m-2)(m-2) = 0$$

$$m = 2$$

So $y_c(x) = (A + Bx)e^{2x}$.

The particular integral is guessed as

$$y_p(x) = C_2 x^2 + C_1 x + C_0 + C_4 \sin x + C_5 \cos x \qquad (10)$$

Next, find the first and second derivatives of $y_p(x)$:

$$y_p' = 2C_2 x + C_1 + C_4 \cos x - C_5 \sin x$$

$$y_p'' = 2C_2 - C_4 \sin x - C_5 \cos x \qquad (11)$$

Substitute (10) and (11) into the left hand side (LHS) of the equation (9), hence

$$
\begin{aligned}
&2C_2 - C_4 \sin x - C_5 \cos x - 4\left(2C_2 x + C_1 + C_4 \cos x - C_5 \sin x\right) \\
&+4\left(C_2 x^2 + C_1 x + C_0 + C_4 \sin x + C_5 \cos x\right) \\
&= 4C_2 x^2 + (-8C_2 + 4C_1)x + (2C_2 - 4C_1 + 4C_0) \\
&\quad + \cos x(-C_5 - 4C_4 + 4C_5) + \sin x(-C_4 + 4C_5 + 4C_4)
\end{aligned}
\qquad (12)
$$

Now compare the coefficients of x^n, $\cos x$ and $\sin x$ in equation (12) with the right hand side (RHS) of the equation (9):

$$4C_2x^2+(-8C_2+4C_1)x+(2C_2-4C_1+4C_0)$$
$$+\cos x(-C_5-4C_4+4C_5)+\sin x(-C_4+4C_5+4C_4)$$
$$=x^2-x+2\sin x$$

Then we have

coefficient for x^2:

$$4C_2 = 1 \quad \Rightarrow \quad C_2 = \frac{1}{4}.$$

coefficient for x^1 :

$$-8C_2 + 4C_1 = -1 \qquad \Rightarrow \quad -8\left(\frac{1}{4}\right) + 4C_1 = 0 \Rightarrow \quad C_1 = \frac{1}{4}$$

coefficient for x^0 :

$$2C_2 - 4C_1 + 4C_0 = 0$$

$$2\left(\frac{1}{4}\right) + 4\left(\frac{1}{4}\right) + 4C_0 = 0 \Rightarrow \quad C_0 = \frac{1}{4}\left(4\left(\frac{1}{4}\right) - 2\left(\frac{1}{4}\right)\right) = \frac{1}{8}$$

coefficient for $\cos x$:

$$3C_5 - 4C_4 = 0 \quad \Rightarrow \quad C_5 = \frac{4}{3}C_4.$$

coefficient for $\sin x$:

$$3C_4 + 4C_5 = 2 \qquad \Rightarrow \quad 3C_4 + \frac{16}{3}C_4 = \frac{25}{3}C_4 = 2 \Rightarrow \quad C_4 = \frac{6}{25}$$

Then

$$C_5 = \frac{4}{3}C_4 = \frac{4}{3}\left(\frac{6}{25}\right) = \frac{8}{25}$$

$$\therefore y_p = \frac{1}{4}x^2 + \frac{1}{4}x + \frac{1}{8} + \frac{6}{25}\sin x + \frac{8}{25}\cos x$$

$$\begin{aligned} y(x) &= y_c + y_p \\ &= (A + Bx)e^{2x} + \frac{1}{4}x^2 + \frac{1}{4}x + \frac{1}{8} + \frac{6}{25}\sin x + \frac{8}{25}\cos x \end{aligned}$$

Given $y(0) = 0$,

$$0 = Ae^0 + B \cdot 0 \cdot e^0 + \frac{1}{4}(0)^2 + \frac{1}{4}(0) + \frac{1}{8} + \frac{6}{25}\sin 0 + \frac{8}{25}\cos 0$$

$$0 = A + \frac{1}{8} + \frac{8}{25}$$

$$A = -\frac{89}{200}$$

Given $y'(0) = 1$,

$$y'(x) = 2Ae^{2x} + Be^{2x} + 2Bxe^{2x} + \frac{1}{2}x + \frac{1}{4} + \frac{6}{25}\cos x - \frac{8}{25}\sin x$$

$$1 = 2Ae^{0} + Be^{0} + 2B \cdot 0 \cdot e^{0} + \frac{1}{2}(0) + \frac{1}{4} + \frac{6}{25}\cos 0 - \frac{8}{25}\sin 0$$

$$1 = 2A + B + \frac{1}{4} + \frac{6}{25}$$

$$1 = 2\left(-\frac{89}{200}\right) + B + \frac{49}{100}$$

$$B = \frac{7}{5}$$

Hence

$$y(x) = -\frac{89}{200}e^{2x} + \frac{7}{5}xe^{2x} + \frac{1}{4}x^2 + \frac{1}{4}x + \frac{1}{8} + \frac{6}{25}\sin x + \frac{8}{25}\cos x.$$

Exercise 2.6

Solve

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = e^{-x}\left(x+1\right)$$

where $y(0) = 0,\ y'(0) = 2.$

Answer: $$y(x) = \frac{22}{9}e^{2x} - \frac{11}{4}e^{x} + \left(\frac{1}{6}x + \frac{11}{36}\right)e^{-x}$$

2.2.2 Solving Non-homogeneous Equations with Variation of Parameters method

The method of undetermined coefficient cannot be used if $f(x)$ has the either forms $\tan(x)$, $\cot(x)$, $\sec(x)$, $\operatorname{cosec}(x)$, $\ln(x)$, etc. In such cases, the method of variation of parameters can be used to get the solution for the nonhomogeneous equations.

Variation of Parameters method

For a nonhomogeneous equation $a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=f(x)$

- where a, b, c are some constants,

Step 1: Get the solution for the homogeneous equation, that is, y_1 and y_2.

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Step 2: Compute the Wronskian value,

$$W=\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}=y_1y_2'-y_2y_1'$$

Step 3: Compute

$$u=-\int\frac{y_2 f(x)}{aW}dx \quad \text{and} \quad v=\int\frac{y_1 f(x)}{aW}dx$$

Step 4: Find the particular integral $y_p(x)$ by using the formula

$$y_p(x) = uy_1 + vy_2.$$

Step 5: The general solution of the equation is

$$\begin{aligned} y(x) &= y_c(x) + y_p(x) \\ &= Ay_1 + By_2 + uy_1 + vy_2. \end{aligned}$$

Example 2.10

Solve the differential equation

$$y'' - 4y' + 4y = xe^{2x} + e^{2x}.$$

Solution

Step 1

The characteristic equation of the equation is

$$m^2 - 4m + 4 = 0$$

$$(m-2)(m-2) = 0$$

$$m = 2$$

$$\therefore \quad y_c(x) = (A + Bx)e^{2x}.$$

Then we have

$$y_1 = e^{2x}, \quad y_2 = xe^{2x}$$

$$y_1' = 2e^{2x}, \quad y_2' = 2xe^{2x} + e^{2x}$$

Step 2

$$W = \begin{vmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & 2xe^{2x} + e^{2x} \end{vmatrix} = 2xe^{4x} + e^{4x} - 2xe^{4x} = e^{4x}$$

Step 3

$$u = -\int \frac{xe^{2x}(xe^{2x} + e^{2x})}{e^{4x}} dx$$

$$= -\int \frac{xe^{4x}(x+1)}{e^{4x}} dx$$

$$= -\int x(x+1)dx$$

$$= -\frac{x^3}{3} - \frac{x^2}{2}$$

$$v = \int \frac{e^{2x}(xe^{2x} + e^{2x})}{e^{4x}} dx$$

$$= \int \frac{e^{4x}(x+1)}{e^{4x}} dx$$

$$= \frac{x^2}{2} + x$$

Step 4

$$y_p(x) = \left(-\frac{x^3}{3} - \frac{x^2}{2}\right)e^{2x} + \left(\frac{x^2}{2} + x\right)xe^{2x}$$

Step 5

$$y(x) = y_c + y_p$$

$$= (A + Bx)e^{2x} + \left(-\frac{x^3}{3} - \frac{x^2}{2}\right)e^{2x} + \left(\frac{x^2}{2} + x\right)xe^{2x}$$

Exercise 2.7

Find the solution of

$$y'' + y = \sec x$$

by using the variation of parameters method.

Answer:

$$y(x) = A\cos x + B\sin x + \cos x \ln\left|\cos x\right| + x\sin x$$

☞ Are you able to

i. describe the basic concept for methods of undetermined coefficients and variation of parameters?

ii. find the solution for second order linear nonhomogeneous equations now?

References

Edwards C. H., Penny D.E. & Calvis D. (2016). Differential Equations and Boundary Value Problems, 5thEdition. Pearson Education Inc..

Zill, D. G. (2017). Differential Equations with Boundary-Value Problems. Cengage Learning Inc

Thank You

Questions & Answer?