BEKG 2433
ENGINEERING MATHEMATICS 2

DIVERGENCE THEOREM

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Lesson Outcomes

Upon completion of this lesson, students should be able to:

- Solve surface integrals using Divergence Theorem

In this section we are going to relate surface integrals to triple integrals.

The theorem is called Divergence Theorem

also known as

Gauss' Theorem

The history of Divergence Theorem

- 1760: Joseph-Louis Lagrange introduced the notion of surface integrals.
- 1762: Lagrange employed surface integrals in his work on fluid mechanics. He discovered the divergence theorem.
- 1813: Carl Friedrich Gauss used surface integrals while working on the gravitational attraction of an elliptical spheroid, when he proved special cases of the divergence theorem.

From Stokes' theorem to divergence theorem

- This is all about a surface in space.
- But unlike, Stokes' theorem, the divergence theorem only applies to *closed surfaces*, meaning surfaces **without a boundary**.
- For example, a hemisphere is not a closed surface, it has a circle as its boundary, hence divergence theorem cannot be applied.
- However, if we add on the disk on the bottom of this hemisphere, and consider the disk and the hemisphere is now a single closed surface.
- In this case, given some vector field, the divergence theorem can be used on this two-part surface.

From Stokes' theorem to divergence theorem

- It allows us to be able to deal with three-dimensional volume *enclosed* by surfaces, where the theorem can be applied for.

- The similarity of divergence theorem to Stokes' theorem that it relates the integral of a derivative of a function over a region to the integral of the original function F over the boundary of the region.

Usefulness of divergence theorem

- Both surface integrals and triple integrals can be very tedious to compute.
- However the divergence theorem gives a tool for translating back and forth between them, and it can help turn a particularly difficult surface integral into an easier volume integral.
- This is especially effective if the volume V is some familiar shape (like sphere, cone, paraboloid) and if the divergence turns out to be a simple function.

Usefulness of divergence theorem

- It is also a powerful theoretical tool, especially for physics.
- For example: In electrodynamics, it can express various fundamental rules like Gauss's law either in terms of divergence, or in terms of a surface integral.
- Sometimes a situation is easier to think about locally, e.g. what individual charges at individual points in space are generating an electric field. But other times you want a more global view, perhaps asking how an electric field passes through an entire surface.

Divergence Theorem

- Think of the divergence of a vector field as the extent to which it behaves like a sink or a source.
- To what extent is there more exiting a region then entering it
- Thus, the expansion of a fluid flowing with a velocity field of $\mathbf{F}$ is captured by the divergence of $\mathbf{F}$.
- The divergence is a scalar which at a given point it is a single number that represents how much of the flow is expanding at that point.

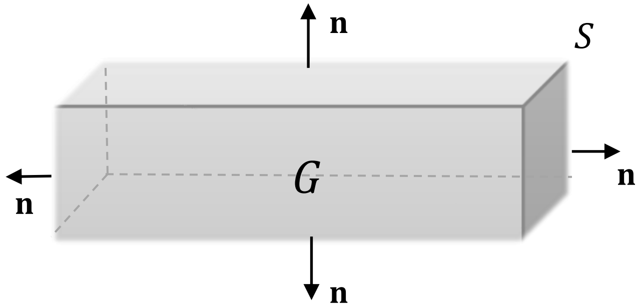
Divergence Theorem – Definition

Suppose G is bounded by a closed surface S and $\hat{\boldsymbol{n}}$ is a unit normal vector outward from G. If $\mathbf{F}(x,y,z) = f(x,y,z)\mathbf{i} + g(x,y,z)\mathbf{j} + h(x,y,z)\mathbf{k}$ is a vector field which $f(x,y,z), g(x,y,z), h(x,y,z)$ have continuous partial derivatives in G, then

$$\iint_S (\mathbf{F} \cdot \hat{\mathbf{n}})dS = \iiint_G \nabla \cdot \mathbf{F}\, dV$$

where $\nabla \cdot \mathbf{F}$ (sometimes written as div $\mathbf{F}$) =$\frac{\partial f}{\partial x} + \frac{\partial g}{\partial y} + \frac{\partial h}{\partial z}$.

Note: The theorem gives a relationship between a triple integral over a solid region G and a surface integral over the surface S of G. Surface S is closed that it forms the complete boundary of the solid G.

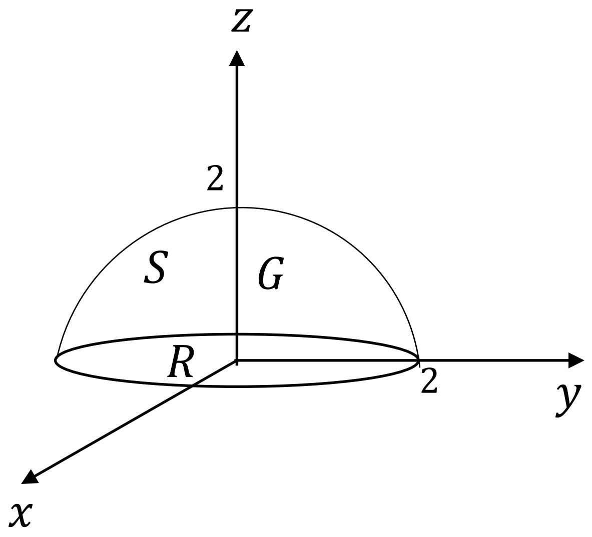


Example 14.1:

Let S be the hemisphere $z = \sqrt{4 - x^2 - y^2}$ oriented by outward normal and let $\mathbf{F}(x, y, z) = 2y\mathbf{j}$. Use Divergence Theorem to evaluate

$$\iint_S (\mathbf{F} \cdot \hat{\mathbf{n}})\, dS\,.$$

Solution:



Solution:

Let G be the spherical solid enclosed by S and

$$\nabla \cdot \mathbf{F} = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right) \cdot 2y\,\mathbf{j}$$

$$= \frac{\partial}{\partial y}(2y) = 2.$$

Solution:

Hence,

$$\iint_S (\mathbf{F}\cdot\hat{\mathbf{n}})dS = \iiint_G \nabla\cdot\mathbf{F}\,dV = 2\iiint_G dV$$

$$= 2\int_0^{2\pi}\int_0^{\frac{\pi}{2}}\int_0^{2} \rho^2 \sin\phi\, d\rho d\phi d\theta$$

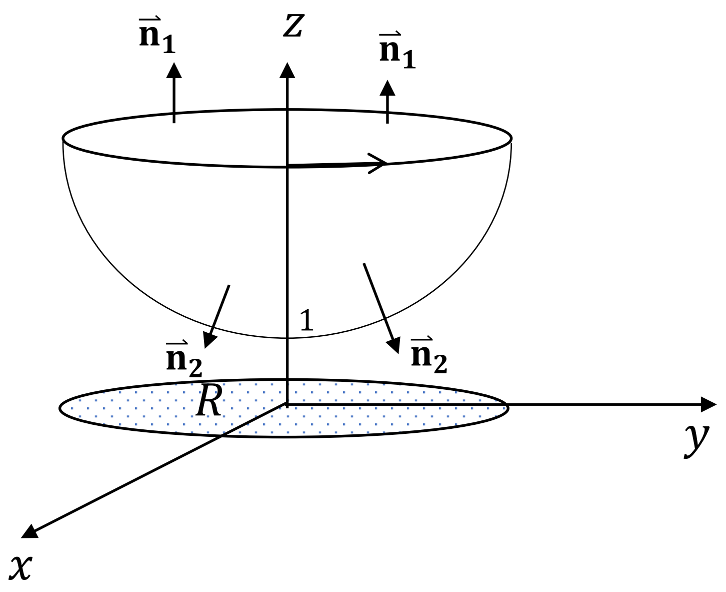
$$= 2\int_0^{2\pi}\int_0^{\frac{\pi}{2}} \left[\frac{\rho^3}{3}\right]_0^2 \sin\phi d\phi d\theta = \frac{16}{3}\int_0^{2\pi} [-\cos\phi]_0^{\frac{\pi}{2}}\, d\theta$$

$$= \frac{16}{3}\int_0^{2\pi} d\theta = \frac{16}{3}[\theta]_0^{2\pi} = \frac{32}{3}\pi$$

Example 14.2:

Use the Divergence theorem to evaluate $\iint_S (\mathbf{F}\cdot\hat{\mathbf{n}})\,dS$, where $\vec{\mathbf{F}}$ is the vector field $\vec{\mathbf{F}} = 2x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and S is the portion of the paraboloid $z = 1 + x^2 + y^2$ bounded by planes $z = 1$ and $z = 5$ and $\vec{\mathbf{n}}$ is the outward unit.

Solution:



Solution:

The divergence of **F** is

$$\nabla \cdot \mathbf{F} = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right) \cdot (2x\mathbf{i} + y\mathbf{j} + z\mathbf{k})$$

$$= \frac{\partial}{\partial x}(2x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z)$$

$$= 2 + 1 + 1 = 4.$$

Solution:

Hence,

$$\iint_S (\mathbf{F}\cdot\hat{\mathbf{n}})dS = \iiint_G \nabla\cdot\mathbf{F}\,dV = 4\iiint_G dV$$

$$= 4\int_0^{2\pi}\int_0^{2}\int_{1+r^2}^{5} dzrdrd\theta = 4\int_0^{2\pi}\int_0^{2}[z]_{1+r^2}^{5}\,rdrd\theta$$

$$= 4\int_0^{2\pi}\int_0^{2}(4r - r^3)drd\theta = 4\int_0^{2\pi}[2r^2 - \frac{r^4}{4}]_0^2\,d\theta$$

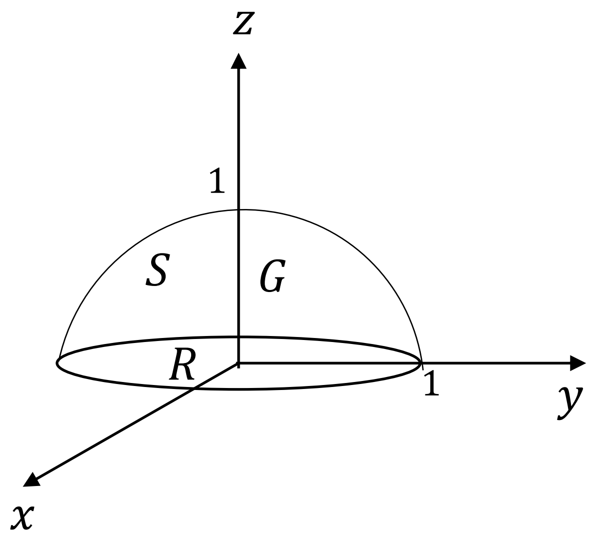
$$= 4\int_0^{2\pi} 4d\theta = 16[\theta]_0^{2\pi} = 32\pi$$

Example 14.3:

Calculate the flux of $\vec{\mathbf{F}} = x\mathbf{i} - y\mathbf{j} + z^2\mathbf{k}$ of the surface S which is a paraboloid of

$$z = 1 - (x^2 + y^2); \quad x^2 + y^2 \leq 1$$

Solution:



Solution:

Let G be the spherical solid enclosed by S and

$$\nabla \cdot \mathbf{F} = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right) \cdot (x\mathbf{i} - y\mathbf{j} + z^2\mathbf{k})$$
$$= \frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z^2)$$
$$= 1 - 1 + 2z = 2z.$$

Solution:

$$\iint_S (\mathbf{F}\cdot\hat{\mathbf{n}})dS = \iiint_G \nabla\cdot\mathbf{F}\,dV = 2\iiint_G z dV$$

$$= 2\int_0^{2\pi}\int_0^{1}\int_0^{1-r^2} zrdzdrd\theta = 2\int_0^{2\pi}\int_0^{1}\left[\frac{z^2}{2}\right]_0^{1-r^2} r\,drd\theta$$

$$= \int_0^{2\pi}\int_0^{1}(1-r^2)^2\,r\,drd\theta = \int_0^{2\pi}\int_0^{1}(r-2r^3+r^5)\,drd\theta$$

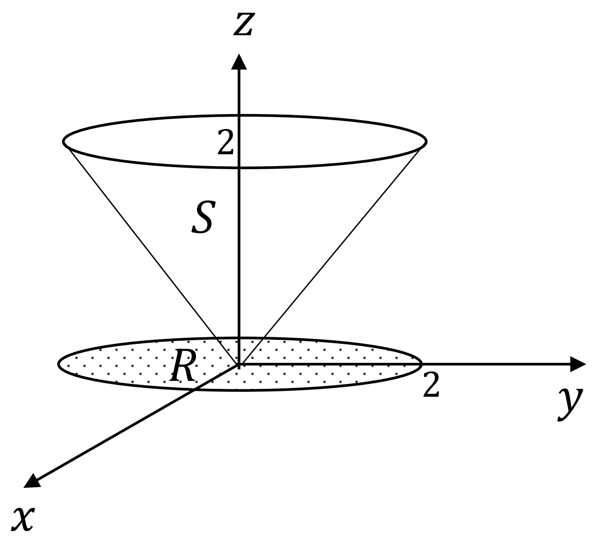
$$= \int_0^{2\pi}\left[\frac{r^2}{2}-\frac{r^4}{2}+\frac{r^6}{6}\right]_0^1 d\theta = \int_0^{2\pi}\left(\frac{1}{2}-\frac{1}{2}+\frac{1}{6}\right)d\theta$$

$$= \int_0^{2\pi}\frac{1}{6}d\theta = \frac{1}{6}[\theta]_0^{2\pi} = \frac{1}{3}\pi$$

Example 14.4:

Verify the divergence theorem for vector field $\vec{\mathbf{F}} = (x-y)\mathbf{i} + (x+z)\mathbf{j} + (z-y)\mathbf{k}$ and surface S that consists of cone $z = \sqrt{x^2+y^2}, 0 \le z \le 2$, and the circular top of the cone (see the following figure). Assume this surface is positively oriented.

Solution:



Solution:

Let G be the solid cone enclosed by S. To verify the theorem, we need to show that

$$\iint_S (\mathbf{F} \cdot \hat{\mathbf{n}})\, dS = \iiint_G \nabla \cdot \mathbf{F}\, dV$$

by calculating each integral separately.

Let's solve the triple integral

$$\iiint_G \nabla \cdot \mathbf{F}\, dV .$$

Solution:

To compute the triple integral, we determine $\nabla \cdot \mathbf{F}$;

$$\nabla \cdot \mathbf{F} = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right) \cdot \left((x-y)\mathbf{i} + (x+z)\mathbf{j} + (z-y)\mathbf{k}\right)$$

$$= 1 + 1 = 2$$

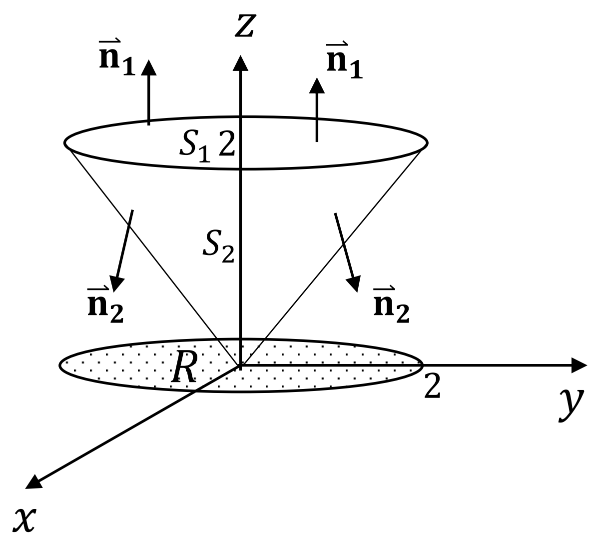
$$\iiint_G \nabla \cdot \mathbf{F}\, dV = 2\iiint_G dV = 2\int_0^{2\pi}\int_0^2\int_r^2 r\, dz\, dr\, d\theta = 2\int_0^{2\pi}\int_0^2 rz\Big|_r^2 dr\, d\theta$$

$$= 2\int_0^{2\pi}\int_0^2 (2r - r^2)\, dr\, d\theta = 2\int_0^{2\pi} r^2 - \frac{r^3}{3}\Big|_0^2 d\theta = 2\int_0^{2\pi} 4 - \frac{8}{3} d\theta$$

$$= 2\int_0^{2\pi} \frac{4}{3} d\theta = \frac{8}{3}\theta\Big|_0^{2\pi} = \frac{16}{3}\pi$$

Solution:

Now to compute the flux integral, first note that S is piecewise smooth. Therefore, the flux integral breaks into two pieces: one flux integral across the circular top of the cone S_1 and one flux integral across the surface of the cone S_2.



Solution:

Let’s start by calculating the flux across the circular top of the cone.

$S_1: z = 2$ (point upwards towards z-positive)

$$\vec{\mathbf{n}} = \boldsymbol{\nabla}\phi = \mathbf{k}$$

$$\vec{\mathbf{F}} \cdot (\boldsymbol{\nabla}\phi) = \big((x-y) + (x+z)\mathbf{j} + (z-y)\mathbf{k}\big) \cdot (\mathbf{k}) = z - y$$

$$\iint_{S_1} \vec{\mathbf{F}} \cdot d\vec{\boldsymbol{S}} = \iint_{S_1} \vec{\mathbf{F}} \cdot \vec{\mathbf{n}} dS = \iint_{R_1} \vec{\mathbf{F}} \cdot (\boldsymbol{\nabla}\phi) dA = \iint_{R_1} (z-y) dA$$

$$= \int_0^{2\pi} \int_0^2 (2 - r\sin\theta) r\,dr\,d\theta = \int_0^{2\pi} r^2 - \frac{r^3}{3}\sin\theta \,\Big|_0^2 d\theta = \int_0^{2\pi} 4 - \frac{8}{3}\sin\theta \, d\theta$$

$$= 4\theta - \frac{8}{3}\cos\theta \,\Big|_0^{2\pi} = 8\pi$$

Solution:

$S_2 : z = \sqrt{x^2 + y^2}$. The unit normal vector $\vec{\mathbf{n}}_2$ is oriented downwards. A parameterization of this surface is

$$\mathbf{f}(r, \theta) = r\cos\theta\,\mathbf{i} + r\sin\theta\,\mathbf{j} + r\mathbf{k}$$

The tangent vectors are

$$\mathbf{f}_{\mathrm{r}}(r, \theta) = \cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j} + \mathbf{k}$$
$$\mathbf{f}_{\theta}(r, \theta) = -r\sin\theta\,\mathbf{i} + r\cos\theta\,\mathbf{j}$$

The cross product of the tangent vectors

$$-(\mathbf{f}_{\mathrm{r}} \times \mathbf{f}_{\theta}) = -\begin{vmatrix} i & j & k \\ \cos\theta & \sin\theta & 1 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = r\cos\theta\,\mathbf{i} + r\sin\theta\,\mathbf{j} - r\mathbf{k}$$

Note that, we must take the negative signs on the x and y components to induce the negative (or inward) orientation of the cone.

Solution:

$$F\big(f(r,\theta)\big)\cdot(\mathbf{f}_r\times\mathbf{f}_\theta)$$
$$=\big((r\cos\theta-r\sin\theta)\mathbf{i}+(r\cos\theta+r)\mathbf{j}+(r-r\sin\theta)\mathbf{k}\big)\cdot(r\cos\theta\,\mathbf{i}+r\sin\theta\,\mathbf{j}-r\mathbf{k})$$
$$=r^2\cos^2\theta+2r^2\sin\theta-r^2$$

$$\iint_S \vec{\mathbf{F}}\cdot d\vec{\boldsymbol{S}}=\iint_R \vec{\mathbf{F}}\big(\mathbf{f}(r,\theta)\big)\cdot(\mathbf{f}_r\times\mathbf{f}_\theta)\,dA=\int_0^{2\pi}\int_0^2(r^2\cos^2\theta+2r^2\sin\theta-r^2)drd\theta$$
$$=\int_0^{2\pi}\left(\frac{r^3}{3}\cos^2\theta+\frac{2r^3}{3}\sin\theta-\frac{r^3}{3}\right)\Big|_0^2 d\theta=\frac{8}{3}\int_0^{2\pi}\frac{\cos 2\theta+1}{2}+2\sin\theta-1d\theta$$
$$=\frac{8}{3}\left[\frac{\sin 2\theta}{4}+\frac{\theta}{2}-2\cos\theta-\theta\right]_0^{2\pi}=\frac{8}{3}\left[-\frac{1}{2}\theta\right]_0^{2\pi}=-\frac{8}{3}\pi$$

$$\iint_S \vec{\mathbf{F}}\cdot d\vec{\boldsymbol{S}}=\iint_{S_1}\vec{\mathbf{F}}\cdot\vec{\mathbf{n}}dS+\iint_{S_2}\vec{\mathbf{F}}\cdot\vec{\mathbf{n}}dS=8\pi-\frac{8}{3}\pi=\frac{16}{3}\pi$$

Solution:

$$\iint_S \vec{\mathbf{F}} \cdot d\vec{\mathbf{S}} = \iint_{S_1} \vec{\mathbf{F}} \cdot \vec{\mathbf{n}} dS + \iint_{S_2} \vec{\mathbf{F}} \cdot \vec{\mathbf{n}} dS = 8\pi - \frac{8}{3}\pi = \frac{16}{3}\pi$$

Therefore, it verifies the Divergence Theorem.

$$\iint_S (\mathbf{F} \cdot \hat{\mathbf{n}}) dS = \iiint_G \nabla \cdot \mathbf{F} \, dV$$

Exercise 14.1:

Use the Divergence theorem to evaluate $\iint_S (\mathbf{F}\cdot\hat{\mathbf{n}})dS$, where $\vec{\mathbf{F}}$ is the vector field $\vec{\mathbf{F}} = 4xz\mathbf{i} + xyz^2\,\mathbf{j} + 3z\mathbf{k}$ and S is the entire surface of the solid G bounded by the cone $z = \sqrt{x^2+y^2}$ and the plane $z = 4$ and $\vec{\mathbf{n}}$ is the outward unit.

[Ans:320π]

Exercise 14.2:

Use the Divergence theorem to evaluate $\iint_S (\mathbf{F} \cdot \hat{\mathbf{n}}) dS$, where $\vec{\mathbf{F}}$ is the vector field $\vec{\mathbf{F}} = x\mathbf{i} + y\,\mathbf{j} + z\mathbf{k}$ and S is the entire surface of the finite cylinder$x^2 + y^2 = 4, 0 \leq z \leq 8$ and $\vec{\mathbf{n}}$ is the outward unit.

[Ans:96π]

Exercise 14.3:

Verify the Divergence Theorem in the case that $\vec{\mathbf{F}}$ is the vector field $\vec{\mathbf{F}} = 2\mathbf{i} + 2\,\mathbf{j} + 2\mathbf{k}$ and S is the cube that is cut from the first octant by the planes $x = 2, y = 2$ and $z = 2$.

Reference

1) R. Ranom, I.W. Jamaludin, N. A. Razak, N.I.A Apandi, Engineering Mathematics, UTeM Press, 2021.

2) Abd Wahid Md. Raji, Ismail Kamis, Mohd Nor Mohamad & Ong Chee Tiong, Advanced Calculus for Science and Engineering Students, UTM Press, 2021.

THANK YOU