BMCG 1013
DIFFERENTIAL EQUATIONS

FOURIER SERIES
(Periodic Functions & Fourier Series)

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Learning Outcomes

Upon completion of this week lesson, students should be able to:

i. describe periodic functions

ii. determine even, odd and neither even nor odd functions

iii. determine the Fourier Series of full range

CHAPTER 4

Fourier Series

- Periodic Functions
- Odd and Even Functions
- Finding a Fouries Series of a function

4.1 Periodic Functions

Definition

A function $f(t)$ defined on $[-L, L]$ is periodic function with period of $T = 2L$ if

$$f(t+T) = f(t), \quad T > 0$$

The smallest positive value of T is called a period of the function $f(t)$.

Example 4.1

Show that $f(t) = \sin t$ is periodic function and find the period.

Solution:

The function $f(t) = \sin t$, hence $f(t+T) = \sin(t+T)$.

Using the trigonometric identity

$$\sin(t+T) = \sin t \cos T + \cos t \sin T$$

If $\sin T = 0$ and $\cos T = 1$, this is true for $T = 2\pi n$ where $n = 1,2,3,\cdots$.

The smallest value for T is $T = 2\pi$ which is known as fundamental period.

Thus, $\sin t = \sin(t + 2\pi)$ is a periodic function with period of 2π.

Example 4.2

Sketch the periodic function

$$f(t) = |t|, -1 \le t \le 1,$$

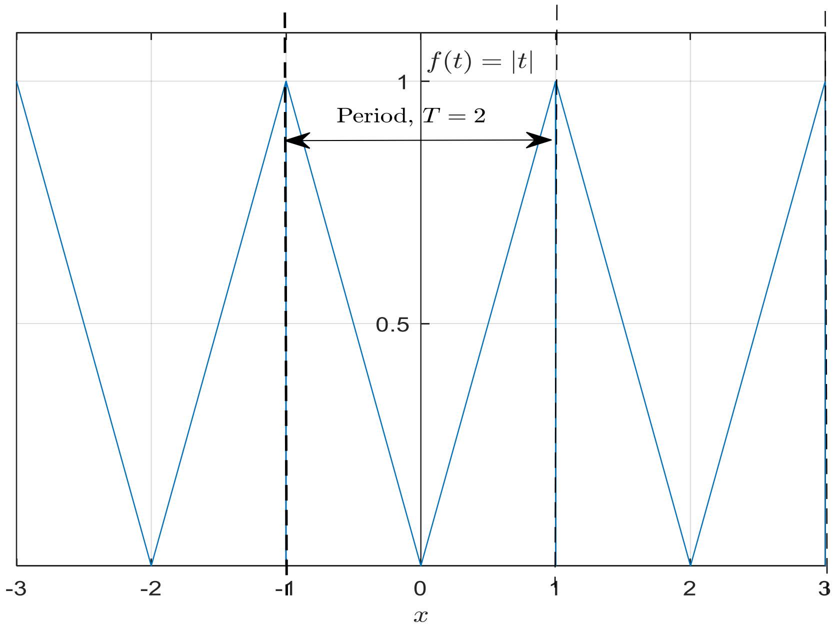
$$f(t+2) = f(t)$$

within the interval $-3 \le t \le 3$.

Solution:

Firstly, sketch the $f(t) = |t|$, $-1 \le t \le 1$.

Then, with $T = 2$, extend the plot within the interval $-3 \le t \le 3$ by repeating itself identically.



Exercise 4.1

1. Determine whether $f(t) = \cos 2t$ is periodic function? If yes, find the period, T.

2. Sketch a periodic function, $f(t) = t^2, f(t+2) = f(t)$.

Answers:
1. Periodic, $T = \pi$

4.2 Odd and Even functions

Properties:

Let $f(t)$ be a real-valued function of a real variable. Then

i. $f(t)$ is an even function if $f(-t) = f(t)$.

The graph of an even function is symmetric about the y-axis.

ii. $f(t)$ is an odd function if $f(-t) = -f(t)$.

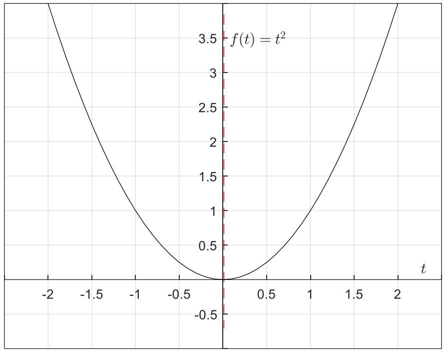
The graph of an odd function is symmetric about the origin.

Example 4.3

i. $f(t) = t^2$ is an even function.

Property 1: $f(-t) = (-t)^2 = t^2 = f(t)$.

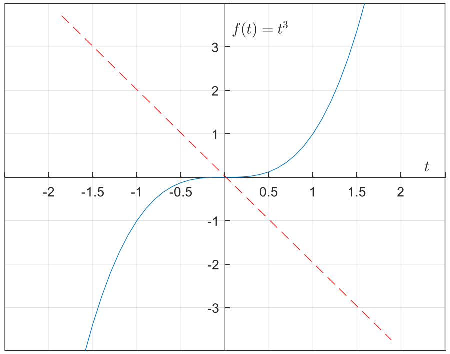
Property 2: The graph is symmetric about y −axis.



ii. $f(t) = t^3$ is an odd function.

Property 1: $f(-t) = (-t)^3 = -t^3 = -f(t)$.

Property 2: The graph is symmetric about the origin.



Example 4.4

Functions	Property 1	Property 2
EVEN $\lvert t\rvert, t^2, t^4, t^6, \cos t, \cosh t$	$f(-t) = f(t)$	Symmetry about y –axis
ODD $t, t^3, t^5, \sin t, \sinh t$	$f(-t) = -f(t)$	Symmetry about the origin

Example 4.5

Given a function;

$$f(t) = \begin{cases} 1+t; & -1 \leq t < 0 \\ 1-t; & 0 \leq t < 1 \end{cases}$$

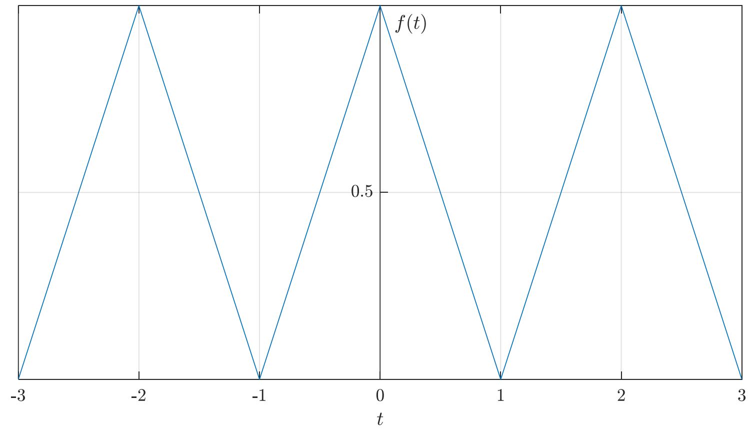
$$f(t+2) = f(t)$$

i. Sketch the function over the interval $[-3,3]$

ii. Hence, determine whether the function is an odd or an even function or neither.

Solution:

i.



ii. Since,

$$f(-t) = \begin{cases} 1-t; & 0 \leq t < 1 \\ 1+t; & -1 \leq t < 0 \end{cases} = f(t)$$

and the sketch is symmetry about y −axis, therefore, the function is an even function.

Exercise 4.2

1. Show that $f(t) = \cos 2t$ is an even function.
2. Determine whether the given function is even, odd or neither both of them.
   a) $f(t) = e^{-t} + e^{t}$
   b) $f(t) = \sqrt{t^2 + 2}$
   c) $f(t) = t^2 \sin t$
   d) $f(t) = t^3 \cos t$
   e) $f(t) = te^{t}$
3. Sketch the following function,

$$f(t) = t^2, \qquad -1 \leq t \leq 1$$
$$f(t+2) = f(t)$$

   over the interval $[-3,3]$. Hence, determine whether the function is an even or an odd function.

Answers: 2. a) even b) even c) odd d) odd e) neither
3. even

4.3 Fourier Series

- Fourier Series is a series representing real periodic functions as an expansion in a series of sine and cosine.
- The Fourier Series method is used to determine the unknown coefficients arise in solving PDE.
- Applications:

Electrical engineering, vibration analysis, optics, signal processing, image processing

Definition – Fourier Series

Let $f(t)$ be a periodic function defined in the interval $[-L, L]$ with period $T = 2L$. If the function $f(t)$ is either continuous or has finite discontinuities at some points in the interval, hence, the Fourier series for $f(t)$ is given by

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\frac{n\pi t}{L} + b_n \sin\frac{n\pi t}{L} \right]$$

where the Fourier coefficients a_0, a_n, b_n are defined by

$$a_0 = \frac{1}{L} \int_{-L}^{L} f(t)\, dt$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(t) \cos\frac{n\pi t}{L}\, dt$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(t) \sin\frac{n\pi t}{L}\, dt$$

- Working steps of Fourier Series

Step 1: Determine parameter L → Step 2: Calculate the Fourier coefficients, a_0, a_n, b_n using formula → Step 3: Substitute Fourier coefficients a_0, a_n, b_n into Fourier series formula

- Some helpful identities for calculating Fourier coefficients.

$$\sin 0 = 0 \quad ; \quad \cos 0 = 1$$

$$\sin(-t) = -\sin t \quad ; \quad \cos(-t) = \cos t$$

$$\sin n\pi = 0 \; ; \; \cos n\pi = (-1)^n$$

$$\sin 2n\pi = 0 \quad ; \quad \cos 2n\pi = 1$$

for n integers.

Example 4.6

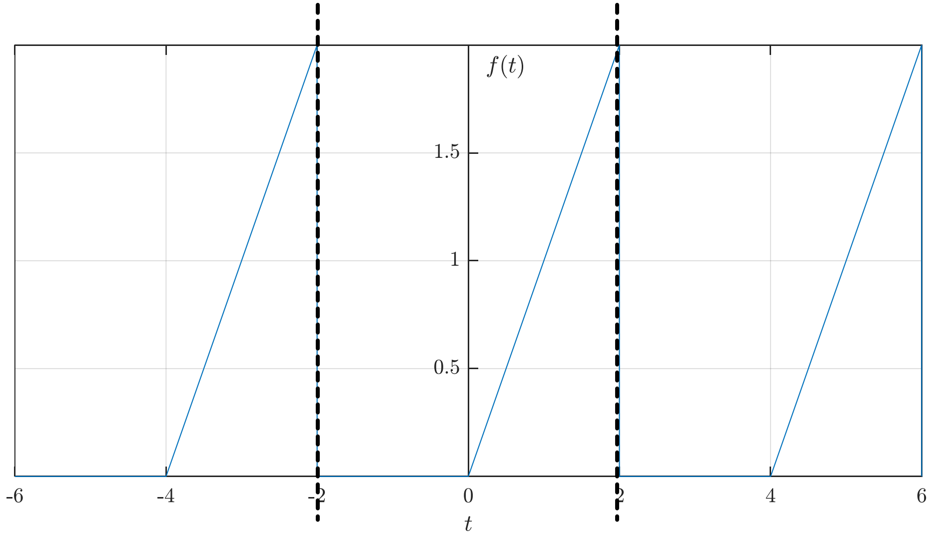
Given a function, $f(t) = \begin{cases} 0\,; & -2 \le t < 0 \\ t\,; & 0 \le t < 2 \end{cases}$, $f(t+4) = f(t)$.

i. Sketch the function over the interval $[-6,6]$. Determine whether the function is even, odd or neither.

ii. Find the Fourier Series of the function.

Solution:

i.



The function is neither even nor odd function.

ii.

Step 1: Based on the function, $L = 2$.

Step 2: Calculate the Fourier coefficients, a_0, a_n, b_n

$$a_0 = \frac{1}{2}\int_{-2}^{2} f(t)\,dt = \frac{1}{2}\left[\int_{-2}^{0} 0dt + \int_{0}^{2} tdt\right]$$

$$= \frac{1}{2}\left[\frac{t^2}{2}\right]_0^2 = \frac{1}{2}\left(\frac{4}{2}\right) = 1$$

$$a_n = \frac{1}{2}\int_{-2}^{2} f(t)\cos\frac{n\pi t}{2}dt = \frac{1}{2}\left[\int_{-2}^{0} 0dt + \int_{0}^{2} t\cos\frac{n\pi t}{2}dt\right]$$

$$= \frac{1}{2}\left[\frac{2t}{n\pi}\sin\frac{n\pi t}{2} + \frac{4}{(n\pi)^2}\cos\frac{n\pi t}{2}\right]_0^2$$

$$= \frac{1}{2}\left[\frac{4}{n\pi}\underbrace{\sin n\pi}_{=0} + \frac{4}{(n\pi)^2}\underbrace{\cos n\pi}_{=(-1)^n} - \frac{4}{(n\pi)^2}\right]$$

$$= \frac{2}{(n\pi)^2}[(-1)^n - 1]$$

Tabular method:

Differentiate	Integrate
$(+)t$	$\cos\frac{n\pi}{2}t$
$(-)1$	$\frac{2}{n\pi}\sin\frac{n\pi t}{2}$
$(+)0$	$\frac{-4}{(n\pi)^2}\cos\frac{n\pi t}{2}$

$$b_n = \frac{1}{2}\int_{-2}^{2} f(t)\sin\frac{n\pi t}{2}\,dt = \frac{1}{2}\left[\int_{-2}^{0} 0dt + \int_{0}^{2} t\sin\frac{n\pi t}{2}\,dt\right]$$

$$= \frac{1}{2}\left[\frac{-2t}{n\pi}\cos\frac{n\pi t}{2} + \frac{4}{(n\pi)^2}\sin\frac{n\pi t}{2}\right]_0^2$$

$$= \frac{1}{2}\left[\frac{-4}{n\pi}\cos \overset{=(-1)^n}{n\pi} + \frac{4}{(n\pi)^2}\sin \overset{=0}{n\pi} - 0\right]$$

$$= \frac{-2}{n\pi}(-1)^n = \frac{2}{n\pi}(-1)^{n+1}$$

Differentiate	Integrate
$(+)t$	$\sin\frac{n\pi}{2}t$
$(-)1$	$\frac{-2}{n\pi}\cos\frac{n\pi t}{2}$
$(+)0$	$\frac{-4}{(n\pi)^2}\sin\frac{n\pi t}{2}$

- Step 3: Substitute Fourier coefficients a_0, a_n, b_n into Fourier series formula.

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\cos\frac{n\pi t}{L} + b_n\sin\frac{n\pi t}{L}\right]$$

$$f(t) = \frac{1}{2} + \sum_{n=1}^{\infty}\left(\frac{2}{(n\pi)^2}[(-1)^n - 1]\cos\frac{n\pi t}{2} + \frac{2}{n\pi}(-1)^{n+1}\sin\frac{n\pi t}{2}\right)$$

Solution to the Example 4.6

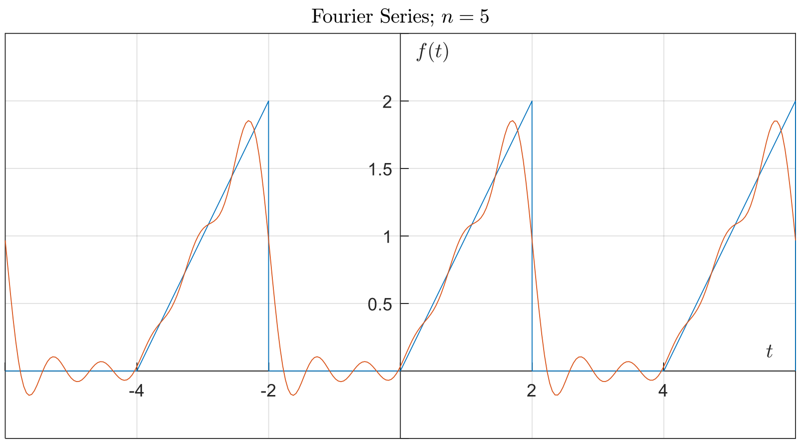
The function, $f(t)$ can be written in series;

$$f(t) = \begin{cases} 0\,; & -2 \le t < 0 \\ t\,; & 0 \le t < 2 \end{cases}, \quad f(t+4) = f(t).$$

$$= \frac{1}{2} + \sum_{n=1}^{\infty} \left(\frac{2}{(n\pi)^2} [(-1)^n - 1] \cos \frac{n\pi t}{2} + \frac{2}{n\pi} (-1)^{n+1} \sin \frac{n\pi t}{2} \right)$$

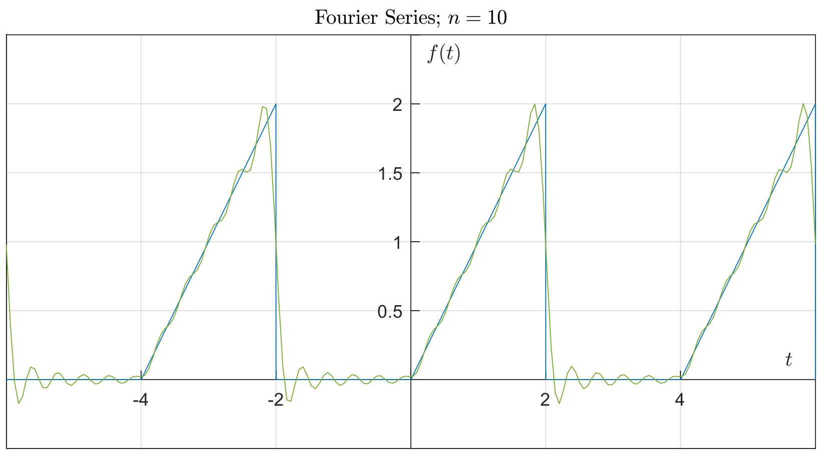
How the Fourier series will look like graphically?

Example 4.6: Graphical Fourier Series for multiple n



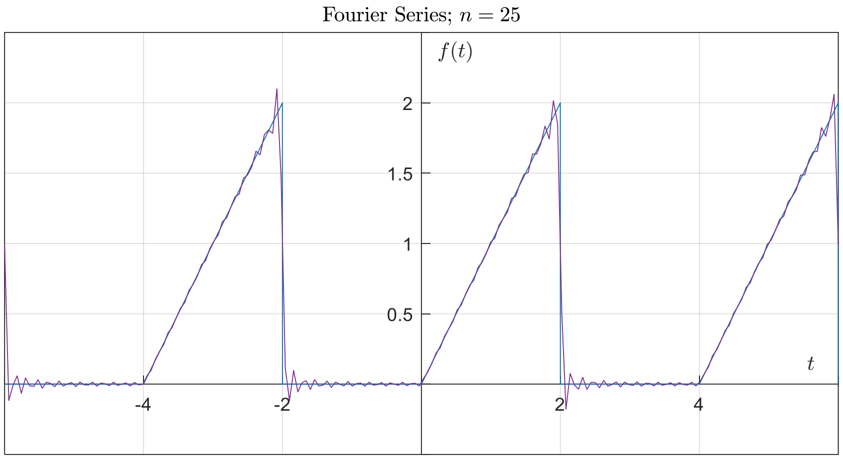
$$f(t) = \frac{1}{2} + \sum_{n=1}^{5} \left(\frac{2}{(n\pi)^2} [(-1)^n - 1] \cos \frac{n\pi t}{2} + \frac{2}{n\pi} (-1)^{n+1} \sin \frac{n\pi t}{2} \right)$$

Example 4.6: Graphical Fourier Series for multiple n



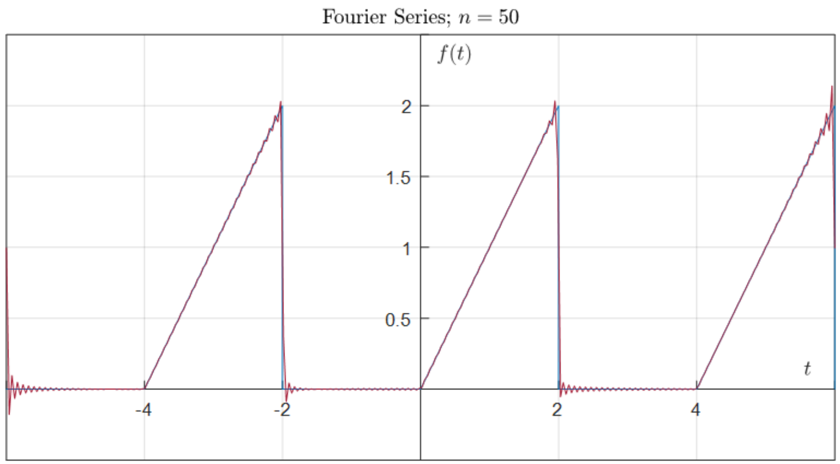
$$f(t) = \frac{1}{2} + \sum_{n=1}^{10}\left(\frac{2}{(n\pi)^2}[(-1)^n - 1]\cos\frac{n\pi t}{2} + \frac{2}{n\pi}(-1)^{n+1}\sin\frac{n\pi t}{2}\right)$$

Example 4.6: Graphical Fourier Series for multiple n



$$f(t) = \frac{1}{2} + \sum_{n=1}^{25} \left(\frac{2}{(n\pi)^2} [(-1)^n - 1] \cos \frac{n\pi t}{2} + \frac{2}{n\pi} (-1)^{n+1} \sin \frac{n\pi t}{2} \right)$$

Example 4.6: Graphical Fourier Series for multiple n



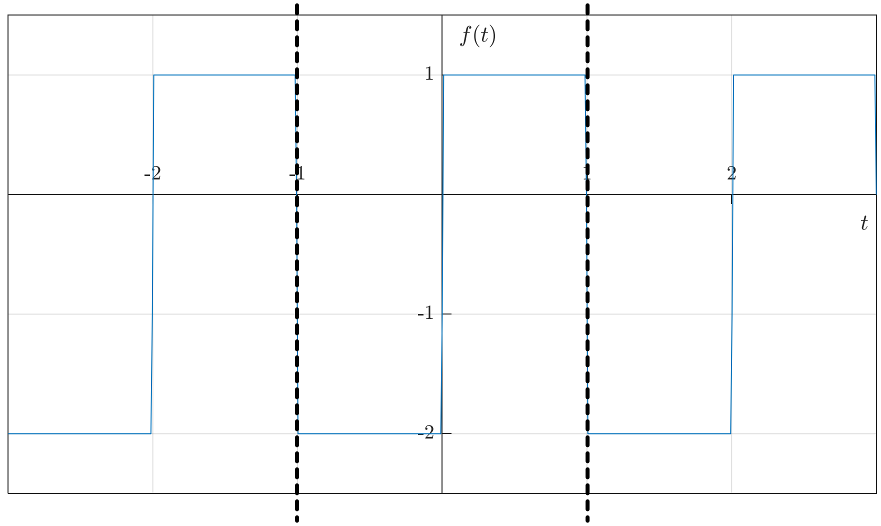
$$f(t) = \frac{1}{2} + \sum_{n=1}^{50} \left(\frac{2}{(n\pi)^2} [(-1)^n - 1] \cos \frac{n\pi t}{2} + \frac{2}{n\pi} (-1)^{n+1} \sin \frac{n\pi t}{2} \right)$$

Example 4.7

Find the Fourier Series of the following function;

$$f(t) = \begin{cases} -2\,; & -1 \le t < 0 \\ 1\,; & 0 \le t < 1 \end{cases}, \quad f(t+2) = f(t).$$

Solution:



The function is neither even nor odd function.

Step 1: Based on the function and sketch, $L = 1$.

Step 2: Calculate the Fourier coefficients, a_0, a_n, b_n

$$a_0 = \frac{1}{1}\int_{-1}^{1} f(t)\,dt = \left[\int_{-1}^{0} -2dt + \int_{0}^{1} 1dt\right]$$

$$= \left([-2t]_{-1}^{0} + [t]_{0}^{1}\right) = -2 + 1 = -1$$

$$a_n = \frac{1}{1}\int_{-1}^{1} f(t)\cos\frac{n\pi t}{1}\,dt = \left[\int_{-1}^{0} -2\cos n\pi t\,dt + \int_{0}^{1} \cos n\pi t\,dt\right]$$

$$= \left[-\frac{2}{n\pi}\sin n\pi t\right]_{-1}^{0} + \left[\frac{1}{n\pi}\sin n\pi t\right]_{0}^{1}$$

$$= \left[-\frac{2}{n\pi}\overset{=0}{\sin(-n\pi)} + \frac{1}{n\pi}\overset{=0}{\sin n\pi}\right] = 0$$

$$b_n = \frac{1}{1}\int_{-1}^{1} f(t)\sin\frac{n\pi t}{1}\,dt = \left[\int_{-1}^{0} -2\sin n\pi t\,dt + \int_{0}^{1} \sin n\pi t\,dt\right]$$

$$= \left[\frac{2}{n\pi}\cos n\pi t\right]_{-1}^{0} + \left[\frac{-1}{n\pi}\cos n\pi t\right]_{0}^{1}$$

$$= \left[\frac{2}{n\pi}\left(\cos 0 - \overset{=(-1)^n}{\cos(-n\pi)}\right) - \frac{1}{n\pi}\left(\overset{=(-1)^n}{\cos n\pi} - \cos 0\right)\right]$$

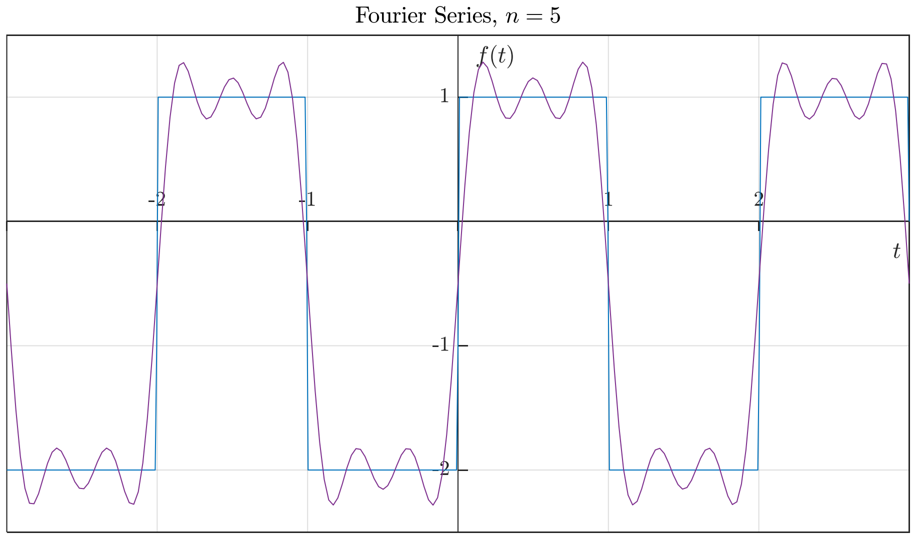
$$= \frac{2}{n\pi}\left(1+(-1)^{n+1}\right) + \frac{1}{n\pi}\left((-1)^{n+1}+1\right) = \frac{3}{n\pi}\left(1+(-1)^{n+1}\right)$$

- Step 3: Substitute Fourier coefficients a_0, a_n, b_n into Fourier series formula.

$$\boxed{f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\cos\frac{n\pi t}{L} + b_n\sin\frac{n\pi t}{L}\right]}$$

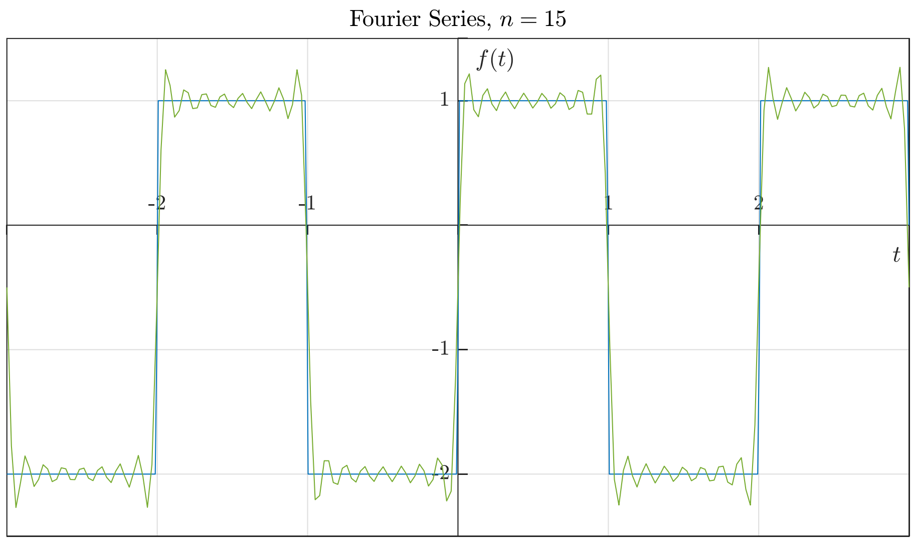
$$f(t) = -\frac{1}{2} + \sum_{n=1}^{\infty}\left(\frac{3}{n\pi}\left(1+(-1)^{n+1}\right)\right)\sin n\pi t$$

Example 4.7: Graphical Fourier Series for multiple n



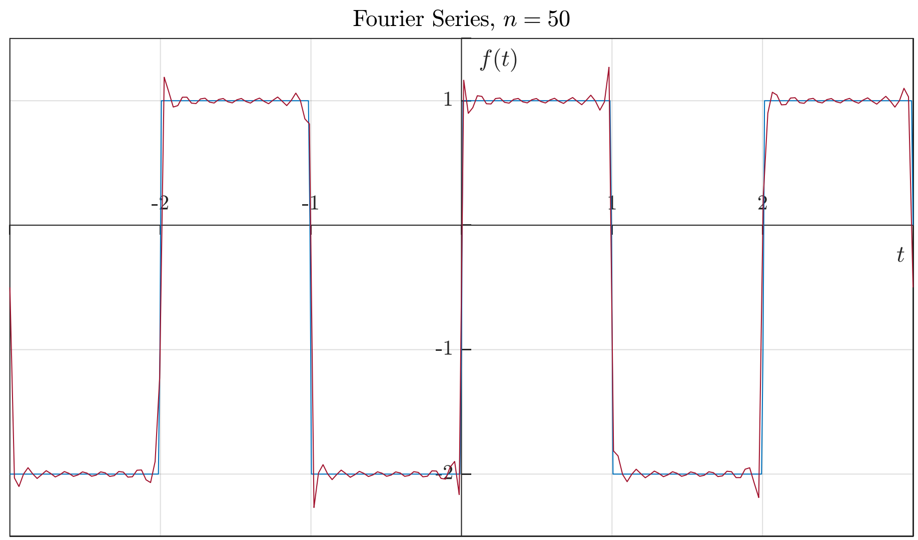
$$f(t) = -\frac{1}{2} + \sum_{n=1}^{5} \left(\frac{3}{n\pi} \left(1 + (-1)^{n+1}\right) \right) \sin n\pi t$$

Example 4.7: Graphical Fourier Series for multiple n



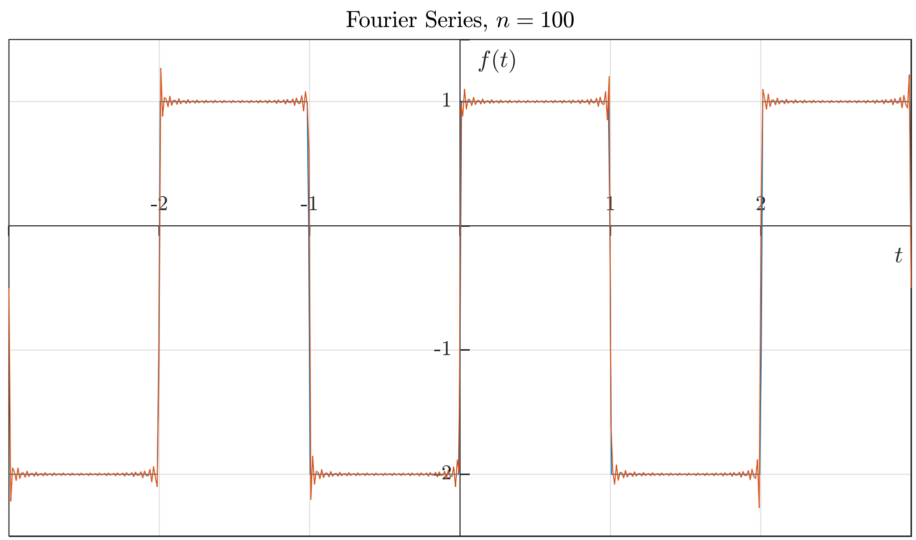
$$f(t) = -\frac{1}{2} + \sum_{n=1}^{15} \left(\frac{3}{n\pi} \left(1 + (-1)^{n+1}\right) \right) \sin n\pi t$$

Example 4.7: Graphical Fourier Series for multiple n



$$f(t) = -\frac{1}{2} + \sum_{n=1}^{50} \left(\frac{3}{n\pi} \left(1 + (-1)^{n+1}\right) \right) \sin n\pi t$$

Example 4.7: Graphical Fourier Series for multiple n



$$f(t) = -\frac{1}{2} + \sum_{n=1}^{100} \left(\frac{3}{n\pi} (1 + (-1)^{n+1}) \right) \sin n\pi t$$

Exercise 4.3

Sketch and find the Fourier series of the following function;

$$f(t) = 1 - t^2; \quad -1 \leq t \leq 1$$
$$f(t+2) = f(t)$$

Answers:

Step 1: Determine the parameter L: $L = 1$.

Step 2: Calculate the Fourier coefficients, a_0, a_n, b_n

$$a_0 = \frac{4}{3}; \quad a_n = \frac{4}{(n\pi)^2}(-1)^{n+1}; \quad b_n = 0$$

Step 3: Substitute Fourier coefficients a_0, a_n, b_n into the Fourier series formula.

$$f(t) = \frac{2}{3} + \sum_{n=1}^{\infty} \frac{4}{(n\pi)^2}(-1)^{n+1} \cos n\pi t$$

Exercise 4.4

1. Given a function

$$f(x) = \begin{cases} x; & -\pi \le x \le 0 \\ 0; & 0 \le x \le \pi \end{cases};$$

$$f(x + 2\pi) = f(x)$$

   a) Sketch the given function.

   b) Determine the Fourier series of the function.

2. Find the Fourier series of the following functions

   a) $f(x) = \begin{cases} 0; & -\frac{1}{2} \le x \le 0 \\ 2; & 0 \le x \le \frac{1}{2} \end{cases}; \quad f(x+1) = f(x)$

   b) $f(t) = \begin{cases} 1 + t; & -1 \le t \le 0 \\ 1; & 0 \le t \le 1 \end{cases}; \quad f(t+2) = f(t)$

Answer:

1. b) $f(x) = -\frac{\pi}{4} + \frac{1}{\pi}\sum_{n=1}^{\infty}\left[\frac{1}{n^2\pi}(1 + (-1)^{n+1})\cos nx + \frac{(-1)^{n+1}}{n}\sin nx\right]$

2. a) $f(x) = 1 + \sum_{n=1}^{\infty}\frac{2}{n\pi}[(-1)^{n+1} + 1]\sin 2n\pi x$

   b) $f(t) = \frac{3}{4} + \sum_{n=1}^{\infty}\frac{1}{(n\pi)^2}[1 + (-1)^{n+1}]\cos n\pi t + \frac{(-1)^{n+1}}{n\pi}\sin n\pi t$

References

Edwards C. H., Penny D.E. & Calvis D. (2016). Differential Equations and Boundary Value Problems, 5thEdition. Pearson Education Inc..

Zill, D. G. (2017). Differential Equations with Boundary-Value Problems, 9thEdition. Cengage Learning Inc

Thank You

Questions & Answer?