BMCG 1013
DIFFERENTIAL EQUATIONS

FOURIER SERIES
(Fourier Sine Series, Fourier Cosine Series & Half-range)

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Learning Outcomes

Upon completion of this week lesson, students should be able to:

i. Find the Fourier Series for odd and even function

ii. Determine the Fourier Series of half range

CHAPTER 4

Fourier Series

- Fourier Sine Series (odd function)
- Fourier Cosine Series (even function)
- Fouries Series of half-range function

4.4 Fourier Odd and Even Series

$f(x)$

Types of functions	Formula Used
Odd	• Fourier Sine Series
Even	• Fourier Cosine Series
Neither	• Fourier Series

Note that: The odd and even function can also be solved using Fourier Series for interval $[-L, L]$

4.4.1 Fourier Sine Series *(also known as Fourier odd series)*

If the function, $f(t)$ is odd, then Fourier Series is given as

$$f(t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi t}{L}$$

for interval $[-L, L]$ where $a_0 = 0, a_n = 0$ and

$$b_n = \frac{2}{L}\int_0^L f(t)\sin\frac{n\pi t}{L}\,dt$$

4.4.2 Fourier Cosine Series *(also known as Fourier even series)*

If the function, $f(t)$ is even, then Fourier Series is given as

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi t}{L}$$

for interval $[-L, L]$ where $b_n = 0$ and

$$a_0 = \frac{2}{L}\int_0^L f(t)\,dt$$

$$a_n = \frac{2}{L}\int_0^L f(t)\cos\frac{n\pi t}{L}\,dt$$

Note that: If the function is neither an odd or an even function, then use Fourier Series

Example 4.8

Sketch the graph of the following function

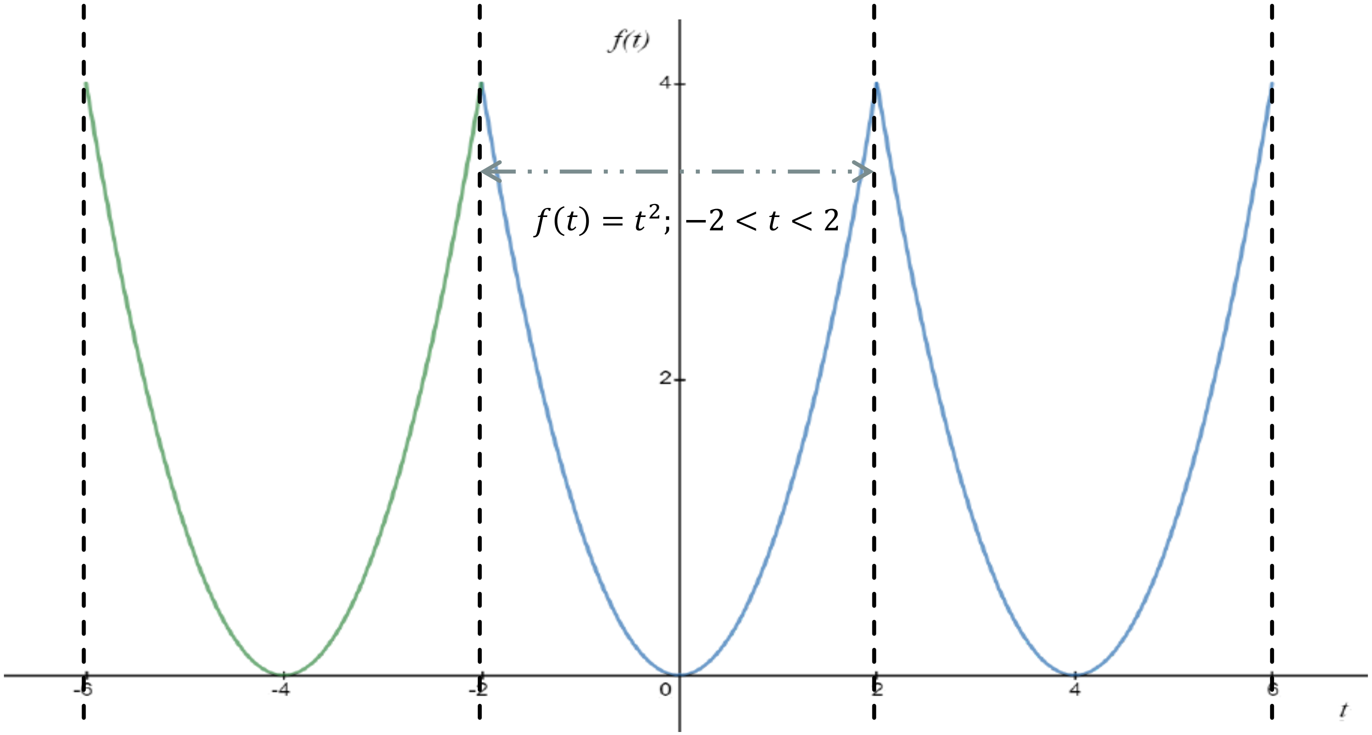
$$f(t) = t^2; \quad -2 < t < 2; \ f(t+4) = f(t)$$

over the interval $-6 < t < 6$. Determine whether the function is odd, even or neither function, hence find the Fourier Series of the function.

Example 4.8 – Solution

Firstly, sketch the $f(t) = t^2, -2 \leq t \leq 2$.

Then, with $T = 4$, extend the plot within the interval $-6 \leq t \leq 6$ by repeating itself identically.



Example 4.8

The sketch is symmetric about y −axis and $f(-t) = f(t)$, therefore this function is an even function. The Fourier Series can be obtained using Fourier Cosine Series.

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi t}{L}$$

Here $L = 2$ and $b_n = 0$;

$$a_0 = \frac{2}{L}\int_0^L f(t)\,dt = \frac{2}{2}\int_0^2 t^2\,dt = \left.\frac{t^3}{3}\right|_0^2 = \frac{8}{3}$$

$$a_n = \frac{2}{L}\int_0^L f(t)\cos\frac{n\pi t}{L}\,dt = \frac{2}{2}\int_0^2 t^2 \cos\frac{n\pi t}{2}\,dt$$

↑ Tabular Method

Example 4.8

$$a_n = \int_0^2 t^2 \cos\frac{n\pi t}{2}\,dt$$

$$= \frac{2t^2}{n\pi}\sin\frac{n\pi t}{2} + \frac{8t}{(n\pi)^2}\cos\frac{n\pi t}{2} - \frac{16}{(n\pi)^3}\sin\frac{n\pi t}{2}\Bigg|_0^2$$

$$= \frac{8}{n\pi}\underbrace{\sin n\pi}_{=0} + \frac{16}{(n\pi)^2}\underbrace{\cos n\pi}_{=(-1)^n} - \frac{16}{(n\pi)^3}\underbrace{\sin n\pi}_{=0}$$

$$-\left(-\frac{16}{(n\pi)^3}\underbrace{\sin 0}_{=0}\right)$$

$$= \frac{16}{(n\pi)^2}(-1)^n$$

Diff	Int
$(+)t^2$	$\cos\frac{n\pi t}{2}$
$(-)2t$	$\frac{2}{n\pi}\sin\frac{n\pi t}{2}$
$(+)2$	$-\frac{4}{(n\pi)^2}\cos\frac{n\pi t}{2}$
0	$-\frac{8}{(n\pi)^3}\sin\frac{n\pi t}{2}$

$$f(t) = \frac{4}{3} + \sum_{n=1}^{\infty}\frac{16}{(n\pi)^2}(-1)^n\cos\frac{n\pi t}{2}$$

Example 4.9

Sketch the periodic function

$$f(t) = |t|, -1 \leq t \leq 1,$$

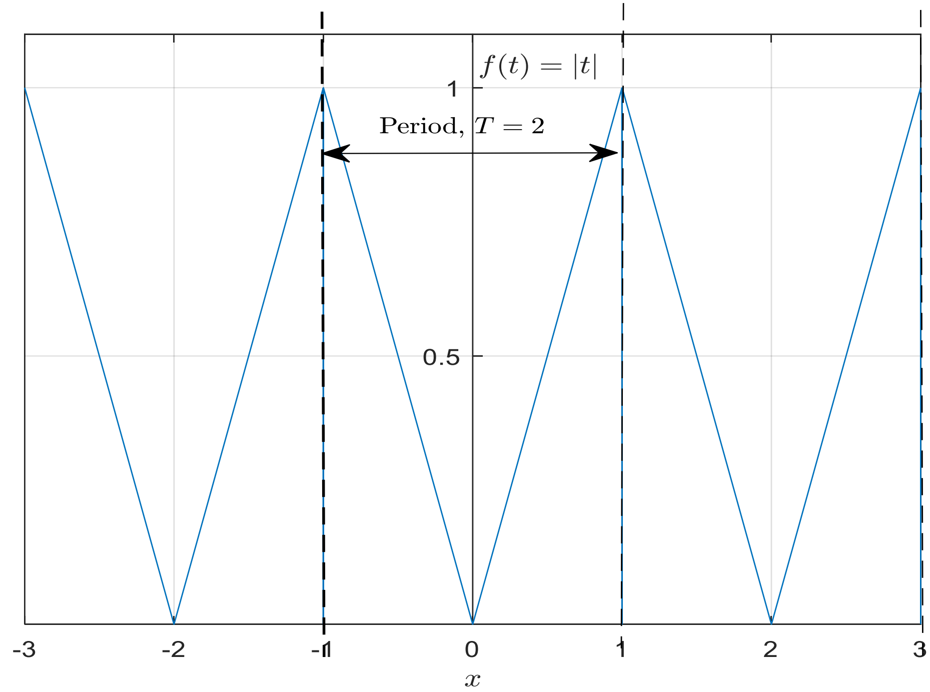
$$f(t+2) = f(t)$$

within the interval $-3 \leq t \leq 3$. Determine whether the function is odd, even or neither function, hence find the Fourier Series of the function.

Example 4.9 - Solution

Firstly, sketch the $f(t) = |t|, -1 \leq t \leq 1$.

Then, with $T = 2$, extend the plot within the interval $-3 \leq t \leq 3$ by repeating itself identically. The sketch is symmetric about y-axis, hence the function is an even function.



Example 4.9

The Fourier Series can be obtained using Fourier Cosine Series.

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi t}{L}$$

Here $L = 1$ and $b_n = 0$;

$$a_0 = \frac{2}{L}\int_0^L f(t)\,dt = \frac{2}{1}\int_0^1 t\,dt = 2\frac{t^2}{2}\bigg|_0^1 = 1$$

$$a_n = \frac{2}{L}\int_0^L f(t)\cos\frac{n\pi t}{L}\,dt = \frac{2}{1}\int_0^1 t\cos\frac{n\pi t}{1}\,dt$$

↑ Tabular Method

Note that: $f(t) = t, 0 \le t \le 1.$

Example 4.9

$$a_n = 2\int_0^1 t\cos\frac{n\pi t}{1}\,dt = 2\left(\frac{t}{n\pi}\sin n\pi t + \frac{1}{(n\pi)^2}\cos n\pi t\Big|_0^1\right)$$

$$= 2\left(\frac{1}{n\pi}\underbrace{\sin n\pi}_{=0} + \frac{1}{(n\pi)^2}\underbrace{\cos n\pi}_{=(-1)^n} - \frac{1}{(n\pi)^2}\underbrace{\cos 0}_{=1}\right)$$

$$= \frac{2}{(n\pi)^2}\left((-1)^n - 1\right)$$

$$f(t) = \frac{1}{2} + \sum_{n=1}^{\infty}\frac{2}{(n\pi)^2}\left((-1)^n - 1\right)\cos n\pi t$$

Diff	Int
$(+)t$	$\cos n\pi t$
$(-)1$	$\frac{1}{n\pi}\sin n\pi t$
0	$-\frac{1}{(n\pi)^2}\cos n\pi t$

Example 4.10

Sketch the periodic function

$$f(t) = -t, -\pi \le t \le \pi,$$

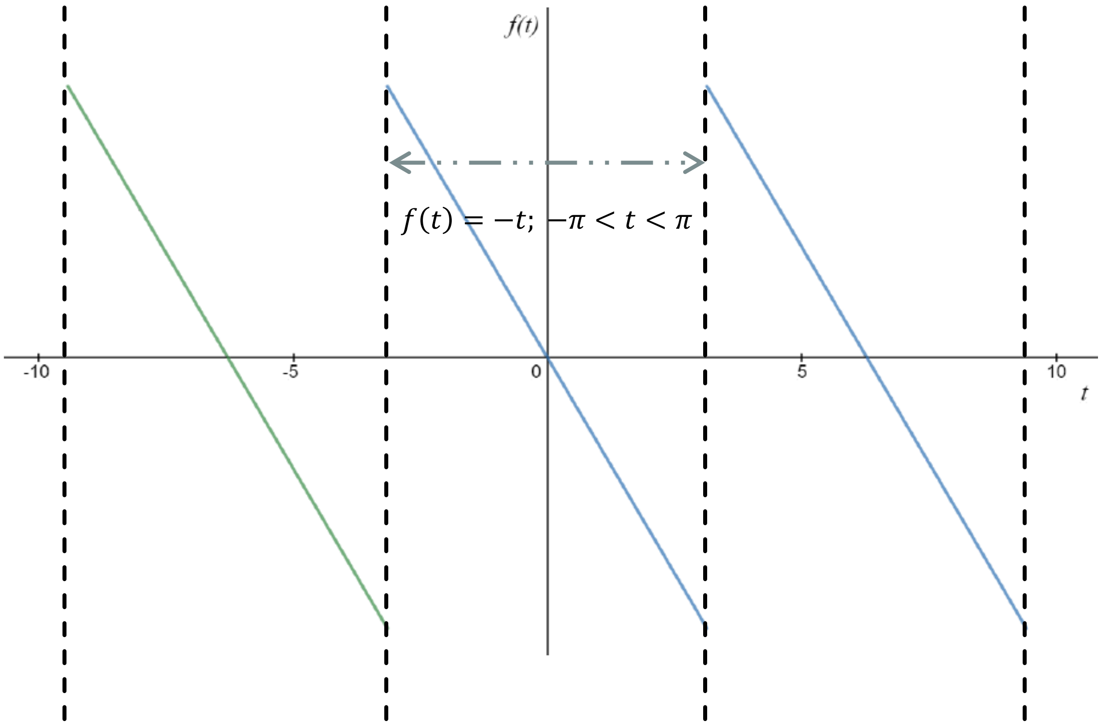
$$f(t + 2\pi) = f(t)$$

within the interval $-3\pi \le t \le 3\pi$. Determine whether the function is odd, even or neither function, hence find the Fourier Series of the function.

Example 4.10 - Solution

Firstly, sketch the $f(t) = -t, -\pi \leq t \leq \pi$.

Then, with $T = 2\pi$, extend the plot within the interval $-3\pi \leq t \leq 3\pi$ by repeating itself identically. The sketch is symmetric about the origin, hence the function is an odd function.



Example 4.10

The Fourier Series can be obtained using Fourier Sine Series.

$$f(t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi t}{L}$$

Here $L = \pi$ and $a_0 = 0, a_n = 0$;

$$b_n = \frac{2}{L}\int_0^L f(t)\sin\frac{n\pi t}{L}dt = \frac{2}{\pi}\int_0^{\pi} -t\sin\frac{n\pi t}{\pi}dt$$

↑ Tabular Method

Example 4.10

$$b_n = \frac{2}{\pi}\int_0^{\pi} -t\sin nt\,dt = \frac{t}{n}\cos nt - \frac{1}{n^2}\sin nt\Big|_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{\pi}{n}\underbrace{\cos n\pi}_{=(-1)^n} - \frac{1}{n^2}\overbrace{\sin n\pi}^{=0} - \frac{1}{n^2}\overbrace{\sin 0}^{=0}\right]$$

$$= \frac{2}{n}(-1)^n$$

$$f(t) = \sum_{n=1}^{\infty}\frac{2}{n}(-1)^n\sin nt$$

Diff	Int
$(+) - t$	$\sin nt$
$(-) - 1$	$-\frac{1}{n}\cos nt$
0	$-\frac{1}{n^2}\sin nt$

Exercise 4.5

1. Sketch the periodic function

$$f(t) = \begin{cases} -1; & -\pi \leq t \leq 0 \\ 1; & 0 \leq t \leq \pi \end{cases},$$

$$f(t + 2\pi) = f(t)$$

within the interval $-3\pi \leq t \leq 3\pi$. Determine whether the function is odd, even or neither function, hence find the Fourier Series of the function.

2. Sketch the periodic function

$$f(t) = \begin{cases} t + 1; & -1 \leq t \leq 0 \\ -t + 1; & 0 \leq t \leq 1 \end{cases}$$

$$f(t + 2) = f(t)$$

within the interval $-3 \leq t \leq 3$. Determine whether the function is odd, even or neither function, hence find the Fourier Series of the function.

Answers:

1. $f(t) = \sum_{n=1}^{\infty} \frac{2}{\pi n} [1 + (-1)^{n+1}] \sin nt$

2. $f(t) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{(n\pi)^2} [1 + (-1)^{n+1}] \cos n\pi t$

4.5 Half-Range Series

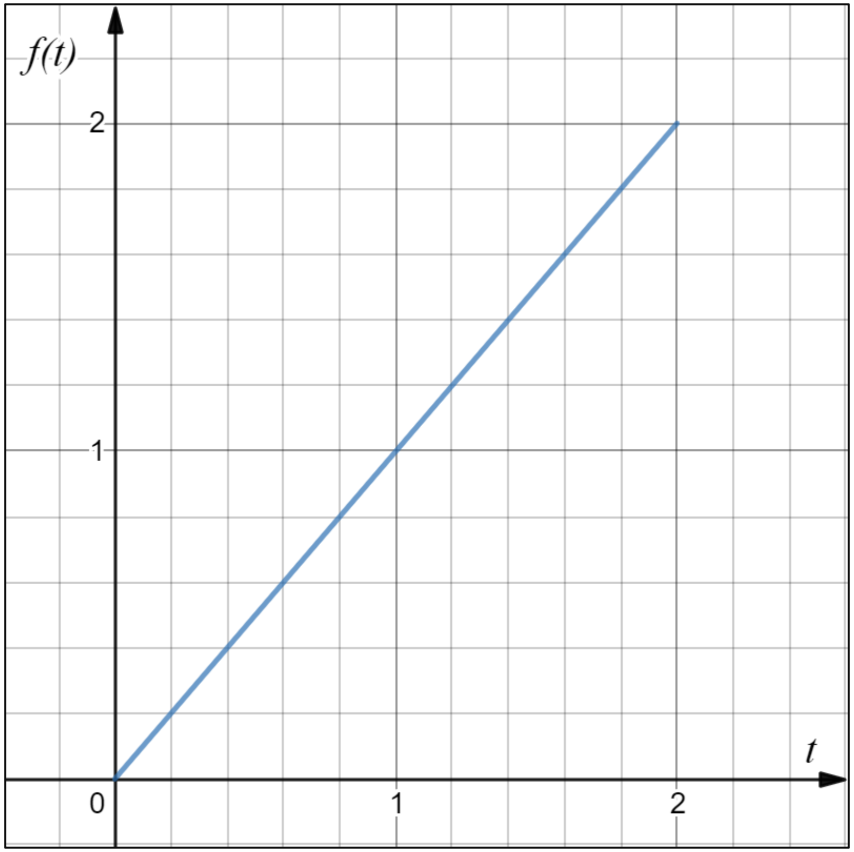
A periodic function with period $T = 2L$ is defined over the half-range $0 < t < L$ instead of the full-range $-L < t < 0$. The function may be expanded in a series of sine terms (odd function) or of cosine terms (even function). The series is called a half-range Fourier series.

The half-range function can be expanded into

- Even Function: Use *Fourier Cosine Series*
- Odd Function: Use *Fourier Sine Series*

Example 4.11

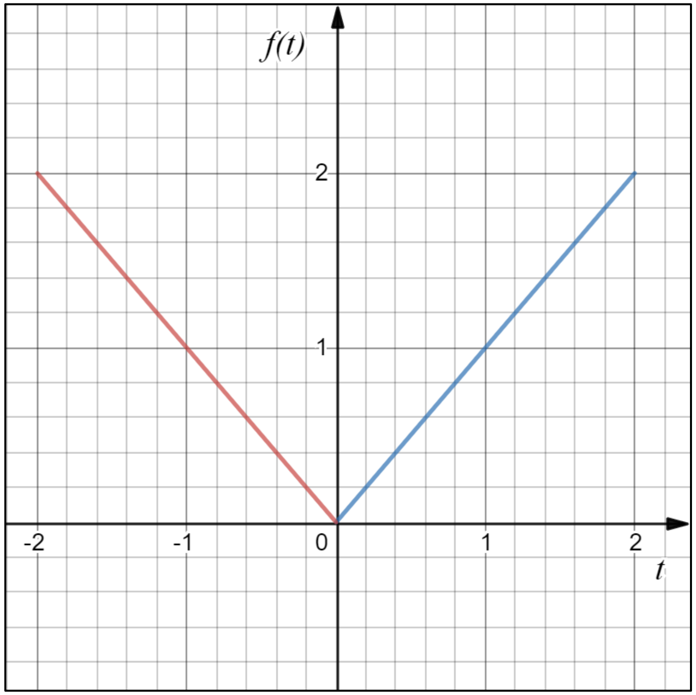
Sketch the function $f(t) = t, \quad 0 < t < 2.$



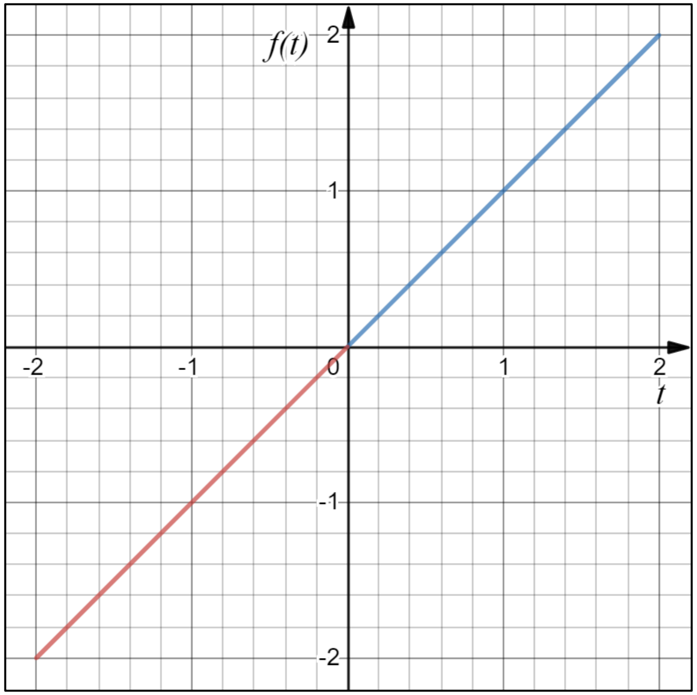
Example 4.11

Expansion of the function $f(t) = t, \quad 0 < t < 2.$

Expand the function to an even function which is symmetric about y-axis.



Expand the function to an odd function which is symmetric about the origin.



Example 4.12

Find the half-range Fourier series for the function

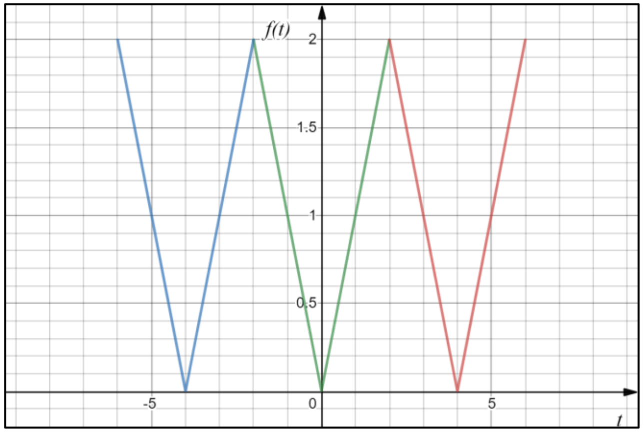
$$f(t) = t, \quad 0 < t < 2$$

using

a) Even Function Expansion
b) Odd Function Expansion

Solution:

a) The plot of even function expansion in the interval $[-6,6]$ where $L = 2;\ T = 4$.
Since it is an even function, use Fourier Cosine Series formula for the series.



$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi t}{L}$$

$$b_n = 0;\ a_0 = \frac{2}{L}\int_0^L f(t)\,dt;\ a_n = \frac{2}{L}\int_0^L f(t)\cos\frac{n\pi t}{L}\,dt$$

Example 4.12

Solution:

$$a_0 = \frac{2}{L}\int_0^L f(t)\,dt = \frac{2}{2}\int_0^2 t dt = \left[\frac{t^2}{2}\right]_0^2 = 2$$

$$a_n = \frac{2}{L}\int_0^L f(t)\cos\frac{n\pi t}{L}\,dt = \frac{2}{2}\int_0^2 t\cos\frac{n\pi t}{2}\,dt$$

$$= \left[\frac{2t}{n\pi}\sin\frac{n\pi t}{2} + \frac{4}{(n\pi)^2}\cos\frac{n\pi t}{2}\right]_0^2$$

$$= \frac{4}{n\pi}\sin n\pi + \frac{4}{(n\pi)^2}\cos n\pi - \frac{4}{(n\pi)^2} = \frac{4}{(n\pi)^2}[(-1)^n - 1]$$

Diff	Int
$(+)t$	$\cos\frac{n\pi t}{2}$
$(-)1$	$\frac{2}{n\pi}\sin\frac{n\pi t}{2}$
0	$-\frac{4}{(n\pi)^2}\cos\frac{n\pi t}{2}$

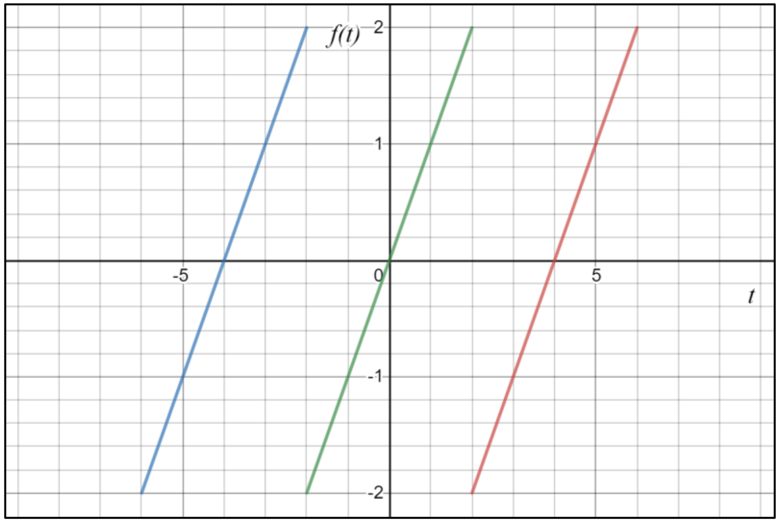
The Fourier Series of the even function expansion is

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\frac{n\pi t}{L} = 1 + \sum_{n=1}^{\infty}\frac{4}{(n\pi)^2}[(-1)^n - 1]\cos\frac{n\pi t}{2}$$

Example 4.12

Solution:

b) The plot of an odd function expansion in the interval $[-6,6]$ where $L = 2;\ T = 4$.



Since it is an odd function, use Fourier Sine Series formula for the series.

$$f(t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi t}{L}$$

$$a_0 = a_n = 0;\ b_n = \frac{2}{L}\int_0^L f(t)\sin\frac{n\pi t}{L}\,dt$$

Example 4.12

Solution:

$$b_n = \frac{2}{L}\int_0^L f(t)\sin\frac{n\pi t}{L}dt = \frac{2}{2}\int_0^2 t\sin\frac{n\pi t}{2}dt$$

$$= \left[-\frac{2t}{n\pi}\cos\frac{n\pi t}{2} + \frac{4}{(n\pi)^2}\sin\frac{n\pi t}{2}\right]_0^2$$

$$= -\frac{4}{n\pi}\cos n\pi + \frac{4}{(n\pi)^2}\sin n\pi = \frac{4}{n\pi}(-1)^{n+1}$$

Diff	Int
$(+)t$	$\sin\frac{n\pi t}{2}$
$(-)1$	$\frac{-2}{n\pi}\cos\frac{n\pi t}{2}$
0	$-\frac{4}{(n\pi)^2}\sin\frac{n\pi t}{2}$

The Fourier Series of the odd function expansion is

$$f(t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi t}{L} = \sum_{n=1}^{\infty}\frac{4}{(n\pi)^2}(-1)^{n+1}\sin\frac{n\pi t}{2}$$

Example 4.13

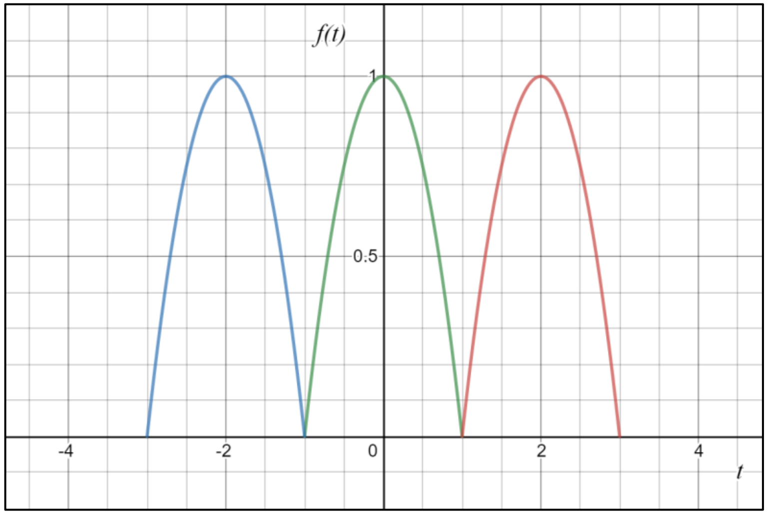
Find the half-range Fourier series for the function

$$f(t) = 1 - t^2, \quad 0 < t < 1$$

using

a) Even Function Expansion
b) Odd Function Expansion

Solution:



a) The plot of even function expansion in the interval $[-3,3]$ where $L = 1;\ T = 2$.
Since it is an even function, use Fourier Cosine Series formula for the series.

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi t}{L}$$

$$b_n = 0;\ a_0 = \frac{2}{L}\int_0^L f(t)\,dt;\ a_n = \frac{2}{L}\int_0^L f(t)\cos\frac{n\pi t}{L}\,dt$$

Example 4.13

Solution:

$$a_0 = \frac{2}{L}\int_0^L f(t)\,dt = \frac{2}{1}\int_0^1 (1-t^2)dt = \left[t - \frac{t^3}{3}\right]_0^1 = 2\left(1 - \frac{1}{3}\right) = \frac{4}{3}$$

$$a_n = \frac{2}{L}\int_0^L f(t)\cos\frac{n\pi t}{L}\,dt = \frac{2}{1}\int_0^1 (1-t^2)\cos n\pi t\,dt$$

$$= 2\left[\frac{(1-t^2)}{n\pi}\sin n\pi t - \frac{2t}{(n\pi)^2}\cos n\pi t + \frac{2}{(n\pi)^3}\sin n\pi t\right]_0^1$$

$$= 2\left(-\frac{2}{(n\pi)^2}\cos n\pi + \frac{2}{(n\pi)^3}\sin n\pi - 0\right) = \frac{4}{(n\pi)^2}(-1)^{n+1}$$

Diff	Int
$(+)(1-t^2)$	$\cos n\pi t$
$(-)-2t$	$\frac{\sin n\pi t}{n\pi}$
$(+)-2$	$-\frac{\cos n\pi t}{(n\pi)^2}$
0	$-\frac{\sin n\pi t}{(n\pi)^3}$

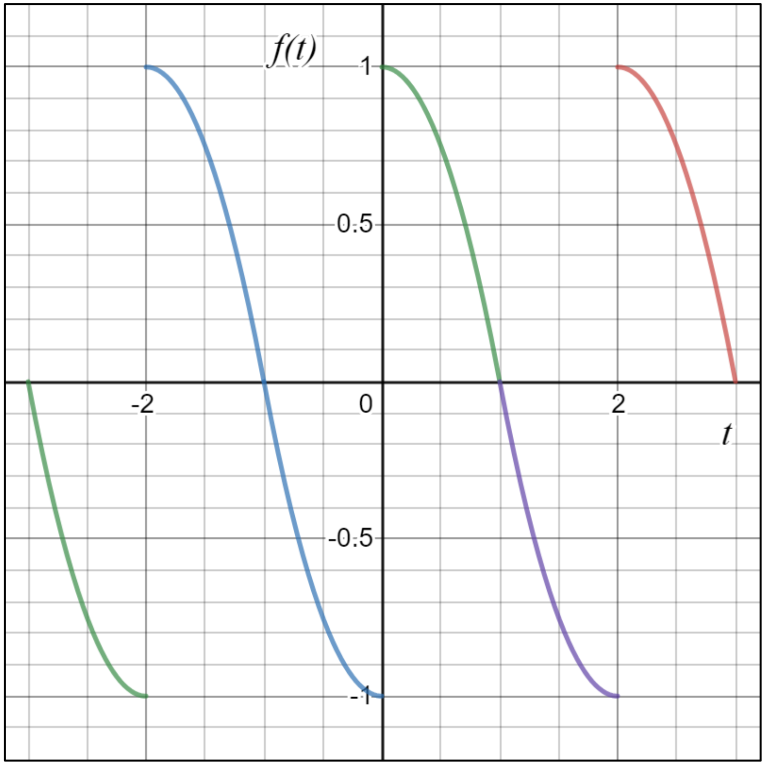
The Fourier Series of the even function expansion is

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi t}{L} = \frac{2}{3} + \sum_{n=1}^{\infty} \frac{4}{(n\pi)^2}(-1)^{n+1}\cos n\pi t$$

Example 4.13

Solution:

b) The plot of an odd function expansion in the interval $[-3,3]$ where $L = 1$; $T = 2$.



Since it is an odd function, use Fourier Sine Series formula for the series.

$$f(t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi t}{L}$$

$$a_0 = a_n = 0; \; b_n = \frac{2}{L} \int_0^L f(t) \sin \frac{n\pi t}{L} dt$$

Example 4.13

Solution:

Diff	Int
$(+)(1-t^2)$	$\sin n\pi t$
$(-)-2t$	$-\dfrac{\cos\ n\pi t}{n\pi}$
$(+)-2$	$-\dfrac{\sin n\pi t}{(n\pi)^2}$
0	$\dfrac{\cos n\pi t}{(n\pi)^3}$

$$b_n = \frac{2}{L}\int_0^L f(t)\sin\frac{n\pi t}{L}\,dt = \frac{2}{1}\int_0^1 \left(1-t^2\right)\sin n\pi t\,dt$$

$$= 2\left[-\frac{\left(1-t^2\right)}{n\pi}\cos n\pi t - \frac{2t}{(n\pi)^2}\sin n\pi t - \frac{2}{(n\pi)^3}\cos n\pi t\right]_0^1$$

$$= 2\left(-\frac{2}{(n\pi)^2}\sin n\pi - \frac{2}{(n\pi)^3}\cos n\pi - \left(-\frac{1}{n\pi} - \frac{2}{(n\pi)^3}\right)\right)$$

$$= 2\left(-\frac{2(-1)^n}{(n\pi)^3} + \frac{1}{n\pi} + \frac{2}{(n\pi)^3}\right) = \frac{1}{n\pi} + \frac{4}{(n\pi)^3}\left[(-1)^{n+1}+1\right]$$

The Fourier Series of the odd function expansion is

$$f(t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi t}{L} = \sum_{n=1}^{\infty}\left(\frac{1}{n\pi} + \frac{4}{(n\pi)^3}\left[(-1)^{n+1}+1\right]\right)\sin n\pi t$$

Exercise 4.6

Given the following function,

$$f(x) = -x; \quad 0 \leq x \leq \pi$$

i. Sketch the even and odd expansion of the function

ii. Find the Fourier cosine and the Fourier sine series, respectively.

Answers:

a) Even: $f(x) = -\frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2}{\pi n^2}[(-1)^{n+1} + 1] \cos nx$

b) Odd: $f(x) = \sum_{n=1}^{\infty} \frac{2}{n}(-1)^n \sin nx$

Exercise 4.7

Given the following functions,

$$f(x) = 1 + x; \quad 0 \leq x \leq 1$$

i. Sketch the even and odd expansion of the function

ii. Find the Fourier cosine and the Fourier sine series, respectively.

Answers:

a) Even: $f(x) = \frac{3}{2} + \sum_{n=1}^{\infty} \frac{2}{(n\pi)^2}[(-1)^n - 1]\cos n\pi x$

b) Odd: $f(x) = \sum_{n=1}^{\infty} \frac{2}{n\pi}[2(-1)^{n+1} + 1]\sin n\pi x$

References

Edwards C. H., Penny D.E. & Calvis D. (2016). Differential Equations and Boundary Value Problems, 5thEdition. Pearson Education Inc..

Zill, D. G. (2017). Differential Equations with Boundary-Value Problems, 9thEdition. Cengage Learning Inc

Thank You

Questions & Answer?