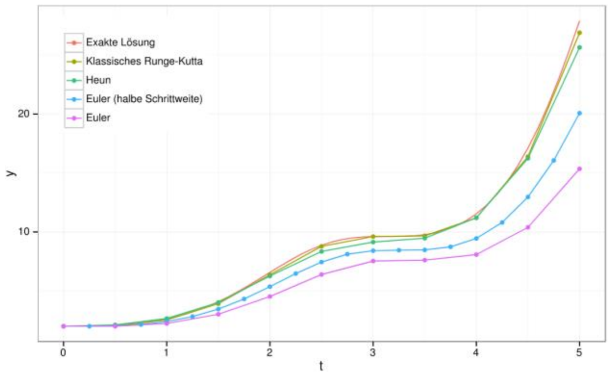
BMCG 1013
DIFFERENTIAL EQUATIONS

Euler's Method & Runge-Kutta Method



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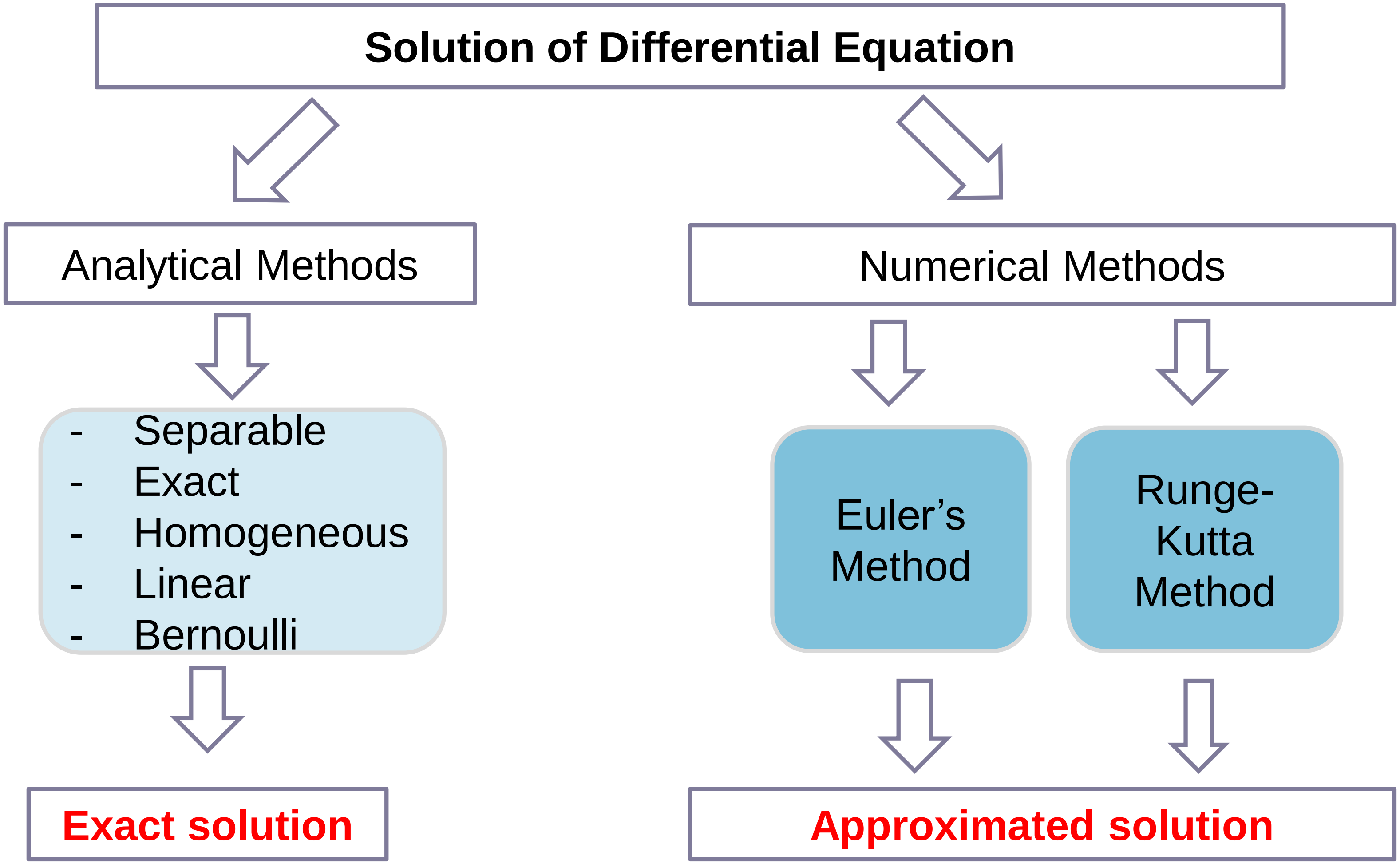
Lesson Outcomes

Upon completion of this lesson, students should be able to:

- solve the first order ordinary differential equation using Euler’s Method
- solve the first order ordinary differential equation using Runge-Kutta Method

1.3 Numerical Methods for DE



1.3.1 Euler's Method

Why is Numerical Method used?

- ✓ Numerical method gives numerical approximation to the solution of ordinary differential equations
- ✓ Numeric approximation is often sufficient for practical purposes, for example, in Engineering.

What is Euler's Method?

- ✓ Most basic explicit method for solving ordinary differential equation (ODE)
- ✓ The simplest Runge-Kutta Method

Formula:

$$y_{i+1} = y_i + hf(x_i, y_i)$$

Basic idea of Euler's Method:

Point $P_i = (x_i, y_i)$ is the numerical approximation for the point $(x_i, y(x_i))$ on the curve $y(x)$.

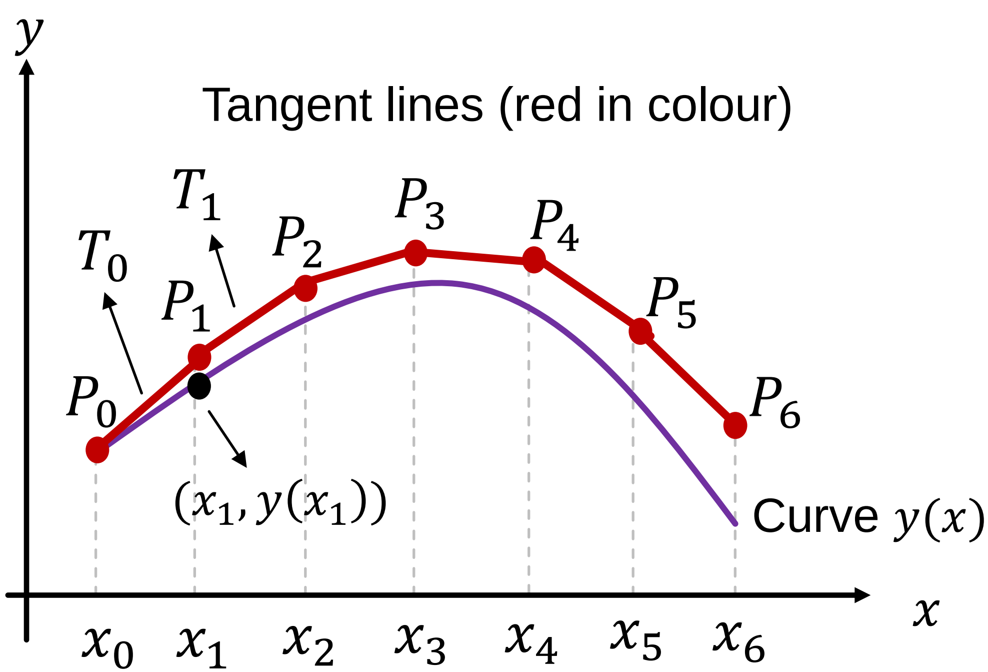


Figure 1.1: Graphical interpretation of Euler's method

- Initially, given $P_0 = (x_0, y_0) = (x_0, y(x_0))$ lying on curve $y(x)$.
- A tangent line, T_0, on the initial point P_0 is used to estimate the next point, P_1, at x_1.
- An error exists between the actual point $(x_1, y(x_1))$ and the estimated point $P_1 = (x_1, y_1)$.
- Next, tangent line, T_1, which is estimated from P_1 with slope $f(x_1, y_1)$, is used to estimate the next point, P_2, at x_2.
- This process is repeated until x_n.

Basic idea of Euler's Method:

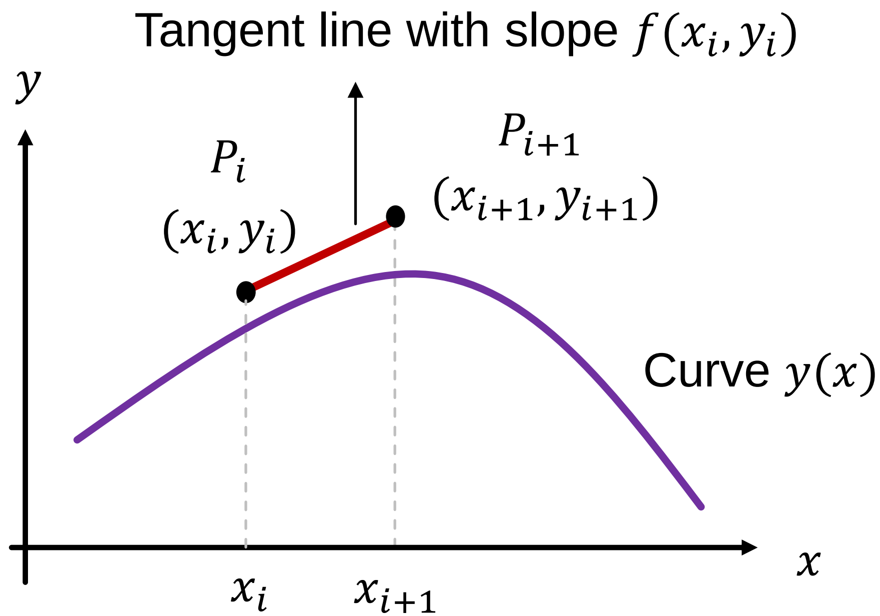


Figure 1.2: Tangent line from Euler's method

Based on the basic idea of Euler's Method:

Point $P_{i+1} = (x_{i+1}, y_{i+1})$ is estimated from the tangent line which is passing through the point P_i and the slope is estimated from y_i, i.e. $f(x_i, y_i)$.

Derivation of Euler's formula:

Now, given an IVP:

$$\frac{dy}{dx} = f(x, y), \qquad y(x_0) = y_0$$

where $y(x)$ is the solution of the ODE.

To obtain the tangent line with gradient

$$\left.\frac{dy}{dx}\right|_{x=x_i} = f(x_i, y_i)$$

that passing through point $P_i = (x_i, y_i)$, we have

$$y - y_i = f(x_i, y_i)[x - x_i].$$

Hence, to estimate point $P_{i+1} = (x_{i+1}, y_{i+1})$,

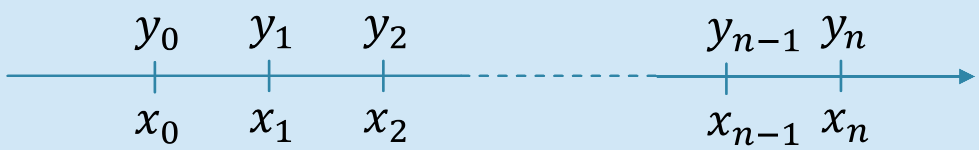
$$y_{i+1} - y_i = f(x_i, y_i)[x_{i+1} - x_i].$$

Since step size, $h = x_{i+1} - x_i$, the formula can be simplified into

$$y_{i+1} = y_i + hf(x_i, y_i) \qquad (1.1)$$

The flow of solving ODE using Euler's method:

Draw a line structure and pull in the initial condition



↓

Rearrange the given ODE in the form of

$$y' = f(x, y)$$

↓

Discretize the equation into nth terms:

$$y_i' = f(x_i, y_i); \quad i = 0,1,2, \dots n-1$$

↓

Compute the next term by using formula

$$y_{i+1} = y_i + hf(x_i, y_i); \quad i = 0,1,2, \dots n-1$$

Example 1.11:

Use Euler's method to numerically integrate

$$\frac{dy}{dx} + y = 3x^3 - 7x^2 + 5x, \qquad y(0) = 1$$

from $x = 0$ to $x = 2$ with a step size of 0.5.

Solution:

$$\frac{dy}{dx} = f(x, y) = 3x^3 - 7x^2 + 5x - y$$
$$0 \leq x \leq 2, \quad h = 0.5, \quad y(0) = 1$$

Reminder:
$y' = f(x, y)$

Construct the time-line:

$1 \,\|\, y_0$	y_1	y_2	y_3	y_4
$x_0 \,\|\, 0$	$x_1 \,\|\, 0.5$	$x_2 \,\|\, 1.0$	$x_3 \,\|\, 1.5$	$x_4 \,\|\, 2.0$

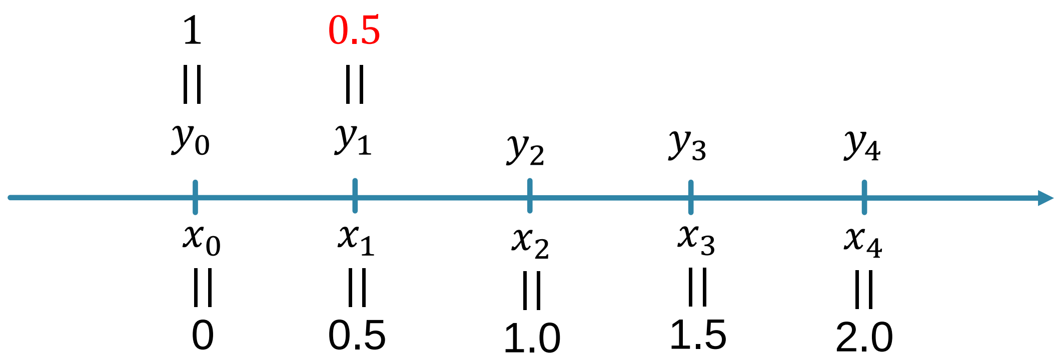
Let $i = 0$,

$$\begin{aligned} y_1 &= y_0 + hf(x_0, y_0) \\ &= 1 + 0.5f(0, 1) \\ &= 1 + 0.5[3(0)^3 - 7(0)^2 + 5(0) - 1] \\ &= 0.5 \end{aligned}$$

$$\frac{dy}{dx} = f(x, y) = 3x^3 - 7x^2 + 5x - y$$
$$0 \le x \le 2, \quad h = 0.5, \quad y(0) = 1$$

Reminder:
$y' = f(x, y)$

Construct the time-line:



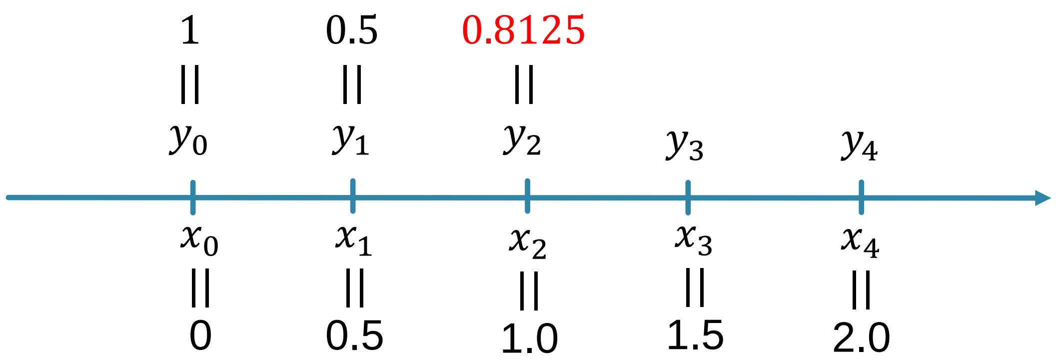
Let $i = 1$,

$$\begin{aligned} y_2 &= y_1 + hf(x_1, y_1) \\ &= 0.5 + 0.5f(0.5, 0.5) \\ &= 0.5 + 0.5[3(0.5)^3 - 7(0.5)^2 + 5(0.5) - 0.5] \\ &= 0.8125 \end{aligned}$$

$$\frac{dy}{dx} = f(x, y) = 3x^3 - 7x^2 + 5x - y$$
$$0 \le x \le 2, \quad h = 0.5, \quad y(0) = 1$$

Reminder:
$y' = f(x, y)$

Construct the time-line:



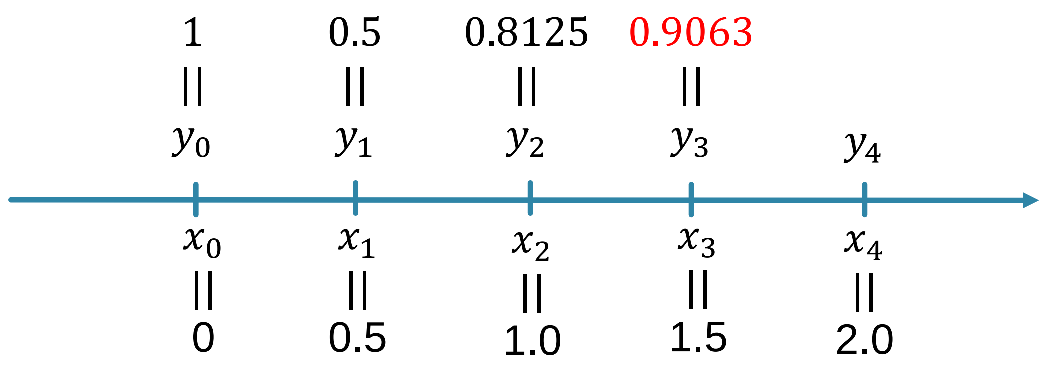
Let $i = 2$,

$$y_3 = y_2 + hf(x_2, y_2)$$
$$= 0.8125 + 0.5f(1, 0.8125)$$
$$= 0.8125 + 0.5[3(1)^3 - 7(1)^2 + 5(1) - 0.8125]$$
$$= 0.9063$$

$$\frac{dy}{dx} = f(x,y) = 3x^3 - 7x^2 + 5x - y$$
$$0 \le x \le 2, \quad h = 0.5, \quad y(0) = 1$$

Reminder:
$y' = f(x,y)$

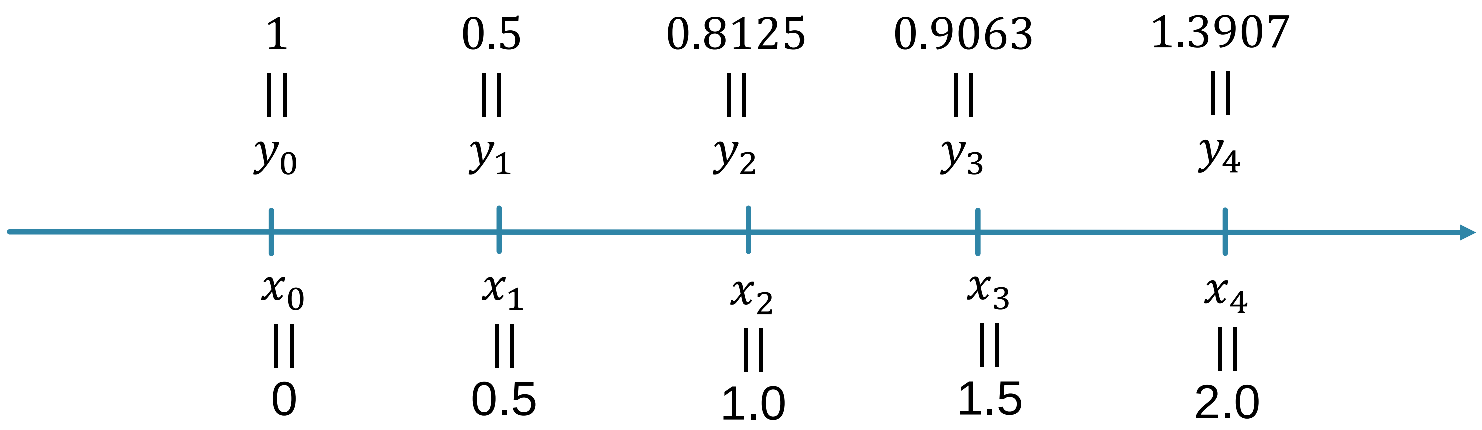
Construct the time-line:



Let $i = 3$,

$$\begin{aligned} y_4 &= y_3 + hf(x_3, y_3) \\ &= 0.9063 + 0.5f(1.5, 0.9063) \\ &= 0.9063 + 0.5[3(1.5)^3 - 7(1.5)^2 + 5(1.5) - 0.9063] \\ &= 1.3907 \end{aligned}$$

Update the time-line:



Example 1.12:

Use Euler's method to solve the initial value problem

$$\frac{dy}{dt} = \frac{t - y}{2}$$

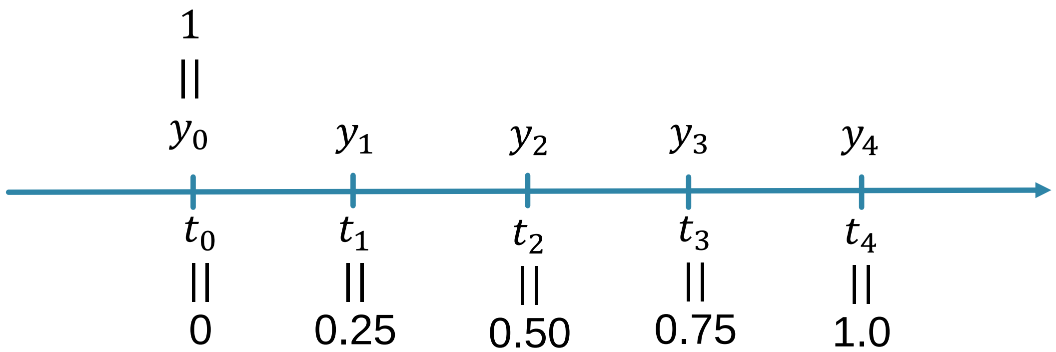
on $0 \leq t \leq 1$ with $h = 0.25$. The initial condition at $t = 0$ is $y = 1$.

Solution:

$$\frac{dy}{dt} = f(t,y) = \frac{t-y}{2} \quad 0 \le t \le 1,$$
$$h = 0.25, \quad y(0) = 1$$

Reminder:
$y' = f(t,y)$

Construct the time-line:

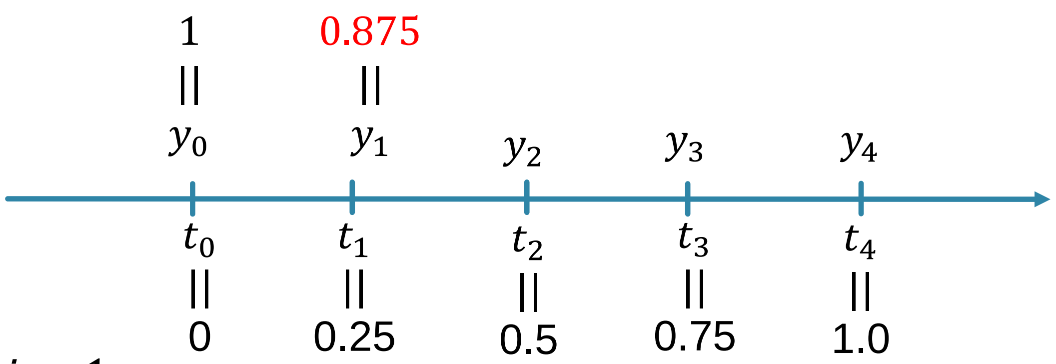


Let $i = 0$,

$$\begin{aligned} y_1 &= y_0 + hf(t_0, y_0) \\ &= 1 + 0.25f(0,1) \\ &= 1 + 0.25\left[\frac{0-1}{2}\right] \\ &= 0.875 \end{aligned}$$

$$\frac{dy}{dt} = f(t,y) = \frac{t-y}{2} \quad 0 \le t \le 1,$$

Construct the time-line:



Let $i = 1$,

$$y_2 = y_1 + hf(t_1, y_1)$$

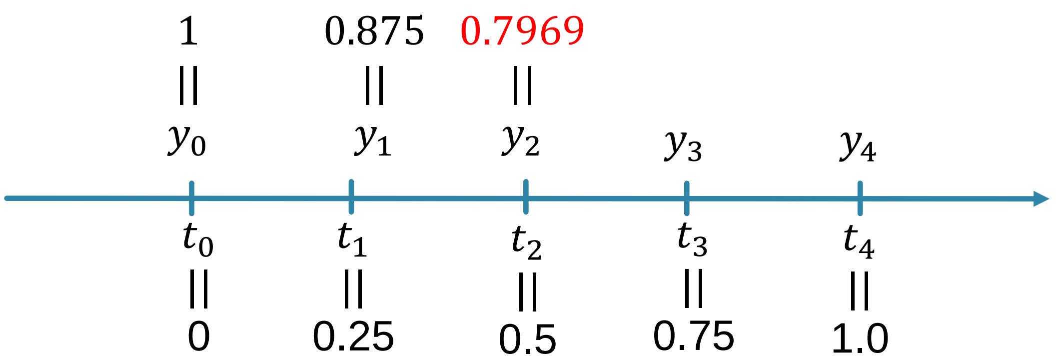
$$= 0.875 + 0.25 f(0.25, 0.875)$$

$$= 0.875 + 0.25\left[\frac{0.25-0.875}{2}\right]$$

$$= 0.7969$$

$$\frac{dy}{dt} = f(t,y) = \frac{t-y}{2} \quad 0 \le t \le 1,$$

Construct the time-line:

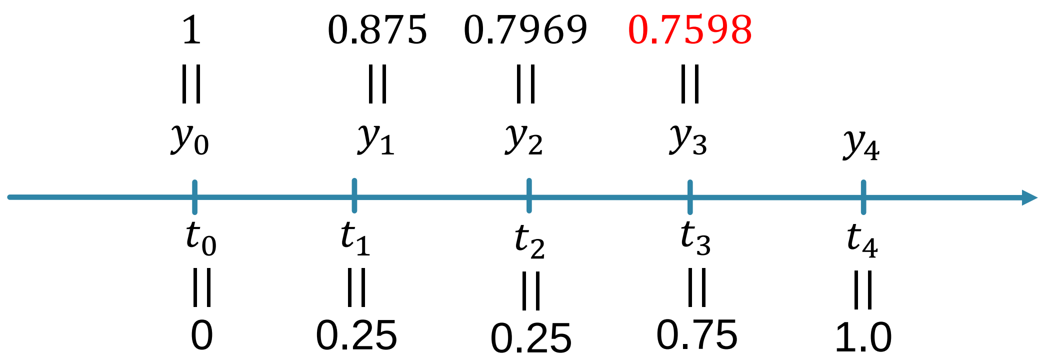


Let $i = 2$,

$$y_3 = y_2 + hf(t_2, y_2)$$
$$= 0.7969 + 0.25f(0.5, 0.7969)$$
$$= 0.7969 + 0.25\left[\frac{0.5-0.7969}{2}\right]$$
$$= 0.7598$$

$$\frac{dy}{dt} = f(t, y) = \frac{t-y}{2} \quad 0 \leq t \leq 1,$$

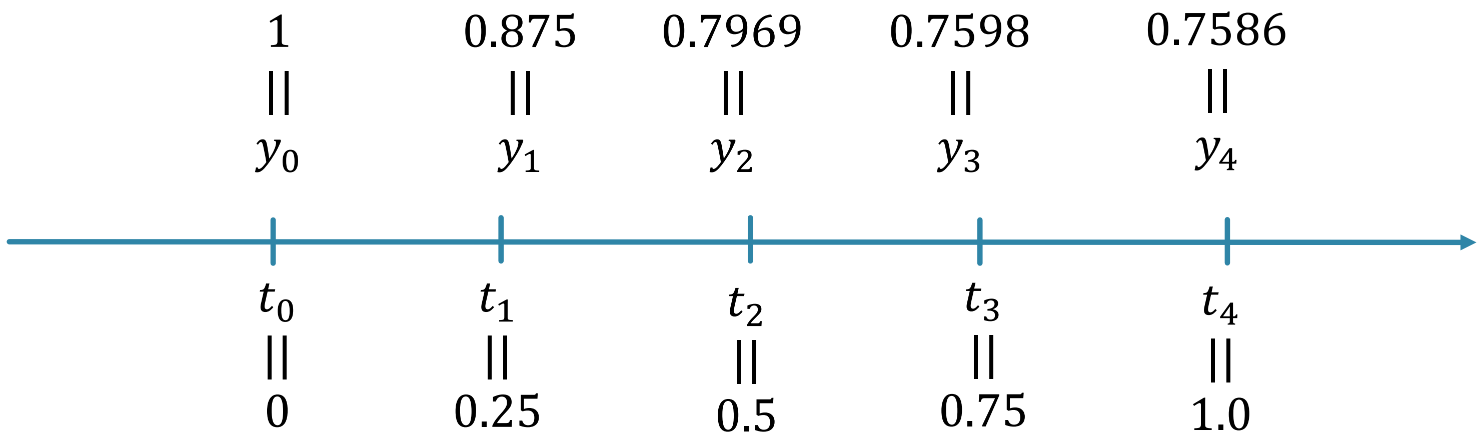
Construct the time-line:



Let $i = 3$,

$$\begin{aligned} y_4 &= y_3 + hf(t_3, y_3) \\ &= 0.7598 + 0.25 f(0.75, 0.7598) \\ &= 0.7598 + 0.25 \left[\frac{0.75 - 0.7598}{2}\right] \\ &= 0.7586 \end{aligned}$$

Update the time-line:



Exercise 1.3:

1) Use Euler's method to numerically integrate

$$\frac{dy}{dx} = 3y^2 - 5x - 1, \qquad y(0) = 0$$

from $x = 0$ to $x = 1$ with a step size of 0.25.

[Ans: $(0,0), (0.25, -0.25), (0.5, -0.7656), (0.75, -1.2010), (1, -1.3067)$]

2) Use Euler's method to numerically integrate

$$\frac{dy}{dx} = 1 - \sin x^2, \qquad y(0) = 1$$

from $x = 0$ to $x = 2\pi$ with a step size of $\frac{\pi}{2}$.

[Ans: $(0,1), \left(\frac{\pi}{2}, 2.5708\right), (\pi, 3.1610), \left(\frac{3\pi}{2}, 5.4077\right), (2\pi, 7.3143)$]

1.3.2 Second Order Runge-Kutta Method (RK2)

To solve an IVP,

$$\frac{dy}{dx} = f(x, y), \qquad y(x_0) = y_0$$

Carl Runge and Wilhelm Kutta introduced the second order method:

$$y_{i+1} = y_i + [a_1 k_1 + a_2 k_2]h \tag{1.2}$$

where

$$k_1 = f(x_i, y_i)$$
$$k_2 = f(x_i + p_1 h, y_i + q_{11} k_1 h)$$

which satisfy

$$a_1 + a_2 = 1, \qquad a_2 p_1 = \frac{1}{2}, \qquad a_2 q_{11} = \frac{1}{2} \tag{1.3}$$

In general, the value of a_2 from Eqn. (1.3) is assumed as

$\frac{1}{2}$ (Heun's Method), 1 (the Midpoint Method) or $\frac{2}{3}$ (Ralston's Method)

Heun's Method

By solving Eqn. (1.3) with assumption $a_2 = \frac{1}{2}$, it gives

$$a_1 = \frac{1}{2}, \qquad p_1 = 1 \quad \text{and} \quad q_{11} = 1$$

resulting in

$$y_{i+1} = y_i + \left[\frac{1}{2}k_1 + \frac{1}{2}k_2\right]h$$

where

$$k_1 = f(x_i, y_i)$$

$$k_2 = f(x_i + h, y_i + k_1 h)$$

Graphical Interpretation of Heun's Method:

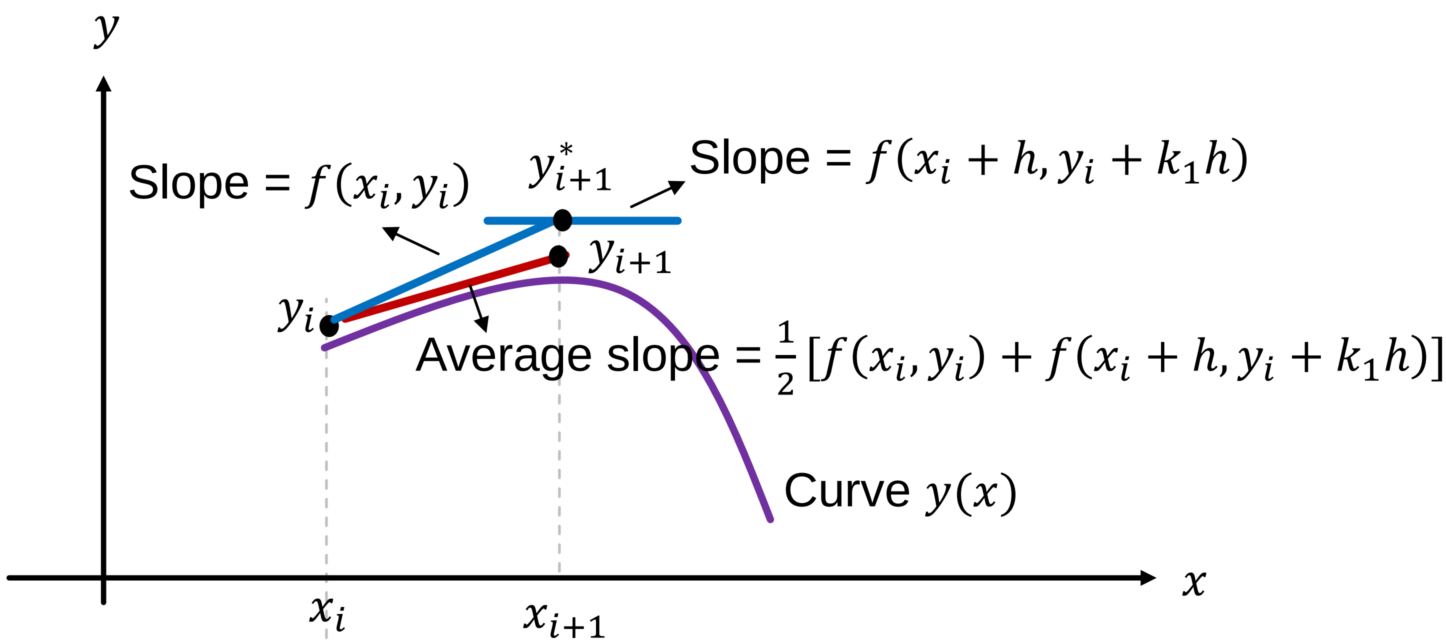


Figure 1.3: Graphical interpretation of Heun's method

$$y_{i+1} = y_i + \left[\frac{1}{2}k_1 + \frac{1}{2}k_2\right]h$$

where

$$k_1 = f(x_i, y_i)$$
$$k_2 = f(x_i + h, y_i + k_1 h)$$

The basic idea of Heun's method is: average slope of y_i and y^*_{i+1} (y^*_{i+1} from Euler's method) is used to form the tangent line to estimate the y_{i+1}.

Midpoint Method

By solving Eqn. (1.3) with assumption $a_2 = 1$, it gives

$$a_1 = 0, \quad p_1 = \frac{1}{2} \quad \text{and} \quad q_{11} = \frac{1}{2}$$

resulting in

$$y_{i+1} = y_i + k_2 h$$

where

$$k_1 = f(x_i, y_i)$$

$$k_2 = f\left(x_i + \frac{1}{2}h, y_i + \frac{1}{2}k_1 h\right)$$

Ralston's Method

By solving Eqn. (1.3) with assumption $a_2 = \frac{2}{3}$, it gives

$$a_1 = \frac{1}{3}, \quad p_1 = \frac{3}{4} \quad \text{and} \quad q_{11} = \frac{3}{4}$$

resulting in

$$y_{i+1} = y_i + \left[\frac{1}{3}k_1 + \frac{2}{3}k_2\right]h$$

where

$$k_1 = f(x_i, y_i)$$

$$k_2 = f\left(x_i + \frac{3}{4}h, y_i + \frac{3}{4}k_1 h\right)$$

The steps of RK2:

Rearrange:
$y' = f(x, y)$

→

Discretizing: $y_i' = f(x_i, y_i)$
& compute values of k_1 and k_2

→

Compute
y_{i+1}

In general, Ralston's method produces the most accurate estimates. Hence, examples will be shown using Ralston's method.

Example 1.13:

Use the second order Runge-Kutta method (Ralston's method) to numerically integrate

$$y' - y = -x^2, \qquad y(0) = 0$$

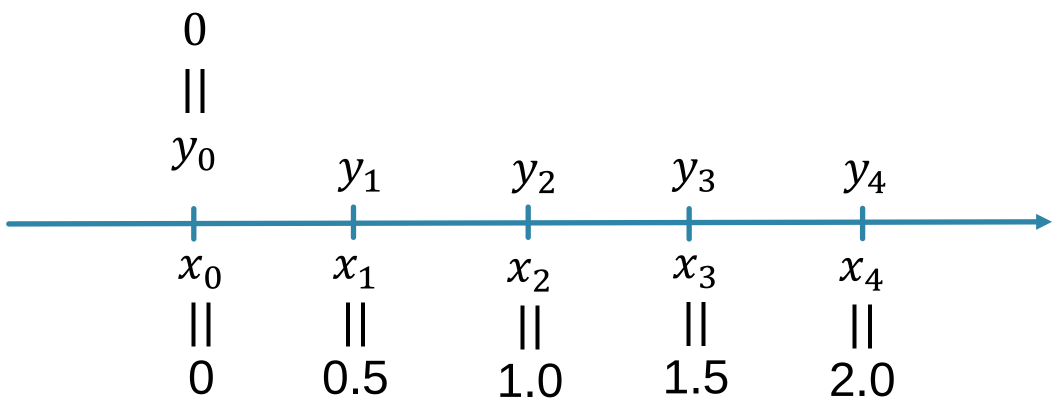
from $x = 0$ to $x = 2$ with a step size of 0.5.

Solution:

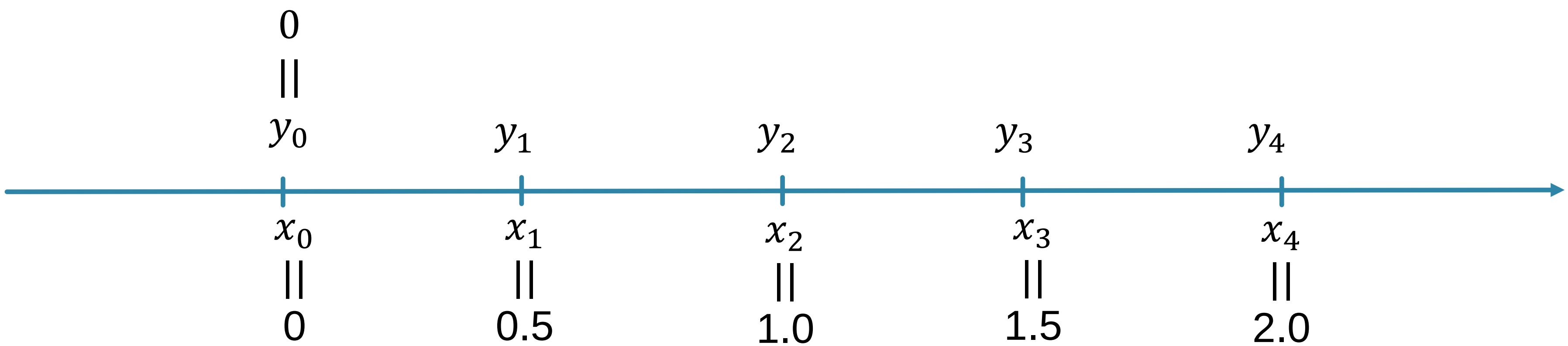
$$y' = f(x, y) = y - x^2$$
$$0 \le x \le 2, \quad h = 0.5, \quad y(0) = 0$$

Reminder:
$y' = f(x, y)$

Construct the time-line:



$$y' = f(x, y) = y - x^2$$



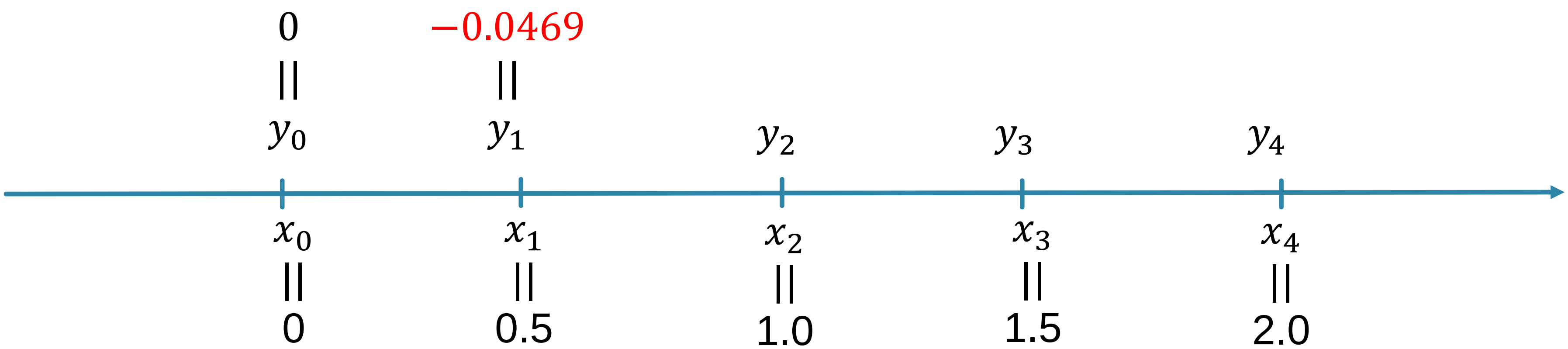
Let $i = 0$,

$$k_1 = f(x_0, y_0) = f(0, 0)$$
$$= 0$$

$$k_2 = f\left(x_0 + \frac{3}{4}h, y_0 + \frac{3}{4}k_1 h\right)$$
$$= f(0.375, 0\ )$$
$$= -0.1406$$

$$y_1 = y_0 + \left[\frac{1}{3}k_1 + \frac{2}{3}k_2\right]h$$
$$= 0 + \left[\frac{1}{3}(0) + \frac{2}{3}(-0.1406)\right](0.5)$$
$$= -0.0469$$

$$y' = f(x, y) = y - x^2$$



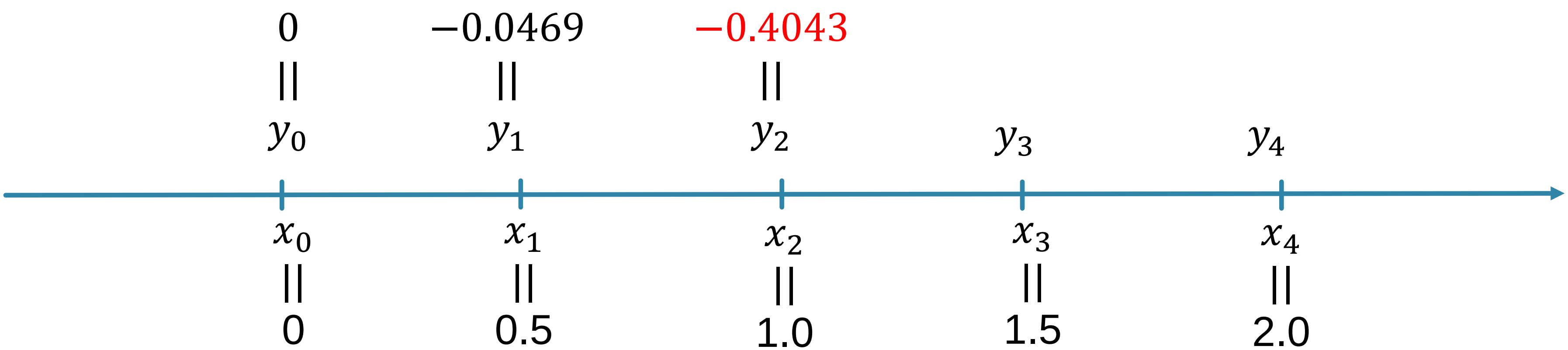
Let $i = 1$,

$$\begin{aligned} k_1 &= f(x_1, y_1) \\ &= f(0.5, -0.0469) \\ &= -0.2969 \end{aligned}$$

$$\begin{aligned} k_2 &= f\left(x_1 + \frac{3}{4}h, y_1 + \frac{3}{4}k_1 h\right) \\ &= f(0.875, -0.1582\,) \\ &= -0.9238 \end{aligned}$$

$$\begin{aligned} y_2 &= y_1 + \left[\frac{1}{3}k_1 + \frac{2}{3}k_2\right]h \\ &= -0.0469 + \left[\frac{1}{3}(-0.2969) + \frac{2}{3}(-0.9238)\right](0.5) \\ &= -0.4043 \end{aligned}$$

$$y' = f(x,y) = y - x^2$$



Let $i = 2$,

$$k_1 = f(x_2, y_2) = f(1, -0.4043) = -1.4043$$

$$k_2 = f\left(x_2 + \frac{3}{4}h, y_2 + \frac{3}{4}k_1 h\right) = f(1.375, -0.9309\ ) = -2.8215$$

$$y_3 = y_2 + \left[\frac{1}{3}k_1 + \frac{2}{3}k_2\right]h$$

$$= -0.4043 + \left[\tfrac{1}{3}(-1.4043) + \tfrac{2}{3}(-2.8215)\right](0.5)$$

$$= -1.5789$$

$$y' = f(x, y) = y - x^2$$

0	-0.0469	-0.4043	-1.5789	
$\|\|$	$\|\|$	$\|\|$	$\|\|$	
y_0	y_1	y_2	y_3	y_4
x_0	x_1	x_2	x_3	x_4
$\|\|$	$\|\|$	$\|\|$	$\|\|$	$\|\|$
0	0.5	1.0	1.5	2.0

Let $i = 3$,

$$k_1 = f(x_3, y_3) = f(1.5, -1.5789) = -3.8289$$

$$k_2 = f\left(x_3 + \frac{3}{4}h, y_3 + \frac{3}{4}k_1 h\right) = f(1.875, -3.0147) = -6.5303$$

$$y_4 = y_3 + \left[\frac{1}{3}k_1 + \frac{2}{3}k_2\right]h$$

$$= -1.5789 + \left[\frac{1}{3}(-3.8289) + \frac{2}{3}(-6.5303)\right](0.5)$$

$$= -4.3938$$

The solution of the following IVP

$$y' = f(x, y) = y - x^2, \qquad y(0) = 0$$

is shown as follows:

y_i	$y_0 = 0$	$y_1 = -0.0469$	$y_2 = -0.4043$	$y_3 = -1.5789$	$y_4 = -4.3938$
x_i	$x_0 = 0$	$x_1 = 0.5$	$x_2 = 1.0$	$x_3 = 1.5$	$x_4 = 2.0$

Example 1.14:

Use the second order Runge-Kutta method (Ralston's method) to numerically integrate

$$y' - \cos 2x - \sin 3y = 0, \qquad y(0) = 1$$

from $x = 0$ to $x = 1$ with a step size of 0.2.

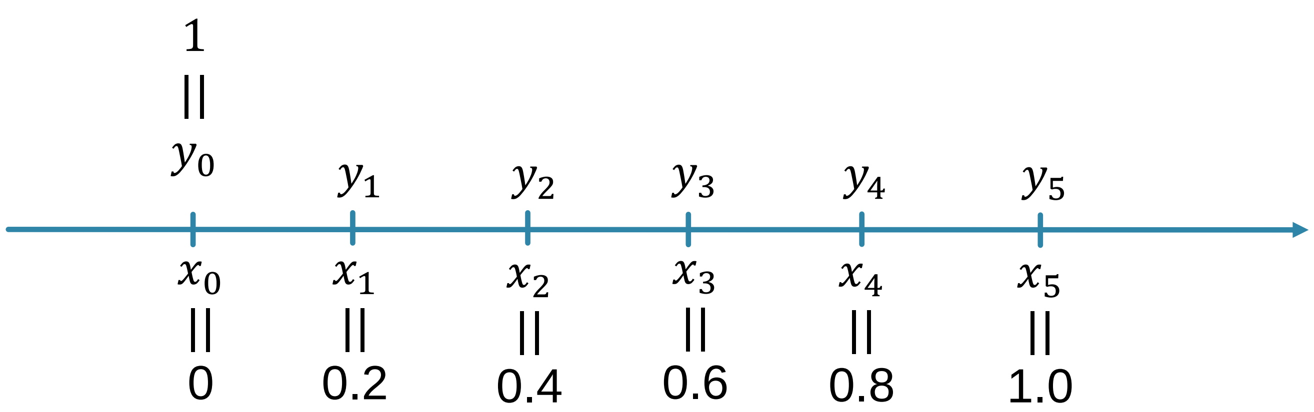
Solution:

$$y' = f(x,y) = \cos 2x + \sin 3y$$

$$0 \le x \le 1, \quad h = 0.2, \quad y(0) = 1$$

Reminder:
$y' = f(x,y)$

Construct the time-line:



$$y' = f(x, y) = \cos 2x + \sin 3y$$

$1 = y_0$	y_1	y_2	y_3	y_4	y_5
$x_0 = 0$	$x_1 = 0.2$	$x_2 = 0.4$	$x_3 = 0.6$	$x_4 = 0.8$	$x_5 = 1.0$

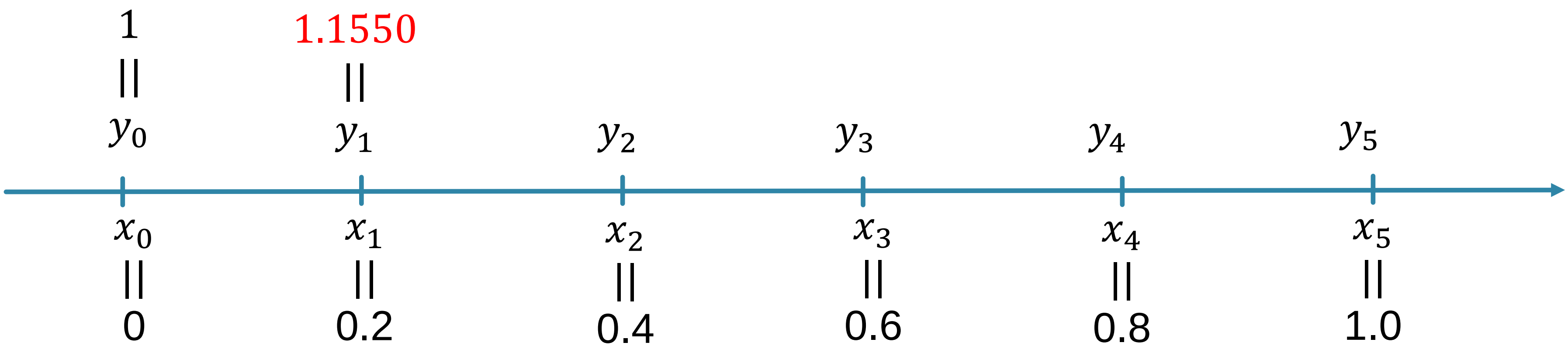
Let $i = 0$,

$$k_1 = f(x_0, y_0) = f(0, 1)$$
$$= 1.1411$$

$$k_2 = f\left(x_0 + \frac{3}{4}h, y_0 + \frac{3}{4}k_1 h\right)$$
$$= f(0.15, 1.1712\ )$$
$$= 0.5919$$

$$y_1 = y_0 + \left[\frac{1}{3}k_1 + \frac{2}{3}k_2\right]h$$
$$= 1 + \left[\frac{1}{3}(1.1411) + \frac{2}{3}(0.5919)\right](0.2)$$
$$= 1.1550$$

$$y' = f(x, y) = \cos 2x + \sin 3y$$



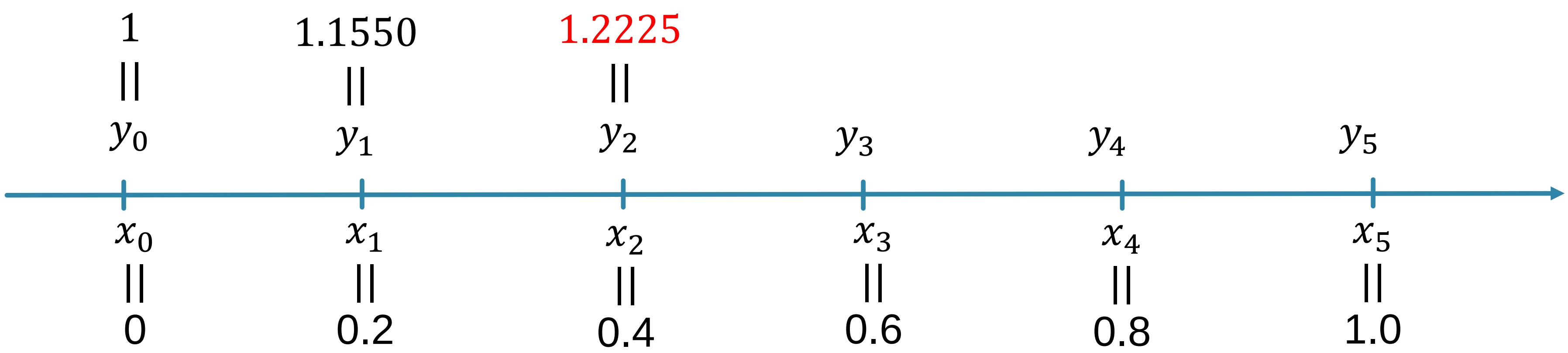
Let $i = 1$,

$$k_1 = f(x_1, y_1)$$
$$= f(0.2, 1.1550)$$
$$= 0.6033$$

$$k_2 = f\left(x_1 + \frac{3}{4}h, y_1 + \frac{3}{4}k_1 h\right)$$
$$= f(0.35, 1.2455\,)$$
$$= 0.2044$$

$$y_2 = y_1 + \left[\frac{1}{3}k_1 + \frac{2}{3}k_2\right]h$$
$$= 1.1550 + \left[\tfrac{1}{3}(0.6033) + \tfrac{2}{3}(0.2044)\right](0.2)$$
$$= 1.2225$$

$$y' = f(x, y) = \cos 2x + \sin 3y$$



Let $i = 2$,

$$\begin{aligned} k_1 &= f(x_2, y_2) \\ &= f(0.4, 1.2225) \\ &= 0.1947 \end{aligned}$$

$$\begin{aligned} k_2 &= f\left(x_2 + \frac{3}{4}h, y_2 + \frac{3}{4}k_1 h\right) \\ &= f(0.55, 1.2517\ ) \\ &= -0.1221 \end{aligned}$$

$$\begin{aligned} y_3 &= y_2 + \left[\frac{1}{3}k_1 + \frac{2}{3}k_2\right]h \\ &= 1.2225 + \left[\frac{1}{3}(0.1947) + \frac{2}{3}(-0.1221)\right](0.2) \\ &= 1.2192 \end{aligned}$$

$$y' = f(x, y) = \cos 2x + \sin 3y$$

1	1.1550	1.2225	1.2192		
$\parallel$	$\parallel$	$\parallel$	$\parallel$		
y_0	y_1	y_2	y_3	y_4	y_5
x_0	x_1	x_2	x_3	x_4	x_5
$\parallel$	$\parallel$	$\parallel$	$\parallel$	$\parallel$	$\parallel$
0	0.2	0.4	0.6	0.8	1.0

Let $i = 3$,

$$k_1 = f(x_3, y_3) = f(0.6, 1.2192) = -0.1311$$

$$k_2 = f\left(x_3 + \frac{3}{4}h, y_3 + \frac{3}{4}k_1 h\right) = f(0.75, 1.1995\,) = -0.3704$$

$$y_4 = y_3 + \left[\frac{1}{3}k_1 + \frac{2}{3}k_2\right]h$$

$$= 1.2192 + \left[\tfrac{1}{3}(-0.1311) + \tfrac{2}{3}(-0.3704)\right](0.2)$$

$$= 1.1611$$

$$y' = f(x, y) = \cos 2x + \sin 3y$$

1	1.1550	1.2225	1.2192	1.1611	
$\|\|$	$\|\|$	$\|\|$	$\|\|$	$\|\|$	
y_0	y_1	y_2	y_3	y_4	y_5
x_0	x_1	x_2	x_3	x_4	x_5
$\|\|$	$\|\|$	$\|\|$	$\|\|$	$\|\|$	$\|\|$
0	0.2	0.4	0.6	0.8	1.0

Let $i = 4$,

$$k_1 = f(x_4, y_4)$$
$$= f(0.8, 1.1611)$$
$$= -0.3643$$

$$k_2 = f\left(x_4 + \frac{3}{4}h, y_4 + \frac{3}{4}k_1 h\right)$$
$$= f(0.95, 1.1065\ )$$
$$= -0.5003$$

$$y_5 = y_4 + \left[\frac{1}{3}k_1 + \frac{2}{3}k_2\right]h$$
$$= 1.1611 + \left[\tfrac{1}{3}(-0.3643) + \tfrac{2}{3}(-0.5003)\right](0.2)$$
$$= 1.0701$$

The solution of the following IVP

$$y' = f(x, y) = \cos 2x + \sin 3y\,, \qquad y(0) = 1$$

is shown as follows:

1	1.1550	1.2225	1.2192	1.1611	1.0701
||	||	||	||	||	||
y_0	y_1	y_2	y_3	y_4	y_5
x_0	x_1	x_2	x_3	x_4	x_5
||	||	||	||	||	||
0	0.2	0.4	0.6	0.8	1.0

Exercise 1.4:

1) Use the second order Runge-Kutta method (Ralston's mehod) to numerically integrate

$$\frac{dy}{dx} + y = \sin x, \qquad y(0) = 1$$

from $x = 0$ to $x = 3$ with a step size of $h = 1$.

[Ans: $(0,1), (1,0.9544), (2,0.9930), (3,0.5994)$]

Exercise 1.4:

2) Use the second order Runge-Kutta method (Ralston's mehod) to numerically integrate

$$\frac{dy}{dx} + x - y = 1, \qquad y(0) = -1$$

from $x = 0$ to $x = 2$ with a step size of $h = 0.5$. Given the exact solution is $y = x - e^x$, find the true percent relative error of $y(2)$.

[Hint: True percent relative error= $\left|\frac{\text{true value} - \text{approximate value}}{\text{true value}}\right| \times 100\%$]

[Ans: $(0, -1), (0.5, -1.125), (1, -1.6406), (1.5, -2.7910), (2, -4.9729); 7.72\%$]

References

1. S.C. Chapra and R.P. Canale. (2015). Numerical Methods for Engineers, 7th Edition. McGraw-Hill Education.

2. R.L. Burden, D.J. Faires and A.M. Burden. (2016). Numerical Analysis, 10th Edition. Cengage Learning.

Thank You

Questions & Answer?