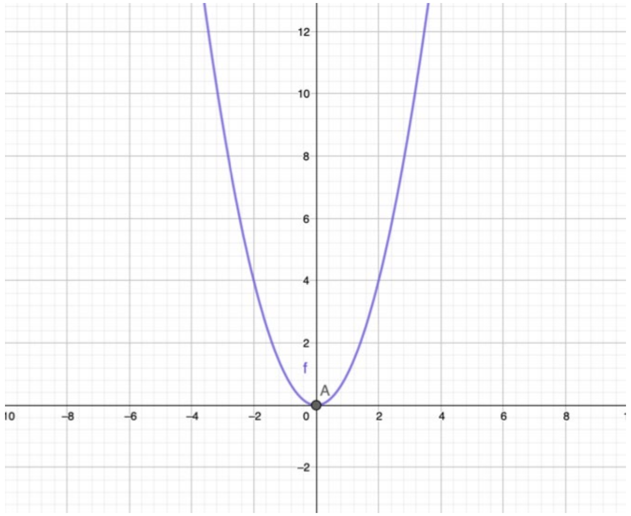
BEKG 2433
ENGINEERING MATHEMATICS 2

LOCAL EXTREMA



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Lesson Outcomes

Upon completion of this lesson, students should be able to:

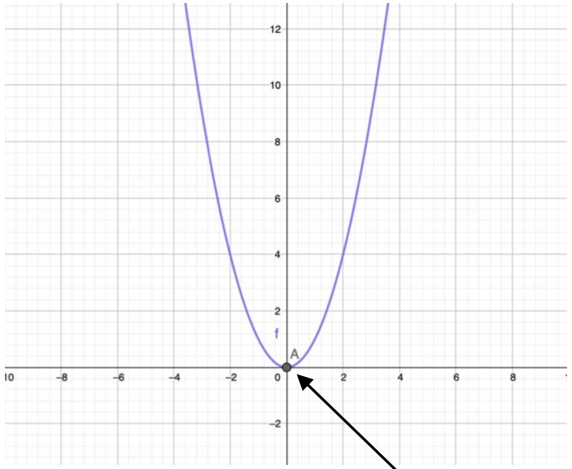
- find the critical point for a given function.
- determine the critical point of a function is a saddle point, local maximum or local minimum value.
- evaluate the local extrema of a given function.

Local Extreme

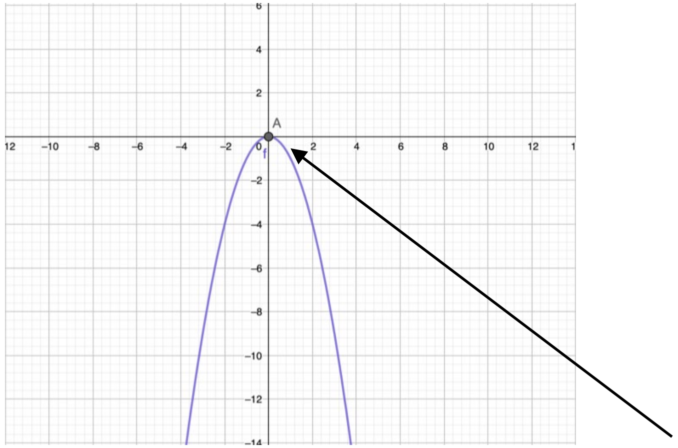
- There are many practical situations in which it is necessary to know the largest and the smallest value of a function of two variables.
- For example, what are the hottest and the coldest points on a metal plate, and where do these extreme temperatures occur?
- We can often answer such questions by examining the partial derivatives of the appropriate function.
- A maximum or minimum value of f is called an **extreme value** or **extremum**.
- The point at which the extremum occurs - **extreme point.**

Local Extrema

- A function $z = f(x, y)$ defines in a domain D and (a, b) in D
  - $f(a, b)$ is said to have a local minimum if $f(a, b) \leq f(x, y)$for all points (x, y) in a region, R where $R \subset D$
  - $f(a, b)$ is said to have a local maximum if $f(a, b) \geq f(x, y)$for all points (x, y) in a region, R where $R \subset D$



Local Minimum



Local Maximum

First Derivative Test for Local Extrema

- The Local extrema can determine by using the idea of first derivatives – set the derivative equal to zero.
- Let a function $z = f(x, y)$ is a local minimum or local maximum at point (a, b). If $f_x(a, b)$ and $f_y(a, b)$ exist, then

$$f_x(a, b) = 0 \text{ and } f_y(a, b) = 0$$

- Both f_x and f_y must set equal to zero simultaneously – use a system of equations.

First Derivative Test for Local Extrema

- Recall that for a function of one variable the condition $f'(a) = 0$ does not guarantee a local extremum at a.
- A similar precaution must be taken with the first derivative test for local extrema.
- The conditions $f_x(a,b) = f_y(a,b) = 0$ do not imply that f has a local extremum at (a,b), as we show momentarily.
- Theorem on the first derivative test for local extrema provides candidates for local extrema.
- These candidates are known as critical points.
- Therefore, the procedure for locating local maximum and minimum values is to find the critical points and then determine whether these critical points correspond to genuine local maximum and minimum values.

Critical Points

An interior point (a, b) in the domain of f is a critical point of f if either

- $f_x(a, b) = 0$ and $f_y(a, b) = 0$, or
- one (or both) of f_x or f_y does not exist at (a, b)

Critical points are candidates for local maximum and local minimum values.

Example 5.1:

Find the critical points of $f(x,y) = xy(x-2)(y+3)$

Solution:

This function is differentiable at all points of $\Re^2$. So the critical points occur only at points where $f_x(x,y) = f_y(x,y) = 0$.

The partial derivatives are:

$$f_x(x,y) = 2y(x-1)(y+3) = 0$$
$$f_y(x,y) = x(x-2)(2y+3) = 0$$

Identify all (x,y) pairs that satisfy both equations.

The first equation is satisfied if and only if $y = 0, x = 1$ or $y = -3$.

Solution continue:

Consider each of this cases.

- Substituting $y = 0$, the second equation is $3x(x-2) = 0$, which has solutions $x = 0$ and $x = 2$. So $(0,0)$ and $(2,0)$ are critical points.
- Substituting $x = 1$, the second equation is $-(2y+3) = 0$, which has the solution $y = -\frac{3}{2}$. So, $\left(1, -\frac{3}{2}\right)$ is critical point.
- Substituting $y = -3$, the second equation is $-3x(x-2) = 0$, which has roots $x = 0$ and $x = 2$. So, $(0,-3)$ and $(2,-3)$ are critical points.

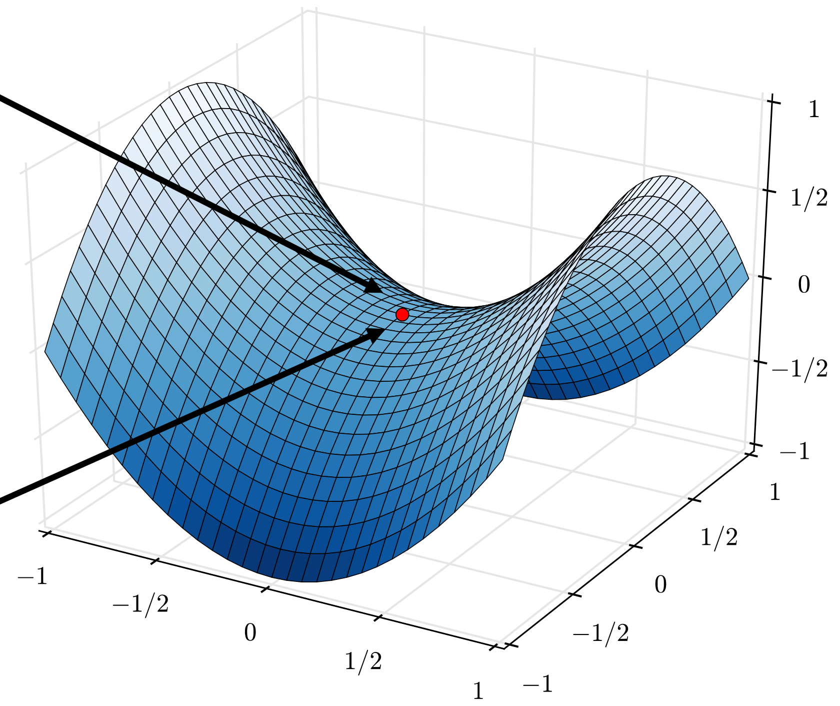
There are five critical points: $(0,0), (2,0), \left(1, -\frac{3}{2}\right), (0,-3)$ and $(2,-3)$.

Second Derivative Test for Local Extrema

- Critical points are candidates for local extreme values.
- With functions of one variable, the second derivative test may be used to determine whether critical points correspond to local maxima or minima (it can also be inconclusive).
- The analogous test for functions of two variables not only detects local maxima and minima, but also identifies another type of point known as a saddle point.

Saddle Point

- But if $f_x(a,b) = 0$ and $f_y(a,b) = 0$, there is no guarantee that a local extrema has located, it may be a saddle point -neither a local minimum or maximum.



The origin is a saddle point of the function. There are no local extreme values.

Both $f_x(0,0) = 0$ and $f_y(0,0) = 0$ but the point (0,0) leads to neither a local minimum nor local maximum for the function.

Saddle Point

Definition

A function f has a saddle point at a critical point (a, b) if, in every open disk centered at (a, b), there are points (x, y) for which $f(x, y) > f(a, b)$ and points for which $f(x, y) < f(a, b)$.

Second Derivative Test for Local Extrema

- For a function $z = f(x, y)$ with f_{xx}, f_{yy} and f_{xy} all exist on a region R and (a, b) critical point in R
- Define $$G(a,b) = \begin{vmatrix} f_{xx}(a,b) & f_{xy}(a,b) \\ f_{xy}(a,b) & f_{yy}(a,b) \end{vmatrix}$$ $$= f_{xx}(a,b)f_{yy}(a,b) - \left[f_{xy}(a,b)\right]^2$$
- Then,
  i. If $G(a,b) > 0$ and $f_{xx}(a,b) > 0$, $f(a,b)$is a local minimum.
  ii. If $G(a,b) > 0$ and $f_{xx}(a,b) < 0$, $f(a,b)$is a local maximum.
  iii. If $G(a,b) < 0$, $f(a,b)$is a saddle point.
  iv. If $G(a,b) = 0$, no conclusion can be made.

Example 5.2:

Find the critical points of $f(x,y) = 3x^2 - 2xy + y^2 - 8y$. Consequently, determine whether the critical points have a local minimum value, local maximum value or a saddle point.

Solution:

$$f(x,y) = 3x^2 - 2xy + y^2 - 8y$$

Step1: determine all the partial derivatives

$$f_x = 6x - 2y; f_y = -2x + 2y - 8$$

$$f_{xx} = 6, f_{xy} = -2, f_{yy} = 2$$

Solution continue:

Obtain the critical points $f_x = 0; f_y = 0$

$$6x - 2y = 0 \;\rightarrow y = 3x$$
$$-2x + 2y - 8 = 0$$
$$-2x + 2y = 8$$
$$\Rightarrow -x + y = 4$$
$$-x + 3x = 4$$
$$x = 2, \rightarrow y = 6$$

Therefore, the critical point is (2,6)

Step 2: Find $G(a, b)$. Compute $f_{xx}(2,6), f_{xy}(2,6), f_{yy}$

$$f_{xx}(2,6) = 6;\;\; f_{xy}(2,6) = -2;\;\;\; f_{yy}(2,6) = 2$$

Solution continue:

Step 3: Compute $G(2,6)$;

$$\begin{aligned} G(2,6) &= \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix} \\ &= \begin{vmatrix} 6 & -2 \\ -2 & 2 \end{vmatrix} \\ &= f_{xx}f_{yy} - \left(f_{xy}\right)^2 \\ &= (6)(2) - (-2)^2 \\ &= 8 \end{aligned}$$

Since $\mathrm{G}(2,6) > 0, f(2,6) > 0$, therefore $f(2,6)$ is a local minimum value.

Example 5.3:

Given that $f(x,y) = 4xy - x^4 - y^4$. Locate all relative extreme and saddle points of $f(x,y)$.

Solution:

Step1: determine all the partial derivatives

$f_x = 4y - 4x^3$; $f_y = 4x - 4y^3$

$f_{xx} = -12x^2$; $f_{yy} = 12y^2$; $f_{xy} = 4$

Obtain the critical points $f_x = 0; f_y = 0$

$f_x = 0 \rightarrow 4y = 4x^3$, so $y = x^3$

$f_y = 0 \rightarrow 4x = 4(x^3)^3$,

Solution continue:

so $4x - 4x^9 = 0 \rightarrow 4x(1 - x^8) = 0$

Therefore; $x = 0, x = 1, x = 0$

The critical points are $(1,1), (-1,-1)$ and $(0,0)$

Step 2: Find $G(a, b)$ for each critical points.

$f_{xx}(0,0) = 0$; $f_{yy}(0,0) = 0$

$f_{xx}(1,1) = -12$; $f_{yy}(1,1) = -12$

$f_{xx}(-1,-1) = -12$; $f_{yy}(-1,-1) = -12$

$G(0,0) = \begin{vmatrix} 0 & 4 \\ 4 & 0 \end{vmatrix} = -16$; $G(1,1) = \begin{vmatrix} -12 & 4 \\ 4 & -12 \end{vmatrix} = 128$

$G(-1,-1) = \begin{vmatrix} -12 & 4 \\ 4 & -12 \end{vmatrix} = 128$

Solution continue:

Step 3: Determine the type of each critical points.

For critical point $(0,0)$;
$G(0,0) = -16 < 0$; therefore $f_{xx}(0,0) = 0$ is a saddle point.

For critical point $(1,1)$;
$G(1,1) = 128 > 0$; $f_{xx}(1,1) = -12 < 0$
hence $f(1,1)$ is a local maximum value.

For critical point $(-1,-1)$;
$G(-1,-1) = 128 > 0$; $f_{xx}(-1,-1) = -12 < 0$
hence $f(-1,-1)$ is a local maximum value.

Example 5.4:

A temperature detector is used to measure the temperature of the underground current carrying conductor. After several readings are taken, it is determined that the reading at arbitrary point (x, y) is given by

$$T = \frac{1}{100}\left[\frac{1}{20}x^2 + 25y^2 - x(y+4)\right]^{\circ} C$$

Determine the lowest temperature and the point (x, y) where the temperature located.

Solution:

$$T_x = \frac{1}{100}\left[\frac{2}{20}x - (y+4)\right] = \frac{1}{100}\left[\frac{1}{10}x - y - 4\right]$$

$$T_y = \frac{1}{100}[50y - x] \qquad ; \qquad T_{yy} = \frac{50}{100} = \frac{1}{2}$$

$$T_{xx} = \frac{1}{100}\left[\frac{1}{10}\right] = \frac{1}{1000} \qquad ; \qquad T_{xy} = \frac{1}{100}[-1] = -\frac{1}{100}$$

$$T_y = 0 \rightarrow \frac{50y}{100} - \frac{x}{100} = 0$$

$$x = 50y$$

$$T_x = 0 \rightarrow \frac{1}{100}\left[\frac{1}{10}(50y) - y - 4\right] = \frac{1}{100}[4y - 4]$$

So $y = 1$; $x = 50$. The critical points is $(50,1)$

Solution:

$$T_{xx}(50,1) = \frac{1}{1000} \quad ; \quad T_{yy}(50,1) = \frac{1}{2} \quad ; \quad T_{xy}(50,1) = -\frac{1}{100}$$

$$G(50,1) = \begin{vmatrix} \frac{1}{1000} & -\frac{1}{100} \\ -\frac{1}{100} & \frac{1}{2} \end{vmatrix} = \frac{1}{2000} - \frac{1}{10000} = 0.0004$$

$$G(50,1) = 0.0004 > 0$$

$$T_{xx} = \frac{1}{1000} > 0$$

Therefore $(50,1)$ is a local minimum point value.

The lowest temperature is;

$$T(50,1) = \frac{1}{100}\left[\frac{1}{20}(50^2) + 25(1^2) - 50(1+4)\right] = -1^0C$$

Example 5.5:

A shipping company handles rectangular boxes provided the sum of the length, width, and height of the box does not exceed 96 inches. Find the dimensions of the box that meets this condition and has the largest volume.

Solution:

Let x, y and z be the dimensions of the box.
So the volume is, $V = xyz$.
The box with the maximum volume satisfies the condition $x + y + z = 96$, which is used to eliminate any one of the variables from the volume function.

From $x + y + z = 96 \rightarrow z = 96 - x - y$
The volume of the function becomes $V(x, y) = xy(96 - x - y)$

The goal is to find the maximum value of V.
The first derivatives of the volume:

$$V_x = 96y - 2xy - y^2$$
$$V_y = 96x - 2xy - x^2$$

Solution continue:

The critical points of V satisfies $V_x = 0$ and $V_y = 0$.

$V_x = 0 \rightarrow \; V_x = 96y - 2xy - y^2 = 0$

$$2xy = 96y - y^2$$

$$x = \frac{96y - y^2}{2y}$$

$$x = 48 - \frac{1}{2}y$$

Substituting $x = 48 - \frac{1}{2}y$ in $V_y = 0$;

$V_y = 0 \rightarrow \; V_y = 96x - 2xy - x^2 = 0$

$$= 96\left(48 - \frac{1}{2}y\right) - 2y\left(48 - \frac{1}{2}y\right) - \left(48 - \frac{1}{2}y\right)^2$$

Solving for y will finally give $y_1 = 96$ and $y_2 = 32$.

Solution continue:

Substitute $y_1 = 96$ and $y_2 = 32$ in $x = 48 - \frac{1}{2}y$ will give $x_1 = 0$ and $x_2 = 32$.
Therefore, the critical points are $(0,96)$ and $(32,32)$.

The required second derivatives are:
$V_{xx} = -2y \quad ; V_{yy} = -2x \quad ; \quad V_{xy} = 96 - 2x - 2y$

Find $G(a, b)$ for critical points $(0,96)$ and $(32,32)$.

At point $(0,96)$;

$$V_{xx}(0,96) = -2(96) = -192$$
$$V_{yy}(0,96) = -2(0) = 0$$
$$V_{xy}(0,96) = 96 - 2(0) - 2(96) = -96$$

Solution continue :

$$G(0,96) = \begin{vmatrix} V_{xx} & V_{xy} \\ V_{xy} & V_{yy} \end{vmatrix} = \begin{vmatrix} -192 & -96 \\ -96 & 0 \end{vmatrix} = 0 - 9216 = -9216$$

$G(0,96) = -9216 < 0$; so $V(0,96)$ is a saddle point.

At point $(32,32)$;

$$V_{xx}(32,32) = -2(32) = -64$$
$$V_{yy}(32,32) = -2(32) = -64$$
$$V_{xy}(32,32) = 96 - 2(32) - 2(32) = -32$$

$$G(0,96) = \begin{vmatrix} V_{xx} & V_{xy} \\ V_{xy} & V_{yy} \end{vmatrix} = \begin{vmatrix} -64 & -32 \\ -32 & -64 \end{vmatrix} = 4096 - 1024 = 3072$$

$G(0,96) = 3072 > 0$; so $V(32,32)$ is a local maximum point.

Solution continue :

The dimensions of the box with maximum volume are $x = 32, y = 32$ and $z = 96 - 32 - 32 = 32$ (it is a cube).

The maximum volume of the box is

$$V = xyz = 32(32)(32) = 32{,}768\ inches^3$$

Exercise 5.1:

Find all the critical points of the following functions.

1. $f(x,y) = 1 + x^2 + y^2$
2. $f(x,y) = (3x - 2)^2 + (y - 4)^2$
3. $f(x,y) = x^4 + y^4 - 16xy$

[Ans: 1. $(0,0)$. 2. $\left(\frac{2}{3}, 4\right)$ 3. $(0,0)$, $(2,2)$ and $(-2,-2)$]

Exercise 5.2:

Find all the critical points of the following functions. Use the Second Derivative Test to determine (if possible) whether each critical point corresponds to a local maximum, local minimum, or saddle point.

1. $f(x,y) = 4 + 2x^2 + 3y^2$
2. $f(x,y) = -4x^2 + 8y^2 - 3$
3. $f(x,y) = x^4 + 2y^2 - 4xy$
4. $f(x,y) = \sqrt{x^2 + y^2 - 4x + 5}$
5. $f(x,y) = \frac{x-y}{1+x^2+y^2}$
6. $f(x,y) = ye^x - e^y$

Exercise 5.2:

[Answer]:

1. Local minimum at $(0,0)$
2. Saddle point at $(0,0)$
3. Saddle point at $(0,0)$, local min at $(1,1)$ and at $(-1,-1)$
4. Local min at $(2,0)$
5. Saddle point at $(0,0)$, local max at $\left(\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\right)$ and $\left(-\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\right)$, local min at $\left(\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\right)$ and $\left(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)$
6. Local max at $\left(\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\right)$, local min at $\left(-\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)$
7. Saddle point at $(0,0)$

Reference

1) Y. Mohammad Yusof, S. Baharun and R. Abdul Rahman. (2012). Multivariable Calculus for Independent Learners, Pearson Revised Second Edition.

2) William L. Briggs and Lyle Cochran. (2011). Calculus Early Transcendentals, Pearson Education Inc.

THANK YOU