BMCG 1013
DIFFERENTIAL EQUATIONS

SOLVING HOMOGENEOUS EQUATIONS WITH CONSTANT COEFFICIENTS

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Lesson outcomes

Upon completion of this week lesson, students should be able to:

i. describe the basic concept of homogeneous equation

ii. describe the three possible solutions of homogeneous equation

iii. find the solution for homogeneous equations

CHAPTER 2

Second Order Linear Differential Equations

- Solving homogeneous equations with constant coefficients
  - real and distinct roots
  - real and repeated roots
  - complex conjugate roots

- Solving nonhomogeneous equations with
  - undetermined coefficients method
  - variation of parameters method

2.1 Solving Homogeneous Equations with Constant Coefficients

General form of the second-order linear differential equation:

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=f(x)$$

where a, b, c are constants.

If $f(x)=0$, the second order linear differential equation is homogeneous.

The homogeneous equation

$$a\frac{d^2y}{dx^2}+b\frac{dy}{dx}+cy=0 \quad \text{or} \quad ay''+by'+cy=0$$

has a **characteristic equation** as follows:

$$am^2+bm+c=0$$

where the derivatives are replaced by

$$y'' \to m^2,\ y' \to m^1,\ y \to m^o.$$

Since the characteristic equation is a quadratic equation, there are 3 possible cases for the roots, i.e. when

i. $b^2 - 4ac > 0,$ real and distinct roots

ii. $b^2 - 4ac = 0,$ real and repeated roots

iii. $b^2 - 4ac < 0,$ complex conjugates roots

The general solution for a characteristic equation with **real and distinct roots** is

$$y = Ae^{m_1 x} + Be^{m_2 x}$$

A and B are some constants.

Here $m_1 \neq m_2$ are two real and distinct roots .

Example 2.1

Find the general solution of

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 6y = 0$$

Solution

The characteristic equation for the homogeneous equation is

$$m^2 - m - 6 = 0$$

This characteristic equation can be factorised as

$(m - 3)(m + 2) = 0.$

Hence $m_1 = 3$, $m_2 = -2$.

As a consequence, the general solution for the homogeneous equation is

$$y = Ae^{3x} + Be^{-2x}.$$

Exercise 2.1

Find the general solution of

$$\frac{d^2y}{dx^2}+5\frac{dy}{dx}+6y=0$$

Answer:

$$y(x)=Ae^{-3x}+Be^{-2x}$$

Example 2.2

Find the solution of

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y = 0$$

where the initial conditions are given as

$$y(0) = 1,\ y'(0) = 1.$$

Solution

Characteristic equation:

$$m^2 - 4m + 3 = 0$$

$$(m-3)(m-1) = 0$$

$$m_1 = 3,\ m_2 = 1$$

$$\therefore\ y(x) = Ae^{3x} + Be^{x}$$

By referring to the initial conditions, we determine the constants A and B :

$$y(0)=1 \quad \Rightarrow \quad Ae^{0}+Be^{0}=1$$

$$A+B=1$$

$$B=1-A$$

Now find the first derivative of $y(x)$, we have

$$y'(x)=3Ae^{3x}+Be^{x}.$$

$$y'(0) = 1 \quad \Rightarrow \quad 3Ae^{0} + Be^{0} = 1$$

$$3A + B = 1$$

$$3A + (1 - A) = 1$$

$$2A = 0$$

$$A = 0,$$

$$B = 1 - 0 = 1$$

$$\therefore \quad y(x) = (0)e^{3x} + (1)e^{x} = e^{x}$$

The general solution for a characteristic equation with **real but repeated roots** is

$$y = (A + Bx)e^{mx}$$

where A and B are some constants.

Here m is real but repeated root, that is, $m = m_1 = m_2$.

Example 2.3

Find the general solution of

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 0$$

Solution

The characteristic equation for the homogeneous equation is

$$m^2 - 2m + 1 = 0$$

This characteristic equation can be factorised as

$$(m - 1)\,(m - 1) = 0.$$

Hence $m = 1$.

As a consequence, the general solution for the homogeneous equation is

$$y = (A + Bx)e^x.$$

Example 2.4

Find the solution of

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 0$$

where the initial conditions are given as

$$y(0) = 3,\ y'(0) = 1.$$

Solution

Characteristic equation:

$$m^2 - 2m + 1 = 0$$

$$(m-1)(m-1) = 0$$

$$m = 1$$

$$\therefore \quad y(x) = (A + Bx)e^x$$

By referring to the initial conditions, we determine the constants A and B :

$$y(0)=3 \quad \Rightarrow \quad \big(A+B(0)\big)e^{0}=3$$

$$A=3$$

Now find the first derivative of $y(x)$, we have

$$y'(x)=e^{x}(B)+(A+Bx)e^{x}$$

$$=(A+B)e^{x}+Bxe^{x}.$$

$$y'(0) = 1 \quad \Rightarrow \quad (3+B)e^{0} + B(0)e^{0} = 1$$

$$3 + B = 1$$

$$B = 1 - 3 = -2$$

$$\therefore \; y(x) = \left(3 - 2x\right)e^{x}$$

Exercise 2.2

Find the solution of

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 0$$

where the initial conditions are given as

$$y(0) = 3,\ y'(0) = 1.$$

Answer: $y(x) = (3 - 5x)e^{2x}$

The general solution for a characteristic equation with **complex conjugates roots** is

$$y = e^{\alpha x}\left(A\cos\beta x + B\sin\beta x\right)$$

where A and B are some constants.

Here $m = \alpha \pm \beta i$.

Example 2.5

Find the general solution of

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 5y = 0$$

Solution

Characteristic equation: $m^2 + 4m + 5 = 0$

Find the roots of the characteristic equation:

$$m = \frac{-4 \pm \sqrt{16 - 4(1)(5)}}{2} = \frac{-4 \pm \sqrt{-4}}{2} = -2 \pm i\,,$$

$$\alpha = -2,\ \beta = 1$$

Hence $y(x) = e^{-2x}\left(A\cos x + B\sin x\right)$

Example 2.6

Find the solution of

$$\frac{d^2y}{dx^2}+4y=0$$

where the boundary conditions are given as

$$y(0)=1,\ y\left(\frac{\pi}{4}\right)=-1.$$

Solution

Characteristic equation: $m^2 + 4 = 0$

Find the roots of the characteristic equation:

$$m^2 + 4 = 0$$

$$m^2 = -4 = 4(\sqrt{-1})^2 = 4i^2$$

$$m = \pm 2i$$

$$\therefore \alpha = 0, \beta = 2$$

Hence $y(x) = e^{0 \cdot x}\left(A\cos 2x + B\sin 2x\right)$

$$= A\cos 2x + B\sin 2x$$

By referring to the boundary conditions:

$$y(0) = 1,$$

$$y(0) = A\cos 0 + B\sin 0 = A \quad \Rightarrow \quad A = 1$$

$$y\left(\frac{\pi}{4}\right) = -1,$$

$$y\left(\frac{\pi}{4}\right) = A\cos\left(2\left(\frac{\pi}{4}\right)\right) + B\sin\left(2\left(\frac{\pi}{4}\right)\right)$$

$$= A\cos\frac{\pi}{2} + B\sin\frac{\pi}{2} = B \Rightarrow \quad B = -1$$

$$\therefore \quad y(x) = \cos 2x - \sin 2x$$

Exercise 2.3

Find the solution of

$$\frac{d^2y}{dx^2} + y = 0$$

where the boundary conditions are given as

$$y(0) = 3,\ y\left(\frac{\pi}{2}\right) = -3.$$

Answer: $$y(x) = 3\cos x - 3\sin x$$

☞ Are you able to

i. describe the basic concept of homogeneous equation?

i. describe the three possible solutions of homogeneous equation?

ii. find the solution for a homogeneous equation now?

References

Edwards C. H., Penny D.E. & Calvis D. (2016). Differential Equations and Boundary Value Problems, 5thEdition. Pearson Education Inc..

Zill, D. G. (2017). Differential Equations with Boundary-Value Problems. Cengage Learning Inc

Thank You

Questions & Answer?