BMCG 1013
DIFFERENTIAL EQUATIONS

PARTIAL DIFFERENTIAL EQUATIONS (INTRODUCTION)

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Lesson Outcomes

Upon completion of this lesson, students should be able to:

- identify the order of a partial differential equation.
- determine whether a PDE is linear or non-linear
- determine whether a PDE is homogeneous or nonhomogeneous
- classify the PDE
- understand the method of separation of variables

5.1 Introduction to PDE

- A partial differential equation is a differential equation that involves several unknown variable functions and their partial derivatives.
- It is widely used in formulating the phenomena such as heat, sound wave, electrostatics, electrodynamics, quantum mechanics and fluid dynamics.
- A partial differential equation for a function $u(x_1, \dots x_n)$ has the following form

$$f\left(x_1, \dots x_n; u, u_{x_1}, \dots u_{x_n}; u_{x_1 x_1}, \dots, u_{x_1 x_n}; \dots\right) = 0$$

5.1 Introduction to PDE

Standard Notation:

Let x and y be independent variables and z be a function of x and y. The first order **partial derivatives**:

$$\frac{\partial z}{\partial x} \quad \text{or} \quad z_x \quad \text{and} \quad \frac{\partial z}{\partial y} \quad \text{or} \quad z_y$$

The second order **partial derivatives** of z with respect to only x or y:

$$\frac{\partial^2 z}{\partial x^2} \quad \text{or} \quad z_{xx} \quad \text{and} \quad \frac{\partial^2 z}{\partial y^2} \quad \text{or} \quad z_{yy}$$

The second order mixed **partial derivatives** of z with respect to x and y:

$$\frac{\partial^2 z}{\partial x \partial y} \quad \text{or} \quad z_{yx} \quad \text{and} \quad \frac{\partial^2 z}{\partial y \partial x} \quad \text{or} \quad z_{xy}$$

Any equation containing at least one of the *partial derivatives* is called a **partial differential equation**.

5.1 Introduction to PDE: Some Practical PDE

1. The one-dimensional **heat equation**

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$$

governs the heat flow (heat transferred by conduction) in a rod. The solution $u(x,t)$ represents the temperature of the rod.

2. The one-dimensional **wave equation**

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}$$

governs the motion of a vibrating string with fixed ends. The solution $u(x,t)$ represents the small displacements of an idealized vibrating string.

3. The **Laplace's equation**

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

describes the steady-state temperature distribution in a thin flat plate.

5.1.1 Order of PDE

The **order** of a PDE is the **order of the highest partial derivative term** occurring in the equation.

Example 5.1:

State the order of each of the following PDEs.

(a) $\frac{\partial z}{\partial y} + 2\frac{\partial^2 z}{\partial x^2} = 0$ (b) $z_{xyx} + yz_y = y^2$

Solution:

(a) Second order because of the term $\frac{\partial^2 z}{\partial x^2}$.

(b) Third order because of the term z_{xyx}.

5.1.1 Order of PDE

Example 5.2:

State the order of each of the following PDEs.

(a) $\left(\frac{\partial w}{\partial r}\right)^4 - \frac{\partial^3 w}{\partial r \partial s^2} = s + r$ (b) $u_{xx} + t^2 u_t = 2xt$

Solution:

(a) Third order because of the term $\frac{\partial^3 w}{\partial r \partial s^2}$.

(b) Second order because of the term u_{xx}.

Exercise 5.1:

State the order of each of the following PDEs.

1) $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 0$

2) $y^2 z_{xx} - x z_{yy} = x + y$

3) $u_y y = u_x\, u_y + u u_{xy}$

4) $y u_{xx} + y^2 u_y - u_{yy} = 0$

5) $u_x (u_{yxx})^2 + u_y u_{xx} = u_{yyx}$

6) $(z_{xx})^4 + 3 z_{yyy} = z_y + z_x$

7) $x^2 \frac{\partial z}{\partial y} + y\frac{\partial^2 z}{\partial x^2} = \frac{\partial z}{\partial x}$

[Ans: 1st; 2nd; 2nd; 2nd; 3rd; 3rd; 2nd]

5.1.2 Linear and Nonlinear PDE

A PDE in u is called **linear** if

a) the **degree** of the dependent variable u and its partial derivatives is **one**, and

b) the coefficients of the u-terms (dependent variable u and its partial derivatives) are **independent of** u.

Otherwise, the equation is called nonlinear.

5.1.2 Linear and Nonlinear PDE

Example 5.3:

Is $2tu_{xt} - 3u_x^2 = t^2 - x$ a linear PDE?

Solution:

This PDE is nonlinear because of the term u_x^2 (degree of 2 on partial derivative of u)

5.1.2 Linear and Nonlinear PDE

Example 5.4:

Is $u\frac{\partial u}{\partial x} + \sqrt{u} = 2x$ a linear PDE?

Solution:

This PDE is nonlinear because of the terms

- $\sqrt{u}$ (degree of ½ on dependent variable), and
- $u\frac{\partial u}{\partial x}$ (coefficient of $\frac{\partial u}{\partial x}$ is not independent of u. In other words, $\frac{\partial u}{\partial x}$ depends on u)

Exercise 5.2:

Classify whether the following PDEs are linear.

1) $x\frac{\partial u}{\partial x} + u^2 = x$

2) $u_t = u_{xx} + u$

3) $uu_t + xu_x = 4$

4) $x^2 u_{xx} - yu_{xy} = 3u$

5) $(u_x)^2 - 3xu_y = 0$

6) $\sqrt{x}u_{yy} + \sqrt{y}u_{xy} = u$

7) $u_t + xu_x = t^2$

[Ans: nonlinear; linear; nonlinear; linear; nonlinear; linear; linear]

5.1.3 Homogeneous and nonhomogeneous PDE

A PDE of any order is called **homogenous** if every term of the PDE depends on the dependent variable. Otherwise, it is called nonhomogeneous.

Example 5.5:

Determine whether the following PDE is homogeneous.

$$uu_x + 5xu_y = 1 + 3u_{xy}$$

Solution:

Nonhomogeneous because of the term 1 (1 is not depending on dependent variable)

5.1.3 Homogeneous and nonhomogeneous PDE

Example 5.6:

Determine the homogeneity of the following PDEs.

1) $u_x - u_y = u + 3x$

2) $\frac{\partial u}{\partial t} = 2\frac{\partial^2 u}{\partial x^2} + u\frac{\partial u}{\partial x}$

Solution:

1) Nonhomogeneous (the term $3x$ does not depend on dependent variable, u)

2) Homogeneous (all terms are depending on dependent variable, u).

Exercise 5.3:

Determine the homogeneity of the following PDEs.

1) $u_x + xu_y = u$

2) $t\frac{\partial u}{\partial t} = u\frac{\partial^2 u}{\partial x^2}$

3) $\frac{\partial u}{\partial t} = x\frac{\partial^2 u}{\partial x^2} + 6$

4) $uu_{xx} + u^2 = (u_t)^3$

5) $u_{xy} + 2y = u^3$

6) $\frac{\partial u}{\partial t} = u\frac{\partial^2 u}{\partial x^2} + 9u$

7) $u_x + xu_y = \sqrt{u}$

[Ans: #1,2,4,6,7 are homogenous, the rest are nonhomogeneous]

5.1.4 Classification of PDE

The second order of linear partial differential equation is given as follows:

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = 0$$

where A, B, C, D, E and F are scalars.

The equation is called

- Hyperbolic if $B^2 - 4AC > 0$
- Parabolic if $B^2 - 4AC = 0$
- Elliptic if $B^2 - 4AC < 0$

5.1.4 Classification of PDE

Example 5.7:

Classify the following PDEs.

1) $3u_{xx} - u_{xy} + 3u_{yy} + u_x = 0$

2) $4u_{tt} + 3u_{xt} + -u_x + 7u = 0$

3) $2u_{xt} + u_{xx} + u_{tt} + u = 0$

Solution:

1) $A = 3, B = -1, C = 3;\ B^2 - 4AC = -35 < 0 \ \Rightarrow$ Elliptic

2) $A = 0, B = 3, C = 4;\ B^2 - 4AC = 9 > 0 \Rightarrow$ Hyperbolic

3) $A = 1, B = 2, C = 1;\ B^2 - 4AC = 0 \Rightarrow$ Parabolic

Exercise 5.4:

Classify the following PDE.

1) $4u_{xx} + 2u_{xy} + u_x = 0$

2) $2u_{xx} + 2u_{yy} - 5u_y + 8u = 0$

3) $\frac{\partial^2 u}{\partial x^2} + 4\frac{\partial^2 u}{\partial y \partial x} + 4\frac{\partial^2 u}{\partial y^2} - \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} - 7u = 0$

4) $3\frac{\partial^2 u}{\partial t \partial x} - \frac{\partial^2 u}{\partial t^2} + 2\frac{\partial u}{\partial x} + 6\frac{\partial u}{\partial t} - 6u = 0$

[Ans: hyperbolic; elliptic; parabolic; hyperbolic]

5.1.5 Solution of a PDE

A solution of a PDE is a function u in such a way it satisfies the PDE and the given condition as well.

Example 5.8:

Show that $u(x,t) = x\sin t$ is a solution of $u_{tt} + u = 0$.

Solution:

Find the necessarily derivatives:

$$u_t = x\cos t\,, u_{tt} = -x\sin t$$

From the LHS of PDE,

$$u_{tt} + u = -x\sin t + x\sin t = 0$$

which is the same to the RHS of the PDE. Hence, $u(x,t) = x\sin t$ is a solution of $u_{tt} + u = 0$.

5.1.5 Solution of a PDE

Example 5.9:

Show that $u(x,t) = \sin x \sin t$ is a solution of $u_{tt} = u_{xx}$.

Solution:

First, rearrange the PDE as follows:

$$u_{tt} - u_{xx} = 0$$

Then find the necessarily derivatives:

$$u_t = \sin x \cos t\,, u_{tt} = -\sin x \sin t\,,$$

$$u_x = \cos x \sin t\,, u_{xx} = -\sin x \sin t$$

Starting from the LHS, $u_{tt} - u_{xx} = -\sin x \sin t - (-\sin x \sin t) = 0$

Hence, $u(x,t) = \sin x \sin t$ is a solution of $u_{tt} = u_{xx}$.

Exercise 5.5:

1) Show that $u(x,t) = e^{-4t}\sin x$ is a solution of $u_t = 4u_{xx}$.
2) Show that $u(x,t) = \cos x \cos t$ is a solution of $u_{tt} = u_{xx}$.
3) Show that $u(x,t) = x^2 + \sin x \sin t$ is a solution of $u_{xx} = u_{tt} + 2$.

5.2 Method of Separation of Variables

In this topic, we are solving separable linear partial differential equations, and Fourier series techniques are involved in obtaining the solution.

By method of separation of variables, we aim to find a solution in the form of a product of 2 functions; a function in x and a function in t,

$$u(x,t) = X(x)T(t) \neq 0.$$

From the assumed solution above, the derivatives are as follows:

$$\frac{\partial u}{\partial x} = X'T, \qquad \frac{\partial u}{\partial t} = XT'$$

$$\frac{\partial^2 u}{\partial x^2} = X''T, \qquad \frac{\partial^2 u}{\partial t^2} = XT''$$

With these derivative representations, a separable linear partial differential equation with two variables can be reduced into two ordinary differential equations.

5.2 Method of Separation of Variables

The partial differential equation is called **separable** if the derivative representations of the PDE can be separated by having a function in single variable on the left and a function in another single variable on the right.

Example 5.10:

Determine whether $u_t = 2u_x$ is separable.

Solution:

Let $u(x,t) = X(x)T(t)$, the heat equation becomes

$$XT' = 2X'T$$

By separating the variables, we get

$$\frac{X'}{X} = \frac{T'}{2T}$$

Hence, the PDE is separable.

5.2 Method of Separation of Variables

Example 5.11:

Determine whether $u_t = u_{xx} - 2u$ is separable.

Solution:

Let $u(x,t) = X(x)T(t)$, the heat equation becomes

$$XT' = X''T - 2XT$$

Rearrange the equation, we get

$$XT' + 2XT = X''T$$

$$X(T' + 2T) = X''T$$

By separating the variables, we get

$$\frac{X''}{X} = \frac{T' + 2T}{T}$$

Hence, the PDE is separable.

Exercise 5.6:

Determine whether the following PDEs are separable.

1) $t^2 u_{tt} = x^2 u_{xx}$
2) $u_x + u_t = 1$
3) $x u_{xx} - 4u_{tt} = 2u_x$
4) $t u_x + x u_t + u = 0$
5) $u_x + u_{xt} = x u_t$

[Ans: Yes; No; Yes; No; Yes]

5.2 Method of Separation of Variables

Example 5.12:

Find the solutions of the heat equation $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$ along a uniform metal rod, using separation of variables.

Solution:

Let $u(x,t) = X(x)T(t)$, the heat equation becomes

$$XT' = kX''T \tag{5.1}$$

By separating the variables, we get

$$\frac{X''}{X} = \frac{T'}{kT}$$

Since the left-hand side of the equation is dependent only on x and it is equal to the right-hand side of the equation where it is dependent only on t, to hold the validity of this equality, both functions must be equal to a constant as follows:

$$\frac{X''}{X} = \frac{T'}{kT} \equiv p$$

where constant p is known as separation constant. Hence X and T must satisfy

$$X'' - pX = 0 \text{ and } T' - pkT = 0. \qquad (5.2)$$

Recall: $X'' - pX = 0$ and $T' - pkT = 0.$ (5.2)

We need to consider three possible cases for p.

Case (1): $p = 0.$

From Eqn (5.2), it follows that $X'' = 0$ and $T' = 0$ which have the solutions

$$X(x) = ax + b \qquad \text{and} \qquad T(t) = c$$

respectively. Hence,

$$u(x,t) = X(x)T(t) = (ax + b)c = Ax + B.$$

Recall: $X'' - pX = 0$ and $T' - pkT = 0.$ (5.2)

We need to consider three possible cases for p.

Case (2): $p > 0$.

Let $p = \lambda^2$ and $\lambda > 0$. From Eqn (5.2), it leads to ODE

$$X'' - \lambda^2 X = 0 \text{ and } T' - \lambda^2 kT = 0$$

which have the solutions

$$X(x) = Ae^{\lambda x} + Be^{-\lambda x} \qquad \text{and} \qquad T(t) = Ce^{\lambda^2 kt}$$

respectively. Hence

$$u(x,t) = X(x)T(t) = \left(Ae^{\lambda x} + Be^{-\lambda x}\right)Ce^{\lambda^2 kt} = De^{\lambda x + \lambda^2 kt} + Ee^{-\lambda x + \lambda^2 kt}$$

Recall: $X'' - pX = 0$ and $T' - pkT = 0.$ (5.2)

Case (3): $p < 0.$

Let $p = -\lambda^2$ and $\lambda > 0$. From Eqn (5.2), it leads to ODE

$$X'' + \lambda^2 X = 0 \text{ and } T' + \lambda^2 kT = 0$$

which have the solutions

$$X(x) = A\cos\lambda x + B\sin\lambda x \quad \text{and} \quad T(t) = Ce^{-\lambda^2 kt}$$

respectively. Hence

$$u(x,t) = X(x)T(t) = (A\cos\lambda x + B\sin\lambda x)Ce^{-\lambda^2 kt}$$

$$= (D\cos\lambda x + E\sin\lambda x)e^{-\lambda^2 kt}$$

NOTE: Notice that there are three possible solutions. However, with the boundary conditions and initial conditions, this problem has a unique solution.

Exercise 5.7:

Find the solutions of the heat equation $\frac{\partial u}{\partial t} = 0.9\frac{\partial^2 u}{\partial x^2}$ along a uniform metal rod, using separation of variables.

[Ans: Case 1: $u(x,t) = Ax + B$;

Case 2: $u(x,t) = De^{\lambda x + 0.9\lambda^2 t} + Ee^{-\lambda x + 0.9\lambda^2 t}$;

Case 3: $u(x,t) = (D\cos\lambda x + E\sin\lambda x)e^{-0.9\lambda^2 t}$]

References

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Thank You

Questions & Answer?