BMCG 1013
DIFFERENTIAL EQUATIONS

INTRODUCTION TO ORDINARY DIFFERENTIAL EQUATION

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Lesson outcomes

Upon completion of this week lesson, students should be able to:

i. describe the basic concept of differential equations

ii. classify the differential equations

iii. solve a separable first order differential equation

iv. solve a first order linear differential equation

CHAPTER 1

Introduction to Ordinary Differential Equations

- Classsification of Differential Equations
  - Ordinary and partial differential equations
  - Independent and dependent variables
  - The order of a differential equations
  - Linear and nonlinear differential equations
  - Initial value problem and boundary value problem

- Solve the First Order Differential Equations
  - Solving the separable first order differential equation
  - Solving the first order linear differential equation

1.1 Classsification of Differential Equations

What is a differential equation???

An equation that consists of one/more unknown function(s) of an independent variables and its derivatives

Example 1.1

$$\frac{dx}{dt} = x + 2e^{-t}$$

is a differential equation for function $x = f\ (t)$, where its derivative is equal to the function $x + 2e^{t}$.

Types of differential equations

Ordinary differential equation (ODE)

consists of ordinary derivatives of one/more dependent variable(s) with respect to an independent variable.

Example:

$$t\frac{d^2x}{dt^2}+\frac{dx}{dt}=e^t \qquad \frac{d^2y}{dx}-\sin x\frac{dy}{dx}-\ln 2x=1$$

Partial differential equation (PDE)

consists of partial derivatives of one/more dependent variable(s) with respect to two or more independent variables.

Example:

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial t^2} - 3t\frac{\partial f}{\partial x} = \frac{2}{t} - e^x \qquad \frac{\partial^2 f}{\partial x^2} + 2\frac{\partial f}{\partial y} - 3y\frac{\partial f}{\partial x} = \sin xy$$

Given a function $y = f(x)$. Here, y is a variable depends on the changes of the independent variable x. So the **dependent variable** y can be differentiated, with respect to the **independent variable** x.

Example 1.2

In this ODE:

$$x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = e^x,$$

the independent variable is x while the dependent variable is y.

Example 1.3

In this PDE:

$$\frac{\partial^2 f}{\partial x^2} - \frac{\partial^2 f}{\partial t^2} = 2t,$$

the independent variables are x and t while the dependent variable is f.

The **order** of a differential equation is determined by the order of the highest derivative which exists in the equation.

Example 1.4

$$\frac{dy}{dx} + \sin x = x^2 \quad \leftarrow \text{First order}$$

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} = x^2 \quad \leftarrow \text{Second order}$$

$$\frac{\partial^3 f}{\partial x^2 \partial y} + \frac{\partial^2 f}{\partial x^2} = x + y \quad \leftarrow \text{Third order}$$

If $y(x)$ is a function, then its derivative with respect to x is denoted as

$$\frac{dy}{dx}, \quad y'(x) \text{ or } y'.$$

The second derivative of $y(x)$ can be written as

$$\frac{d^2y}{dx^2}, \quad y''(x) \text{ or } y''.$$

Hence, the *n*-th derivatives of $y(x)$ is represented by

$$\frac{d^{(n)}y}{dx^{(n)}}, \quad y^{(n)}(x) \text{ or } y^{(n)}$$

Linear equations are defined as those in which the dependent variable(s) and their derivatives do not occur as product, raised to power or in nonlinear functions. **Nonlinear equations** are those that are not linear.

Example of linear equations:

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} = 2x, \qquad \frac{dy}{dx} + 2y = \sin x$$

x is independent variable

Example of nonlinear equation:

$$\frac{dy}{dx} + 2y = \sin y$$

y is dependent variable, $\sin y$ is nonlinear function

If a function $y = f(x)$ is substituted into a differential equation and an identity is obtained, then $y = f(x)$ is called the **solution** for the differential equation.

For example, $y(x) = x - 1 + Ce^{-x}$ is the solution of

$$\frac{dy}{dx} = x - y$$

where C is any constant .

Initial condition : conditions applied on the same value of independent variable that determine the values of the particular solution of a differential equation and its derivative.

Example:

$$y(0) = 0 \text{ and } \frac{dy(0)}{dx} = 2.$$

or in the form

$$y = 0 \quad \text{and} \quad \frac{dy}{dx} = 2$$

when $x = 0$.

Initial value problem (IVP) : differential equation with provided initial condition(s).

Example 1.5

$$\frac{dy}{dt} = ky, \qquad y(0) = 1$$

initial condition

differential equation

is an IVP.

Boundary condition : conditions applied on the different values of independent variable that determine the values of the particular solution of a differential equation and its derivative.

Example:

$$y(0) = 0 \text{ and } \frac{dy(1)}{dx} = 2.$$

or in the form

$$y = 0 \text{ when } x = 0 \text{ and}$$
$$\frac{dy}{dx} = 2 \text{ when } x = 1.$$

Boundary value problem (BVP) : differential equation with provided boundary conditions.

Example 1.6

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} + 4y = 3x, \quad y(0) = 0,\ y'(1) = \frac{3}{2}$$

is an BVP.

1.2.1 Solving the separable first order differential equation

If the function $f(x,y)$ of the first order differential equation

$$\frac{dy}{dx} = f(x, y)$$

can be represented as

$$f(x, y) = u(x)v(y),$$

then the first order differential equation is said to be **separable**.

Example 1.7

Solve the given differential equation:

$$\frac{dy}{dx} = -3y$$

Solution:

Step 1: Separate the variables

$$\frac{1}{y}dy = -3dx$$

Step 2: Integrate

$$\int \frac{1}{y}dy = -\int 3dx$$

$$\ln y = -3x + C, \quad C \text{ is some constant}$$

$$e^{\ln y} = e^{-3x+C} = e^{-3x} \cdot e^{C} = Ae^{-3x}, \quad A = e^{C}$$

$$y = Ae^{-3x}$$

Example 1.8

Solve the given differential equation:

$$\frac{dy}{dx} = \frac{x-1}{3y^2}$$

Solution:

Step 1: Separate the variables

$$3y^2dy = (x-1)dx$$

Step 2: Integrate

$$\int 3y^2dy = \int (x-1)dx$$

$$\frac{3y^3}{3} = \frac{x^2}{2} - x + C, \quad C \text{ is some constant}$$

$$y = \left(\frac{x^2}{2} - x + C\right)^{1/3}$$

Exercise 1.1

Find the solution for

$$x\frac{dy}{dx} = 1 + y.$$

Answer:

$$y = Cx - 1$$

1.2.2 Solving the first order linear differential equation

Any first order differential equation of the form

$$\frac{dy}{dx} + p(x)y = q(x)$$

is a **linear equation**.

Steps in solving a first order linear differential equation:

1. Rewrite the problem in the standard form of linear equation, i.e.

$$\frac{dy}{dx} + p(x)y = q(x)$$

2. Determine the integrating factor

$$\lambda(x) = e^{\int p(x)\,dx}$$

3. Represent the equation in the form

$$\frac{d}{dx}(\lambda y) = \lambda q$$

4. Integrate the above equation with respect to x.

Example 1.9

Solve the following differential equation:

$$\frac{dy}{dx} = x - y$$

Solution:

Step 1: $$\frac{dy}{dx}+y=x, \quad \therefore p(x)=1, q(x)=x$$

Step 2: $$\lambda(x)=e^{\int 1dx}=e^{x}$$

Step 3: $$\frac{d}{dx}(e^{x}y)=xe^{x}$$

Step 4: $$e^{x}y=\int xe^{x}\ dx$$

$$e^{x}y=xe^{x}-e^{x}+C$$

$$y=x-1+Ce^{-x}$$

Integration by parts

$$u=x, \quad dv=e^{x}dx$$

$$du=dx, \quad v=e^{x}$$

$$\int udv=uv-\int vdu$$

$$=xe^{x}-\int e^{x}dx$$

$$=xe^{x}-e^{x}$$

Example 1.10

Solve the given differential equation:

$$\frac{dy}{dx} = -3y$$

Solution:

Step 1: $$\frac{dy}{dx}+3y=0, \quad \therefore\ p(x)=3, q(x)=0$$

Step 2: $$\lambda(x)=e^{\int 3\,dx}=e^{3x}$$

Step 3: $$\frac{d}{dx}(e^{3x}y)=0\cdot e^{3x}$$

Step 4: $$\int d\ e^{3x}y=0$$

$$e^{3x}y+C=0$$

$$y=-Ce^{-3x}=Ae^{-3x}, \quad A=-C$$

Exercise 1.2

Find the solution for

$$\frac{1}{x^2}\frac{dy}{dx} = \frac{2y}{x^3} + \cos x$$

Answer:

$$y = x^2(\sin x + C)$$

☞ Are you able to

i. describe the basic concept of differential equations ?

ii. classify the differential equations?

iii. solve a separable first order differential equation?

iv. solve a first order linear differential equation now?

References

Edwards C. H., Penny D.E. & Calvis D. (2016). Differential Equations and Boundary Value Problems, 5thEdition. Pearson Education Inc..

Zill, D. G. (2017). Differential Equations with Boundary-Value Problems. Cengage Learning Inc

Thank You

Questions & Answer?