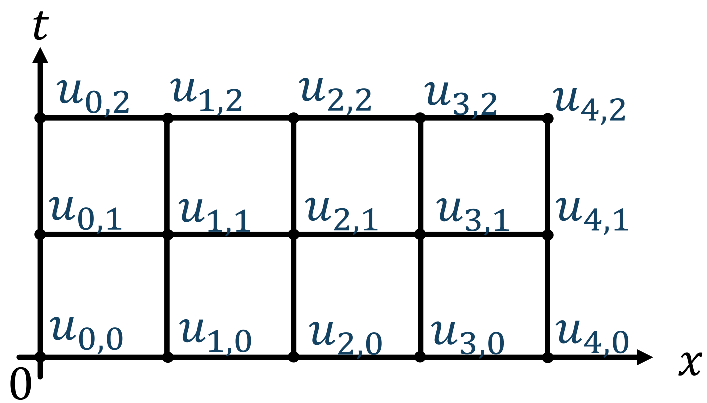
BMCG 1013
DIFFERENTIAL EQUATIONS

FINITE DIFFERENCE METHOD (PARABOLIC EQUATION)



Ser Lee Loh
slloh@utem.edu.my

Lesson Outcomes

Upon completion of this lesson, students should be able to:

- apply finite difference method in solving partial differential equations
- solve parabolic equation using finite difference method

Second Order Linear Partial Differential Equation

Analytical Methods

- Separation of variables
- Integral transform
- Characteristic etc.

Exact solution

Numerical Methods: Finite Differences

Parabolic eqn.

Hyperbolic eqn.

Elliptic eqn.

Approximated solution

5.4 Numerical Methods for PDE (Explicit Finite Difference Method)

Why numerical methods are needed?

- Limitation of analytical method: Analytical methods are limited to highly simplified problems in simple geometries with simple thermal conditions.
- A better modeling: There is always a tendency to oversimplify a problem into a mathematical model to ensure an analytical solution. Hence, a mathematical model for a numerical solution may represent the actual problem better.
- Flexibility: Numerical methods suit the needs of engineering problems that requires extensive parametric studies.

5.4.1 Difference Formulas

Forward-Difference Formula:

$$\left(\frac{\partial u}{\partial t}\right)_{i,j} = \frac{u_{i,j+1} - u_{i,j}}{k} \qquad \left(\frac{\partial u}{\partial x}\right)_{i,j} = \frac{u_{i+1,j} - u_{i,j}}{h}$$

Backward-Difference Formula:

$$\left(\frac{\partial u}{\partial t}\right)_{i,j} = \frac{u_{i,j} - u_{i,j-1}}{k} \qquad \left(\frac{\partial u}{\partial x}\right)_{i,j} = \frac{u_{i,j} - u_{i-1,j}}{h}$$

Central-Difference Formula:

$$\left(\frac{\partial u}{\partial t}\right)_{i,j} = \frac{u_{i,j+1} - u_{i,j-1}}{2k} \qquad \left(\frac{\partial u}{\partial x}\right)_{i,j} = \frac{u_{i+1,j} - u_{i-1,j}}{2h}$$

$$\left(\frac{\partial^2 u}{\partial t^2}\right)_{i,j} = \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{k^2} \qquad \left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2}$$

5.4.2 Parabolic Equation – Heat Equation

Consider the **Heat Equation**:

$$\frac{\partial u}{\partial t} = c\frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < a, \qquad t > 0$$

with boundary conditions

$$u(0,t) = u(a,t) = 0, \qquad t > 0$$

and initial condition

$$u(x,0) = f(x), \qquad 0 \le x \le a.$$

By forward-difference and central-difference formulas,

$$\left(\frac{\partial u}{\partial t}\right)_{i,j} - c\left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j} = 0$$

is approximated to

$$\boxed{\frac{u_{i,j+1} - u_{i,j}}{k} - c\,\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} = 0}$$

where $h = \Delta x$ and $k = \Delta t$.

Example 5.17:

Find approximate solution for the following heat equation

$$\frac{\partial u}{\partial t} = 0.5\frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < 1, \qquad t > 0$$

with boundary conditions

$$u(0,t) = u(1,t) = 0, \qquad 0 \le t \le 0.1$$

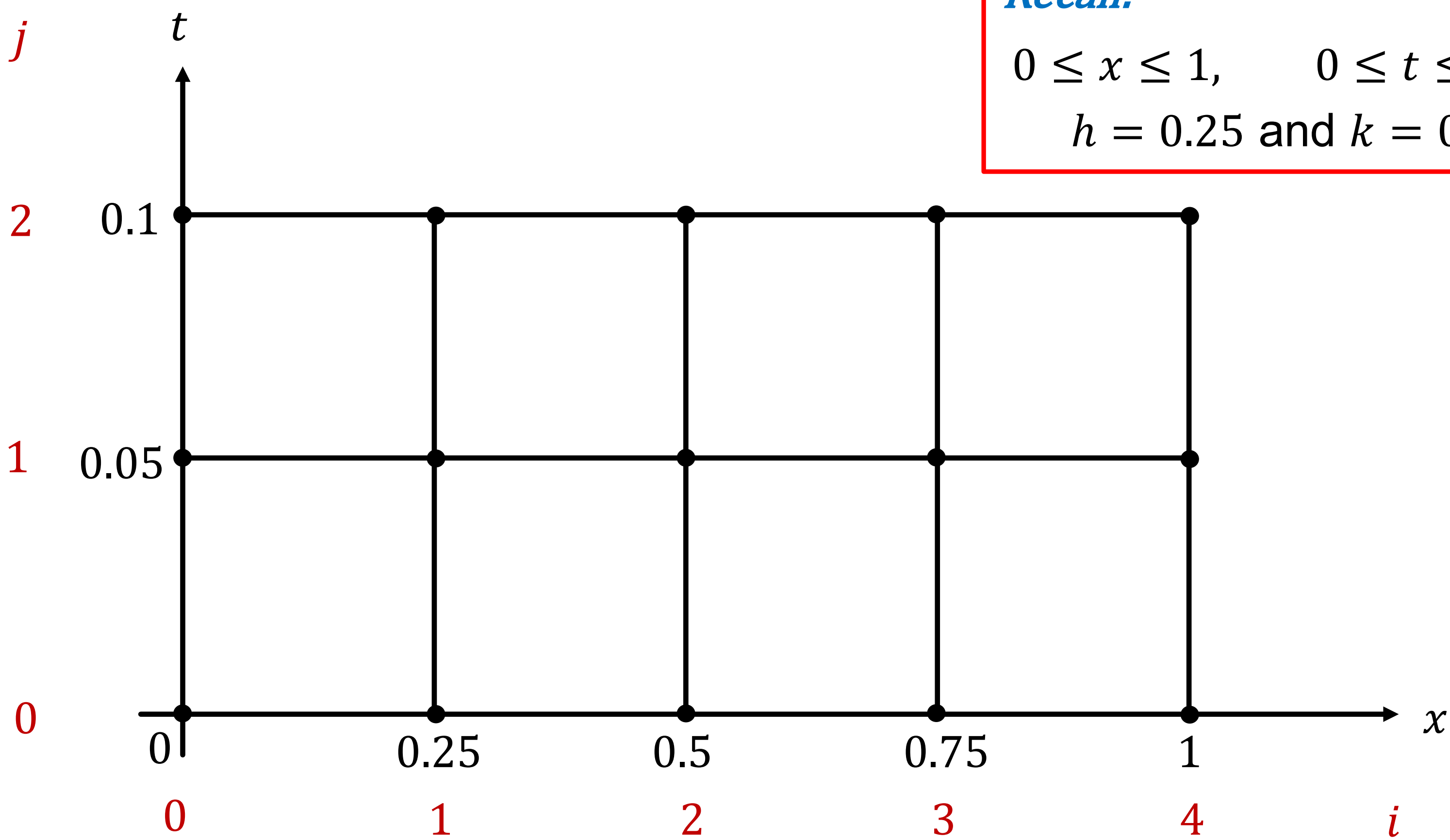
and initial condition

$$u(x,0) = x\cos\left(\frac{\pi x}{2}\right), \qquad 0 \le x \le 1.$$

by using the forward-difference and central-difference formula. Given $h = 0.25$ and $k = 0.05$.

Solution:

Step 1: Sketch the grid points.



Recall:

$0 \le x \le 1, \qquad 0 \le t \le 0.1$

$h = 0.25$ and $k = 0.05$

Step 2: Compute boundary and initial values.

Recall:
$h = 0.25$ and $k = 0.05$

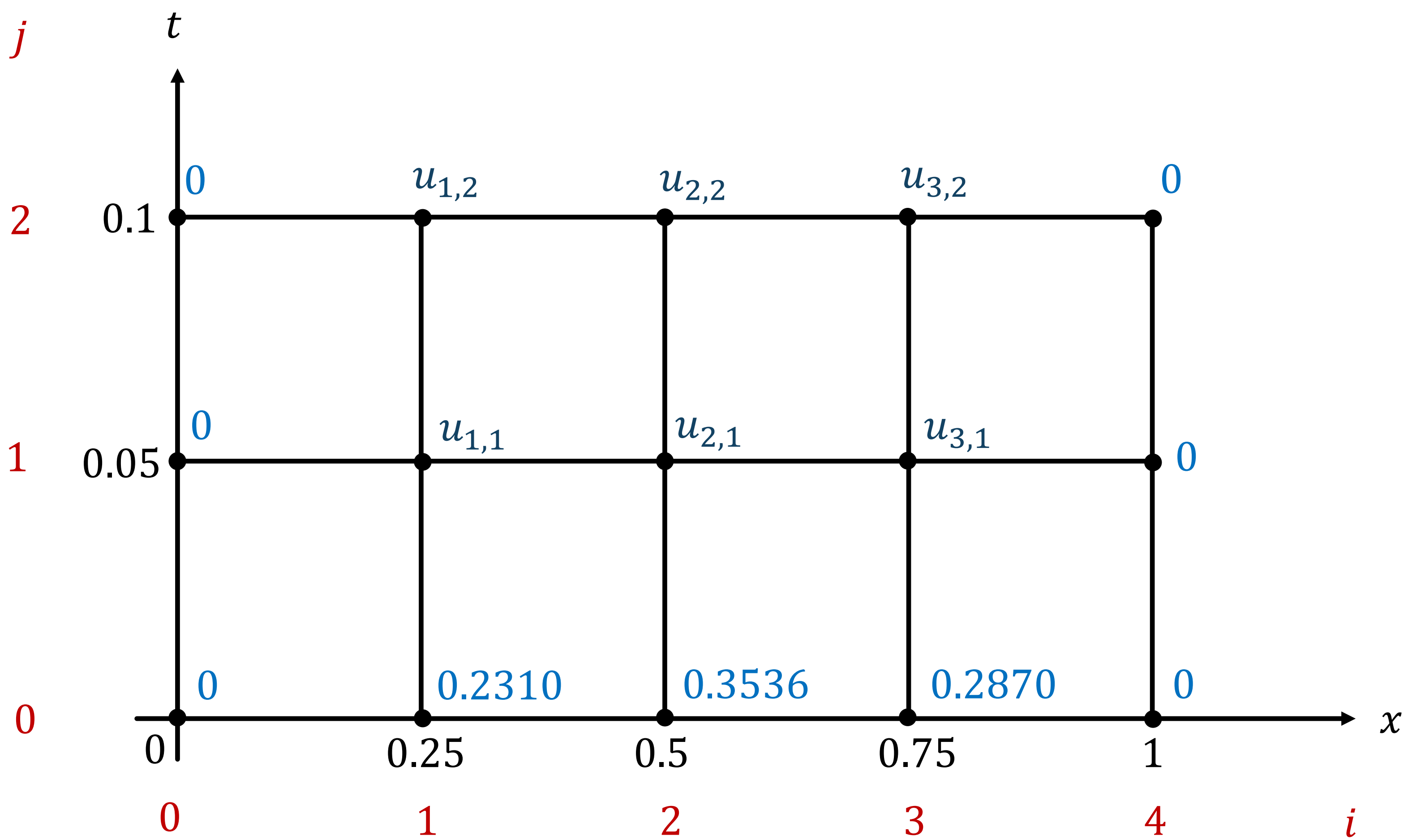
$$\boldsymbol{u(x,0) = x\cos\left(\frac{\pi x}{2}\right), \qquad 0 \le x \le 1}$$

$u(0,0) = 0$
$u(0.25,0) = 0.2310$
$u(0.5,0) = 0.3536$
$u(0.75,0) = 0.2870$
$u(1,0) = 0$

$$\boldsymbol{u(0,t) = 0, \qquad u(1,t) = 0, \qquad 0 \le t \le 0.1}$$

$u(0,0) = 0$	$u(1,0) = 0$
$u(0,0.05) = 0$	$u(1,0.05) = 0$
$u(0,0.1) = 0$	$u(1,0.1) = 0$

Step 3: Fill in the values into the grid.



Step 4: Obtain formula $u_{i,j+1}$ from finite-difference formula.

By forward-difference and central-difference formulas,

$$\boxed{\frac{\partial u}{\partial t} = 0.5\frac{\partial^2 u}{\partial x^2}} \Rightarrow \left(\frac{\partial u}{\partial t}\right)_{i,j} - 0.5\left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j} = 0$$

Choose the required difference formula from the list:

Forward-Difference Formula:

$$\boxed{\left(\frac{\partial u}{\partial t}\right)_{i,j} = \frac{u_{i,j+1} - u_{i,j}}{k}} \qquad \left(\frac{\partial u}{\partial x}\right)_{i,j} = \frac{u_{i+1,j} - u_{i,j}}{h}$$

Backward-Difference Formula:

$$\left(\frac{\partial u}{\partial t}\right)_{i,j} = \frac{u_{i,j} - u_{i,j-1}}{k} \qquad \left(\frac{\partial u}{\partial x}\right)_{i,j} = \frac{u_{i,j} - u_{i-1,j}}{h}$$

Central-Difference Formula:

$$\left(\frac{\partial u}{\partial t}\right)_{i,j} = \frac{u_{i,j+1} - u_{i,j-1}}{2k} \qquad \left(\frac{\partial u}{\partial x}\right)_{i,j} = \frac{u_{i+1,j} - u_{i-1,j}}{2h}$$

$$\left(\frac{\partial^2 u}{\partial t^2}\right)_{i,j} = \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{k^2} \qquad \boxed{\left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2}}$$

By forward-difference and central-difference formulas,

$$\left(\frac{\partial u}{\partial t}\right)_{i,j} - 0.5\left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j} = 0$$

is approximated to

Recall:
$h = 0.25$ and $k = 0.05$

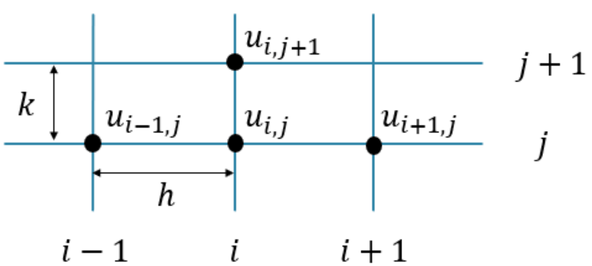
$$\frac{u_{i,j+1} - u_{i,j}}{k} - 0.5\left(\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2}\right) = 0$$

$$u_{i,j+1} - u_{i,j} = \frac{0.5k}{h^2}\left(u_{i+1,j} - 2u_{i,j} + u_{i-1,j}\right)$$

$$u_{i,j+1} = \frac{0.5(0.05)}{(0.25)^2}\left(u_{i+1,j} - 2u_{i,j} + u_{i-1,j}\right) + u_{i,j}$$

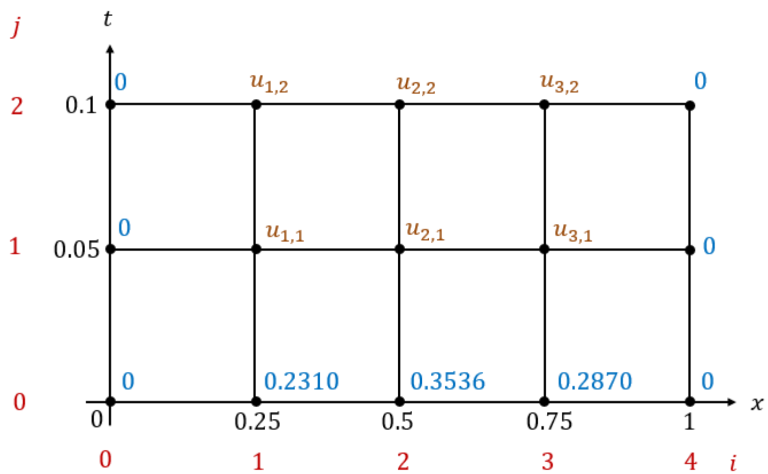
$$u_{i,j+1} = 0.4\left(u_{i+1,j} - 2u_{i,j} + u_{i-1,j}\right) + u_{i,j}$$

$$u_{i,j+1} = 0.4u_{i-1,j} + 0.2u_{i,j} + 0.4u_{i+1,j}$$



Step 5: Compute solutions for internal points $u_{i,j+1}$ (Refer to grid in Step 3).

$$u_{i,j+1} = 0.4u_{i-1,j} + 0.2u_{i,j} + 0.4u_{i+1,j}$$



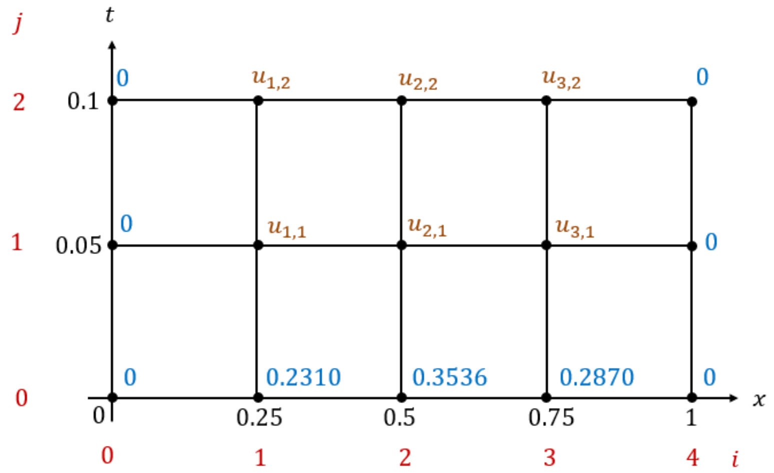
Let $i = 1, j = 0,$

$$u_{1,1} = 0.4u_{0,0} + 0.2u_{1,0} + 0.4u_{2,0}$$

$$u_{1,1} = 0.4(0) + 0.2(0.2310) + 0.4(0.3536)$$

$$u_{1,1} = 0.1876$$

$$u_{i,j+1} = 0.4u_{i-1,j} + 0.2u_{i,j} + 0.4u_{i+1,j}$$



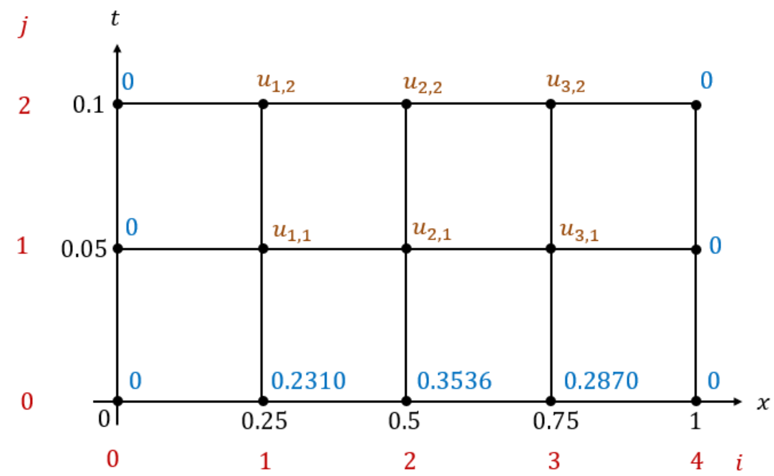
Let $i = 2, j = 0,$

$$u_{2,1} = 0.4u_{1,0} + 0.2u_{2,0} + 0.4u_{3,0}$$

$$u_{2,1} = 0.4(0.2310) + 0.2(0.3536) + 0.4(0.2870)$$

$$u_{2,1} = 0.2779$$

$$u_{i,j+1} = 0.4u_{i-1,j} + 0.2u_{i,j} + 0.4u_{i+1,j}$$



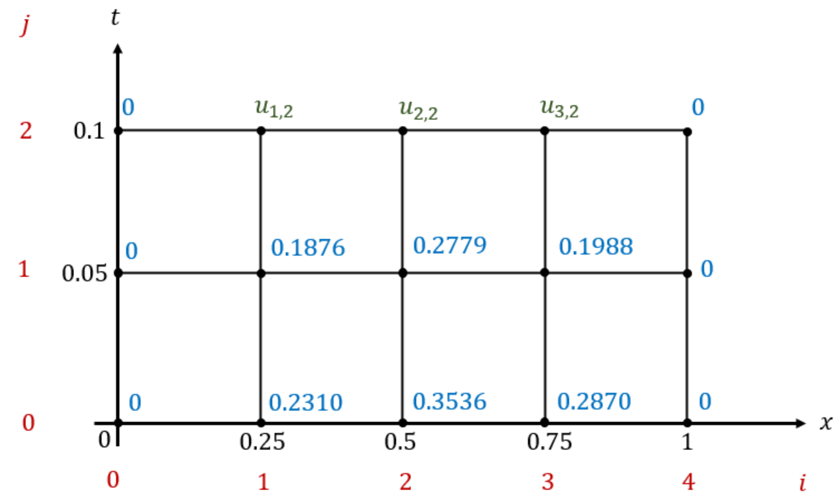
Let $i = 3, j = 0,$

$$u_{3,1} = 0.4u_{2,0} + 0.2u_{3,0} + 0.4u_{4,0}$$

$$u_{3,1} = 0.4(0.3536) + 0.2(0.2870) + 0.4(0)$$

$$u_{3,1} = 0.1988$$

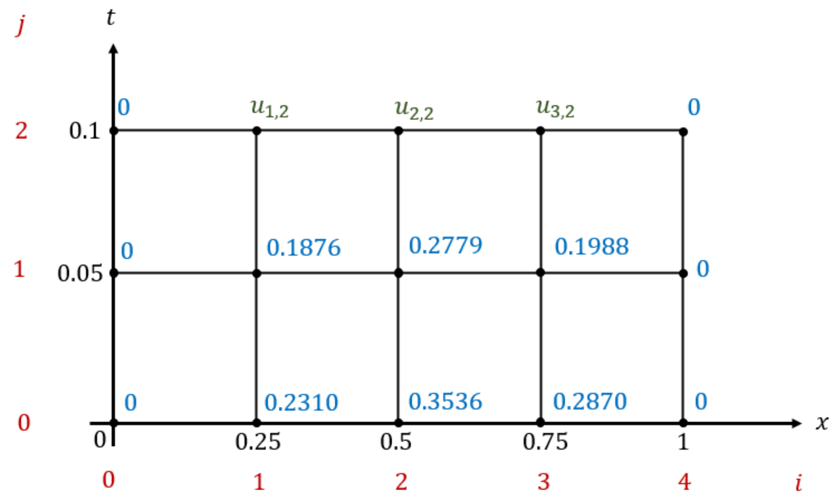
$$u_{i,j+1} = 0.4u_{i-1,j} + 0.2u_{i,j} + 0.4u_{i+1,j}$$



Let $i = 1, j = 1,$

$$u_{1,2} = 0.4u_{0,1} + 0.2u_{1,1} + 0.4u_{2,1}$$
$$u_{1,2} = 0.4(0) + 0.2(0.1876) + 0.4(0.2779)$$
$$u_{1,2} = 0.1489$$

$$u_{i,j+1} = 0.4u_{i-1,j} + 0.2u_{i,j} + 0.4u_{i+1,j}$$



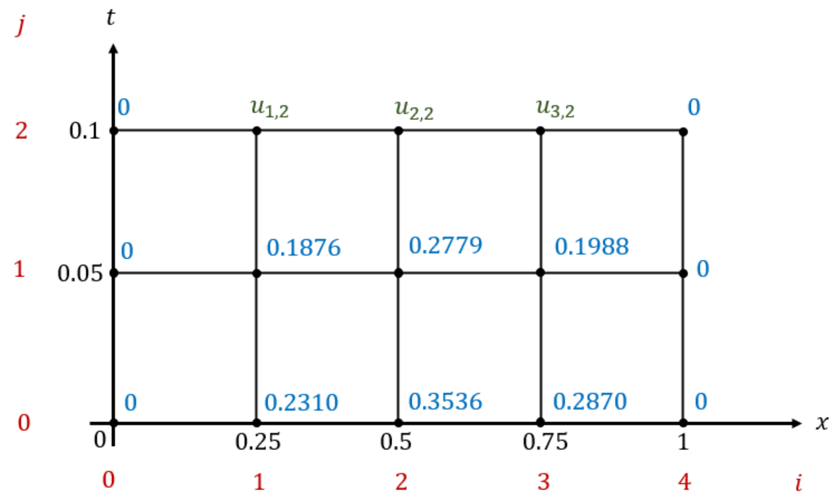
Let $i = 2, j = 1,$

$$u_{2,2} = 0.4u_{1,1} + 0.2u_{2,1} + 0.4u_{3,1}$$

$$u_{2,2} = 0.4(0.1876) + 0.2(0.2779) + 0.4(0.1988)$$

$$u_{2,2} = 0.2101$$

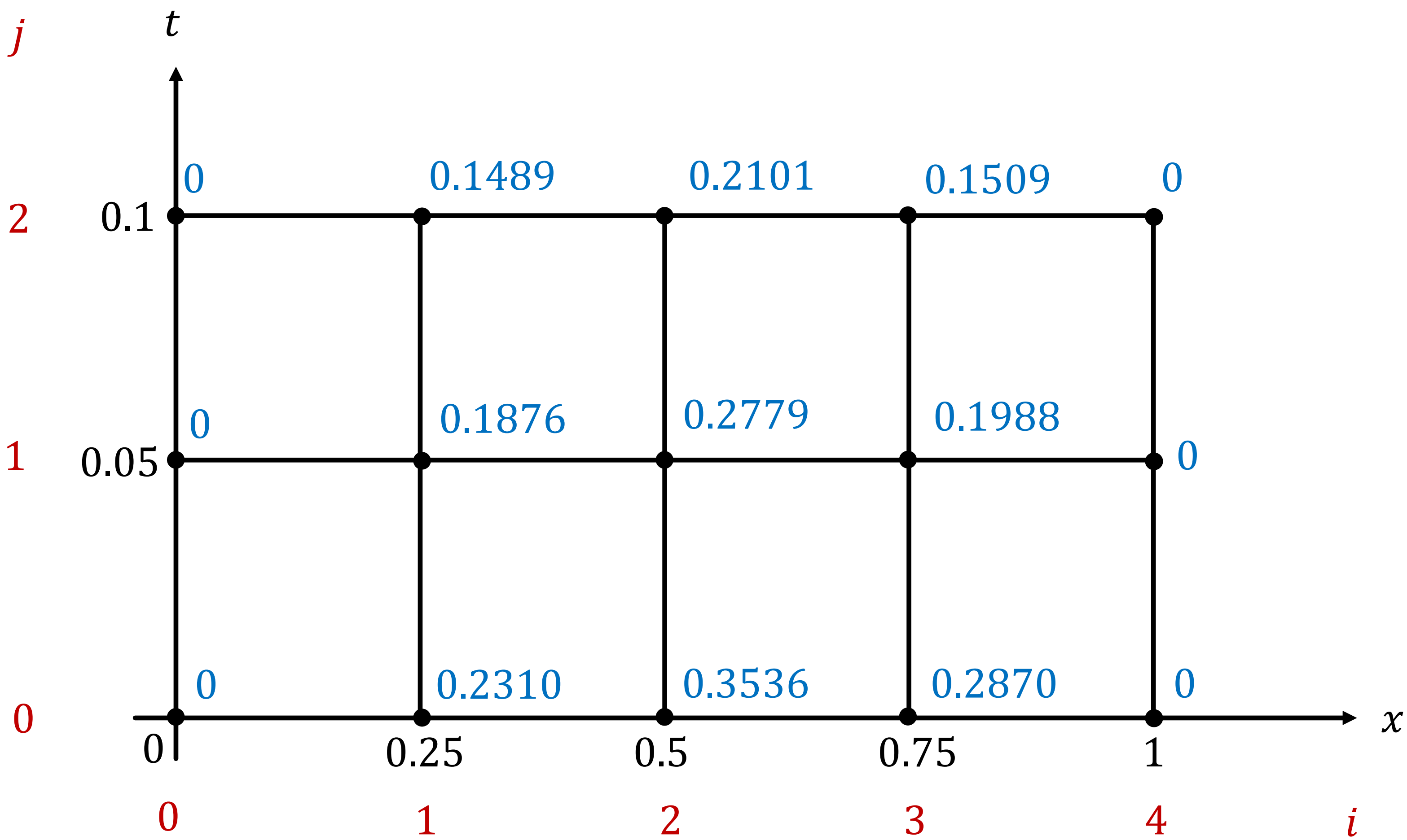
$$u_{i,j+1} = 0.4u_{i-1,j} + 0.2u_{i,j} + 0.4u_{i+1,j}$$



Let $i = 3, j = 1,$

$$u_{3,2} = 0.4u_{2,1} + 0.2u_{3,1} + 0.4u_{4,1}$$
$$u_{3,2} = 0.4(0.2779) + 0.2(0.1988) + 0.4(0)$$
$$u_{3,2} = 0.1509$$

Step 6: Fill in the values into the grid.



Example 5.18:

Find approximate solution for the following heat equation

$$\frac{\partial u}{\partial t} = 0.1\frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < 1, \qquad t > 0$$

with boundary conditions

$$u(0,t) = u(1,t) = 0, \qquad 0 \le t \le 0.4$$

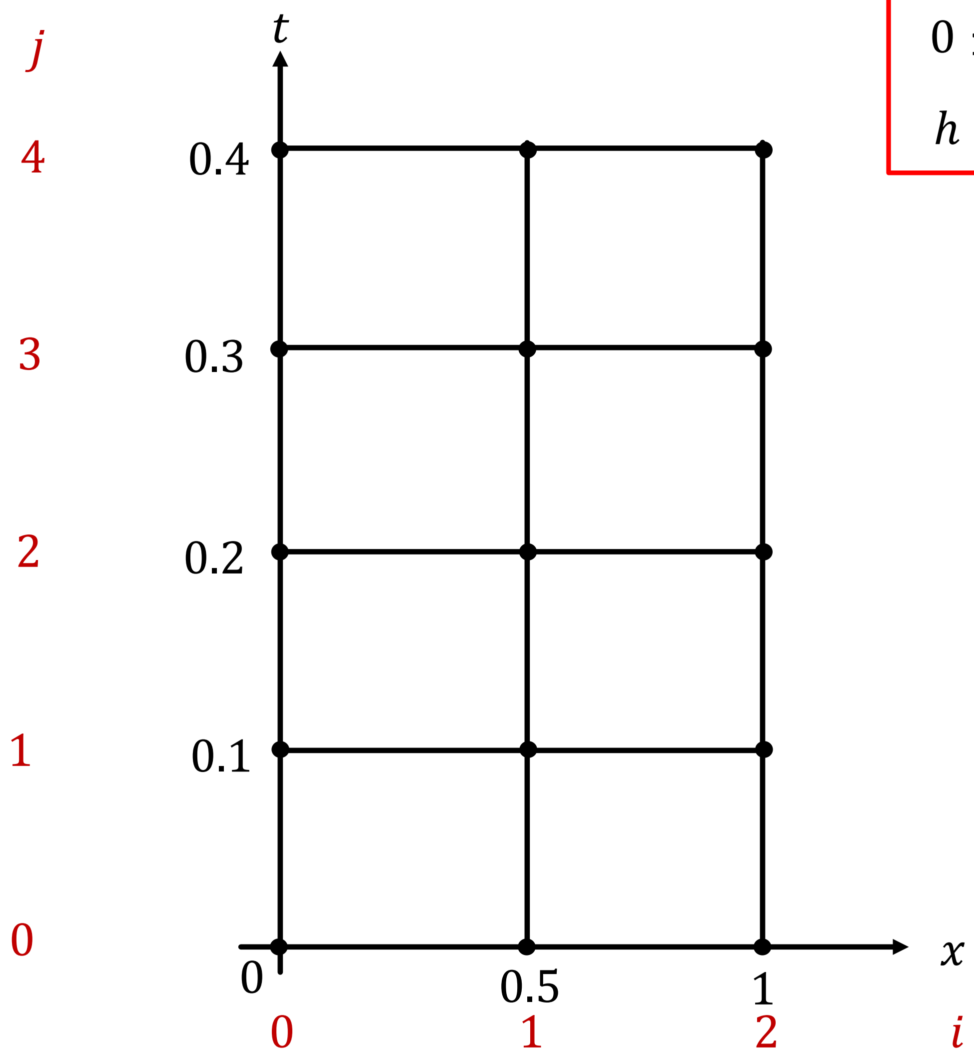
and initial condition

$$u(x,0) = x^2(1-x), \qquad 0 \le x \le 1.$$

by using the forward-difference and central-difference formula. Given $h = 0.5$ and $k = 0.1$.

Solution:

Step 1: Sketch the grid points.



Recall:

$0 \leq x \leq 1, \qquad 0 \leq t \leq 0.4$

$h = 0.5$ and $k = 0.1$

Step 2: Compute boundary and initial values.

Recall:
$h = 0.5$ and $k = 0.1$

$\boldsymbol{u(x,0) = x^2(1-x)}, \qquad \boldsymbol{0 \le x \le 1}$

$u(0,0) = 0$

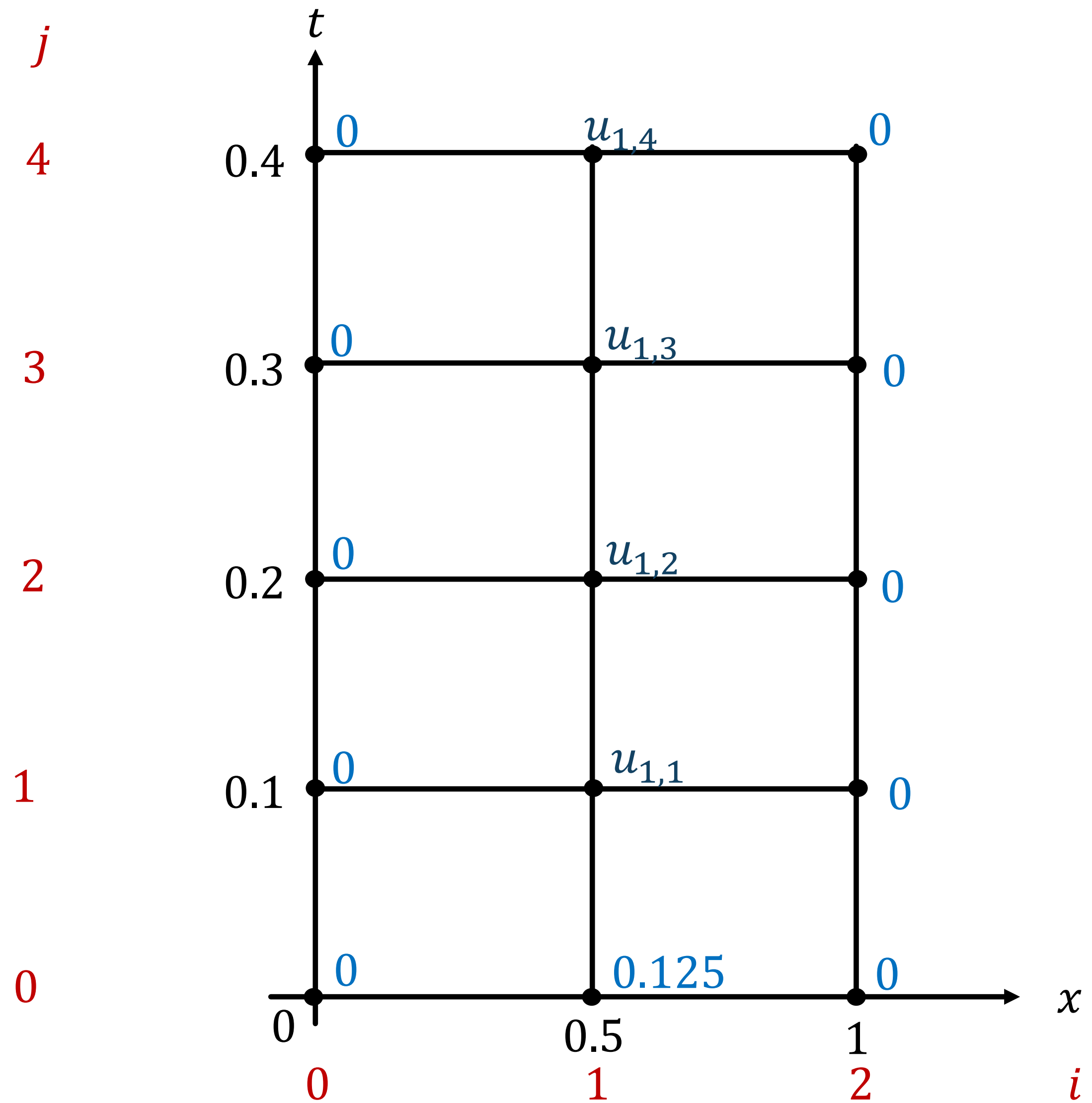
$u(0.5,0) = 0.125$

$u(1,0) = 0$

$\boldsymbol{u(0,t) = 0}, \qquad \boldsymbol{u(1,t) = 0}, \qquad \boldsymbol{0 \le t \le 0.4}$

$u(0,0) = 0$	$u(1,0) = 0$
$u(0,0.1) = 0$	$u(1,0.1) = 0$
$u(0,0.2) = 0$	$u(1,0.2) = 0$
$u(0,0.3) = 0$	$u(1,0.3) = 0$
$u(0,0.4) = 0$	$u(1,0.4) = 0$

Step 3: Fill in the values into the grid.



Step 4: Obtain formula $u_{i,j+1}$ from finite-difference formula.

By forward-difference and central-difference formulas,

$$\left(\frac{\partial u}{\partial t}\right)_{i,j} - 0.1\left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j} = 0$$

is approximated to

> ***Recall:***
> $h = 0.5$ and $k = 0.1$

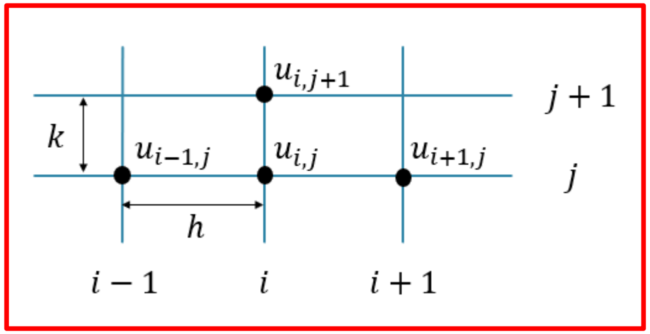
$$\frac{u_{i,j+1} - u_{i,j}}{k} - 0.1\left(\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2}\right) = 0$$

$$u_{i,j+1} - u_{i,j} = \frac{0.1k}{h^2}\left(u_{i+1,j} - 2u_{i,j} + u_{i-1,j}\right)$$

$$u_{i,j+1} = \frac{0.1(0.1)}{(0.5)^2}\left(u_{i+1,j} - 2u_{i,j} + u_{i-1,j}\right) + u_{i,j}$$

$$u_{i,j+1} = 0.04\left(u_{i+1,j} - 2u_{i,j} + u_{i-1,j}\right) + u_{i,j}$$

$$u_{i,j+1} = 0.04u_{i-1,j} + 0.92u_{i,j} + 0.04u_{i+1,j}$$



Step 5: Compute solutions for internal points $u_{i,j}$ (Refer to grid in Step 3).

$$u_{i,j+1} = 0.04u_{i-1,j} + 0.92u_{i,j} + 0.04u_{i+1,j}$$

Let $i = 1, j = 0,$

$u_{1,1} = 0.04u_{0,0} + 0.92u_{1,0} + 0.04u_{2,0}$

$u_{1,1} = 0.04(0) + 0.92(0.125) + 0.04(0)$

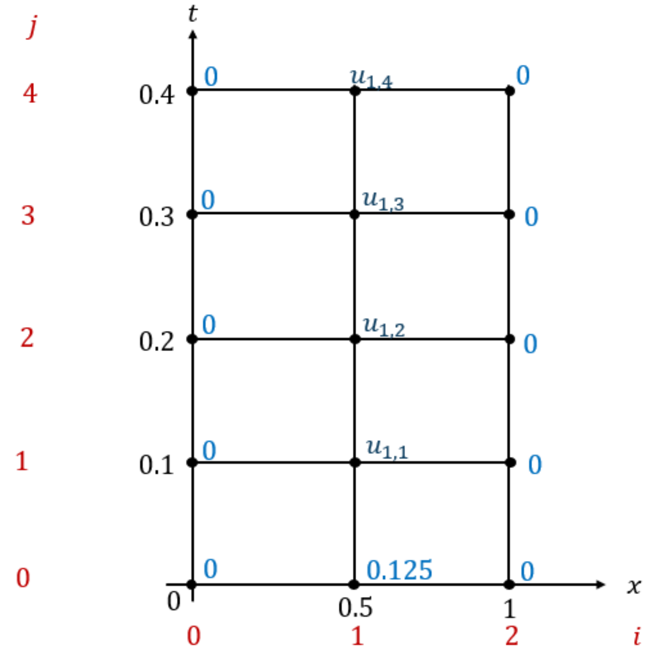
$u_{1,1} = 0.115$

Let $i = 1, j = 1,$

$u_{1,2} = 0.04u_{0,1} + 0.92u_{1,1} + 0.04u_{2,1}$

$u_{1,2} = 0.04(0) + 0.92(0.115) + 0.04(0)$

$u_{1,2} = 0.1058$



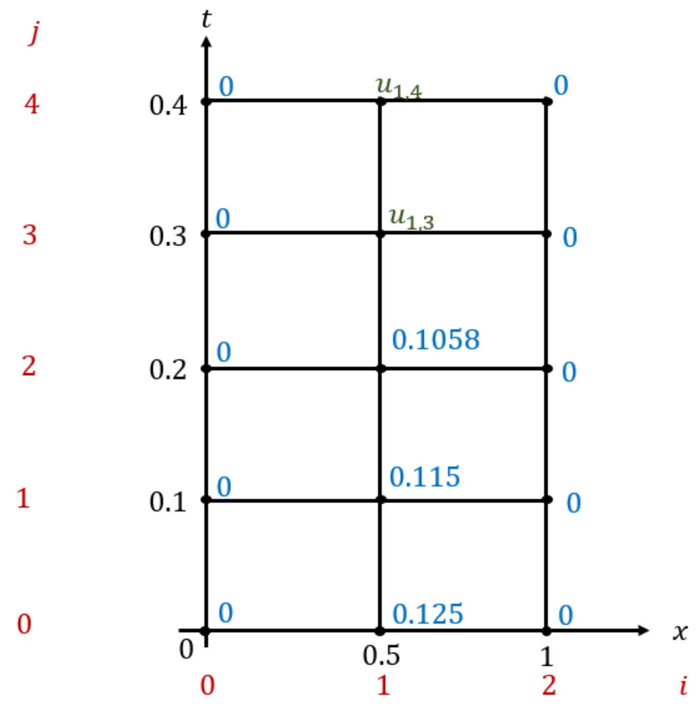
$$u_{i,j+1} = 0.04u_{i-1,j} + 0.92u_{i,j} + 0.04u_{i+1,j}$$

Let $i = 1, j = 2,$

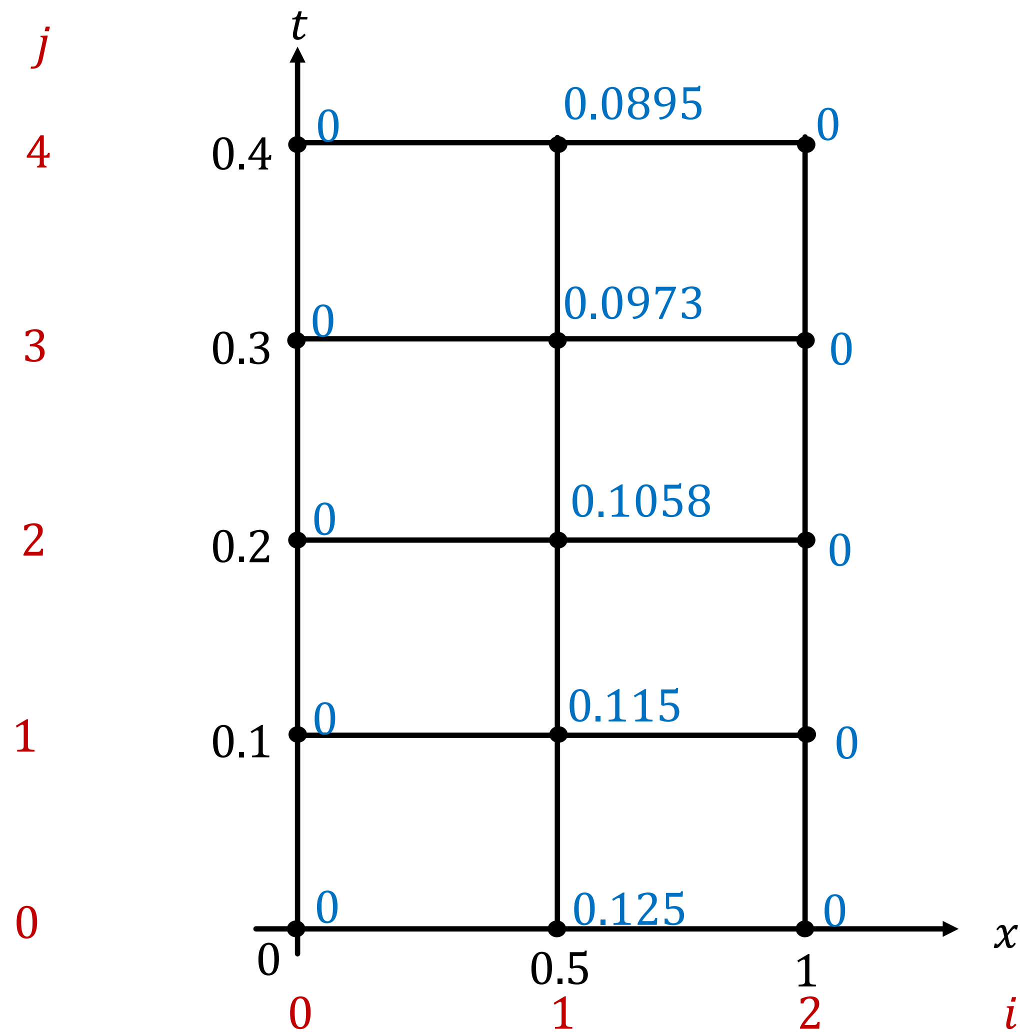
$$u_{1,3} = 0.04u_{0,2} + 0.92u_{1,2} + 0.04u_{2,2}$$
$$u_{1,3} = 0.04(0) + 0.92(0.1058) + 0.04(0)$$
$$u_{1,3} = 0.0973$$

Let $i = 1, j = 3,$

$$u_{1,4} = 0.04u_{0,3} + 0.92u_{1,3} + 0.04u_{2,3}$$
$$u_{1,4} = 0.04(0) + 0.92(0.0973) + 0.04(0)$$
$$u_{1,4} = 0.0895$$



Step 6: Fill in the values into the grid.



Exercise 5.10:

1) Solve the heat equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < 1, \qquad t > 0$$

with boundary conditions

$$u(0,t) = u(1,t) = 0, \qquad 0 \leq t \leq 0.02$$

and initial condition

$$u(x,0) = x(1-x) \;\; 0 \leq x \leq 1.$$

by using the forward-difference and central-difference formula. Given $h = 1/5$ and $k = 1/100$.

[Ans: $u_{0,0} = u_{0,1} = u_{0,2} = u_{5,0} = u_{5,1} = u_{5,2} = 0; u_{1,0} = u_{4,0} = 0.16; u_{2,0} = u_{3,0} = 0.24; u_{1,1} = u_{4,1} = 0.14; u_{2,1} = u_{3,1} = 0.22; u_{1,2} = u_{4,2} = 0.125; u_{2,2} = u_{3,2} = 0.2$]

Exercise 5.10:

2) Solve the heat equation

$$\frac{\partial u}{\partial t} = 0.03\frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < 2, \qquad t > 0$$

with boundary conditions

$$u(0,t) = u(2,t) = 0, \qquad 0 \leq t \leq 1$$

and initial condition

$$u(x,0) = 3\sin\frac{\pi}{2}x, \qquad 0 \leq x \leq 2.$$

by using the forward-difference and central-difference formula. Given $h = 1/2$ and $k = 1/2$.

[Ans: $u_{0,0} = u_{0,1} = u_{0,2} = u_{4,0} = u_{4,1} = u_{4,2} = 0; u_{1,0} = u_{3,0} = 2.1213; u_{2,0} = 3;$
$u_{1,1} = u_{3,1} = 2.0468; u_{2,1} = 2.8946; u_{1,2} = u_{3,2} = 1.9748; u_{2,2} = 2.7928$]

References

1. S.C. Chapra and R.P. Canale. (2015). Numerical Methods for Engineers, 7th Edition. McGraw-Hill Education.

2. R.L. Burden, D.J. Faires and A.M. Burden. (2016). Numerical Analysis, 10th Edition. Cengage Learning.

Thank You

Questions & Answer?