

# MECHANICAL VIBRATION

## BMCG 3233

# CHAPTER 5: MULTI-DEGREE OF FREEDOM SYSTEM

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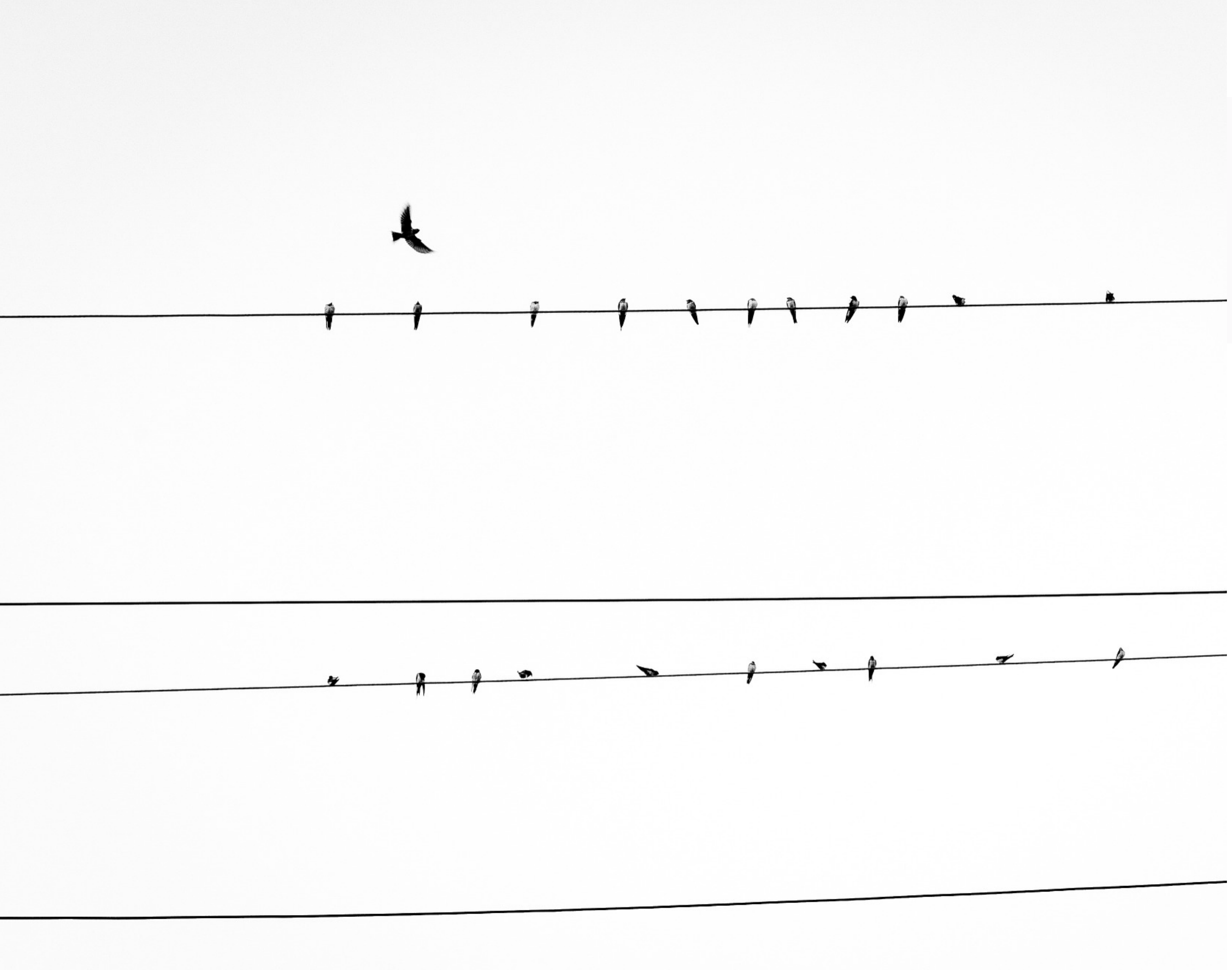


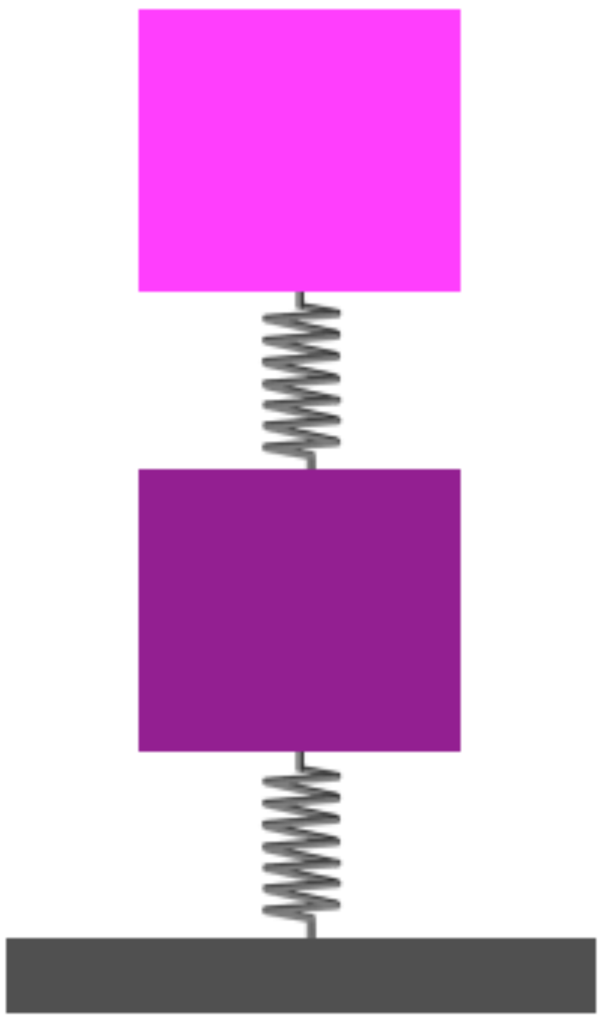
# CONTENTS



# LEARNING OBJECTIVES

1. Derive the equation of motion
2. Calculate the natural frequencies and mode shape functions

System with one or two degrees of freedom is still relatively easy for analysis.



(Discrete system)

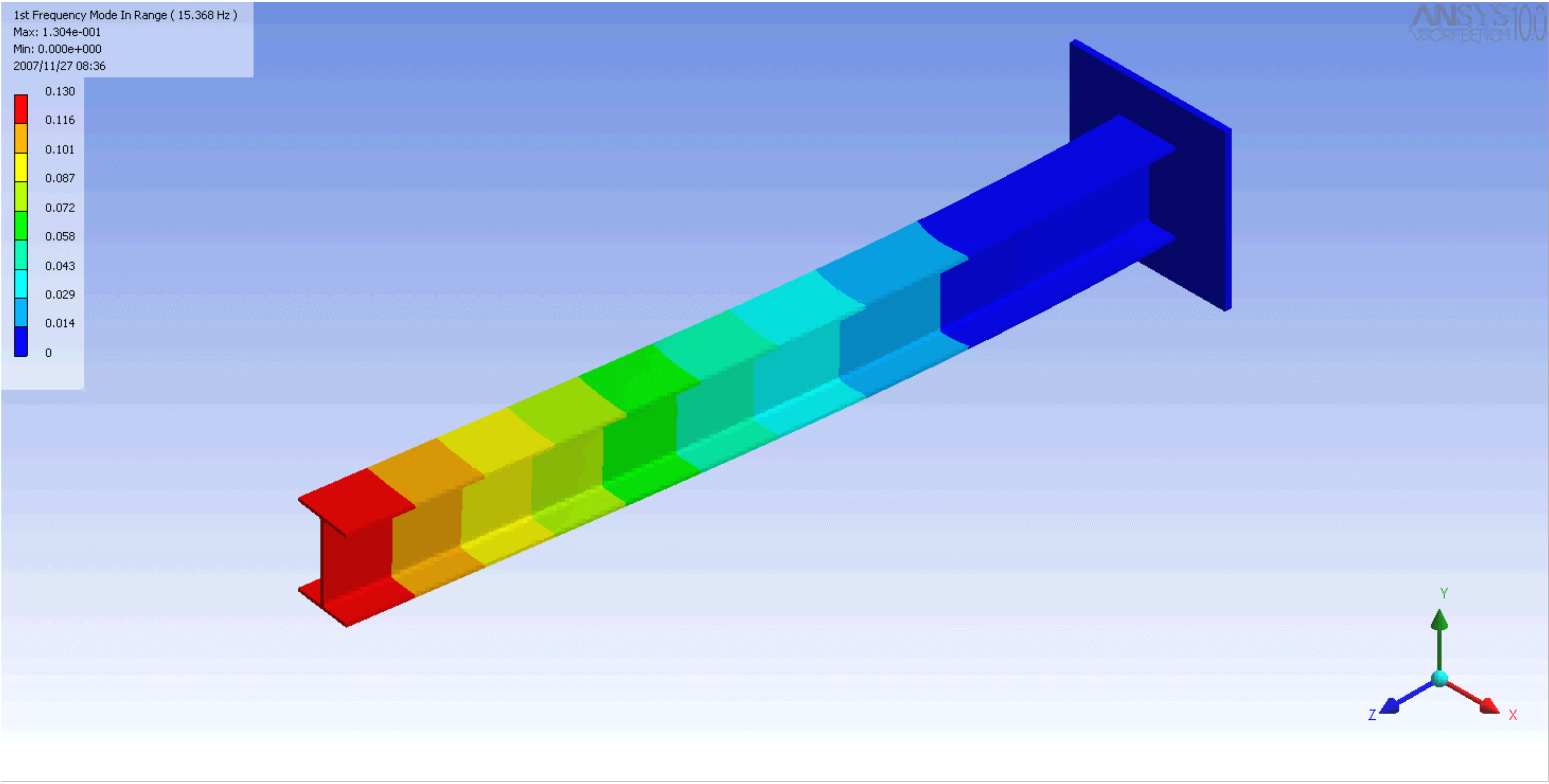
Two degrees of freedom system:

- Two equations of motion
- Two natural frequencies
- Two modes of vibration
- Two resonances

# A more complex system with infinite degrees of freedom:

## Finite Element Analysis

## (Continuous system)

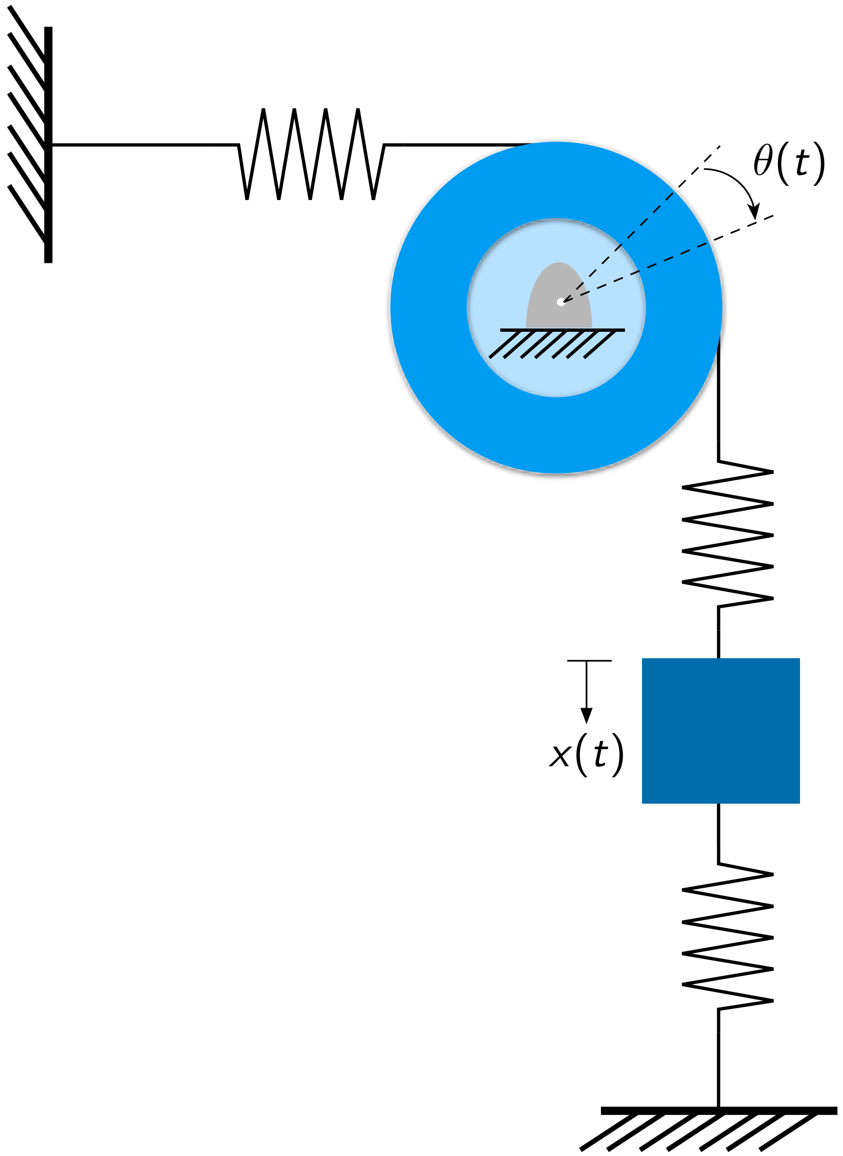


Degree of Freedom (DOF):

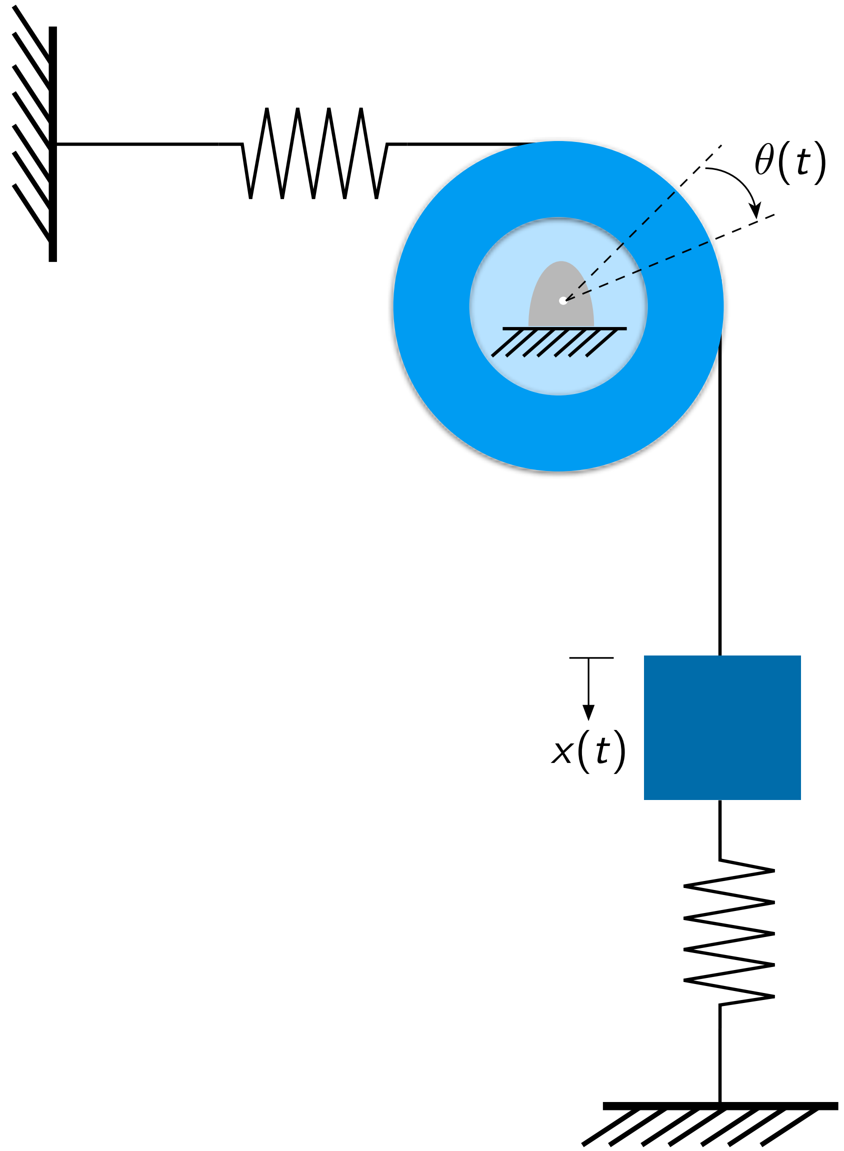
"Minimum number of independent coordinates to specify the motion of a system"

## How many DOFs?

### A.



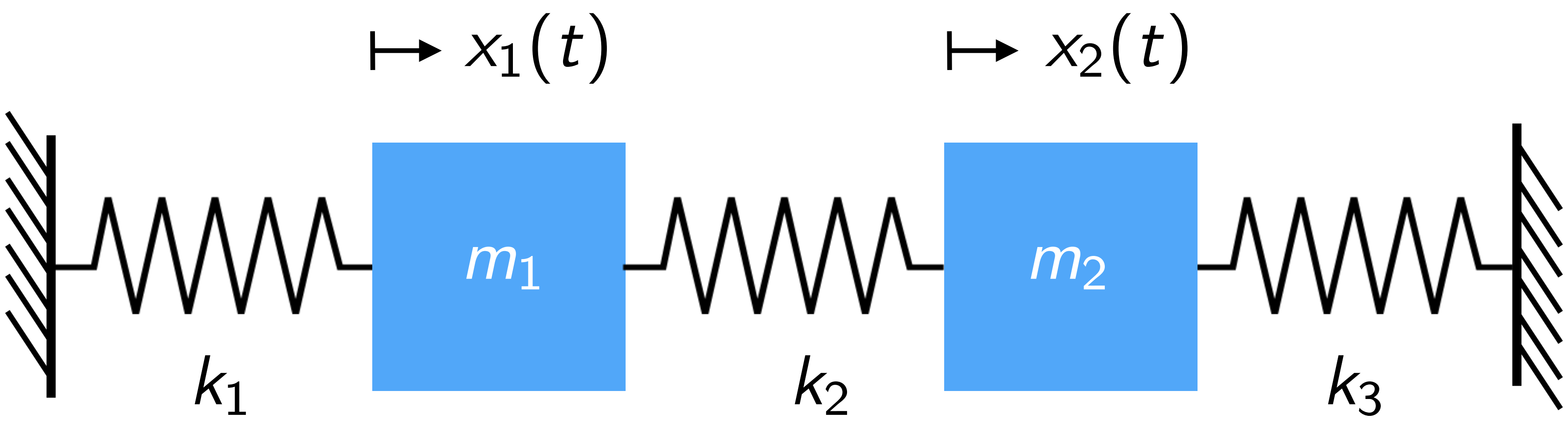

### B.



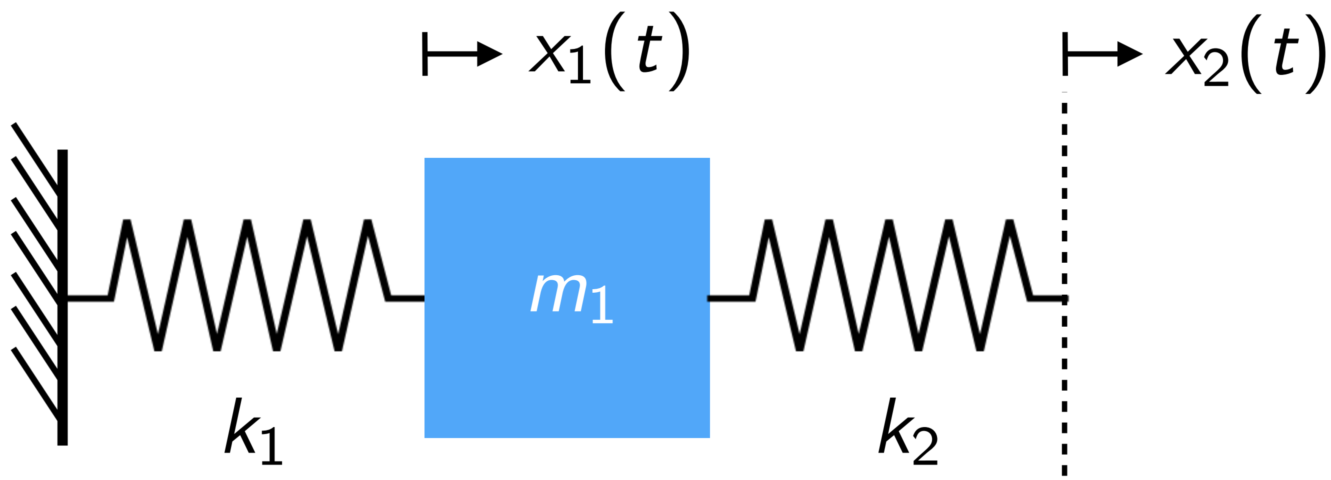
# 5.1

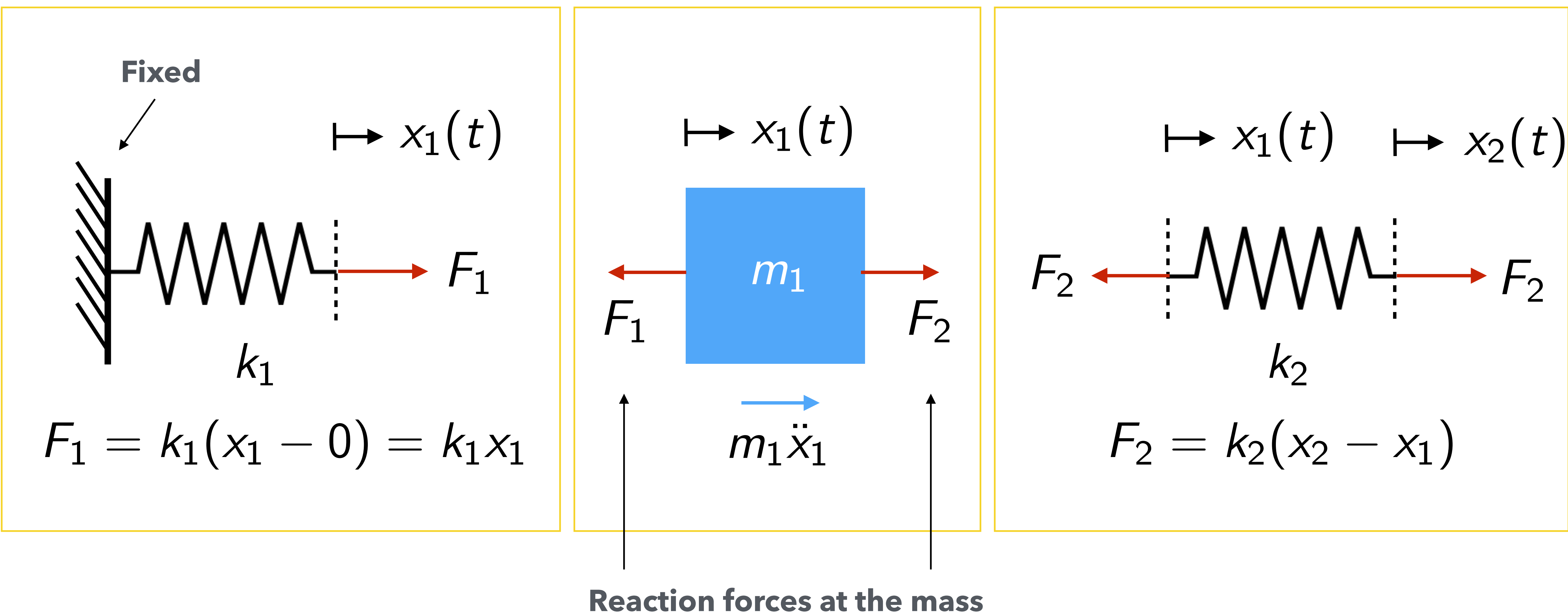
# DERIVATION OF EQUATION OF MOTION

**Derive the equations of motion from the following system.**



## 1. For mass $m_1$




$$F_1 = k_1(x_1 - 0) = k_1 x_1$$

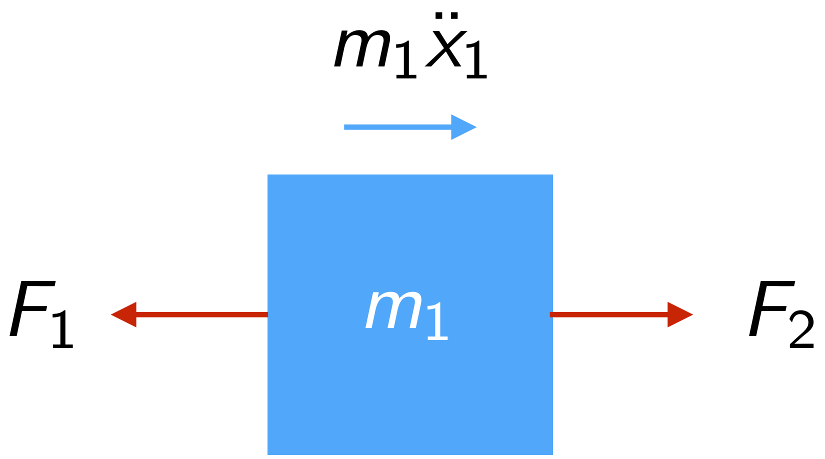
$$F_2 = k_2(x_2 - x_1)$$

## The Newton 2nd Law (at mass $m_1$ )

$$\sum F = m_1\ddot{x}_1$$

$$-F_1 + F_2 = m_1\ddot{x}_1$$

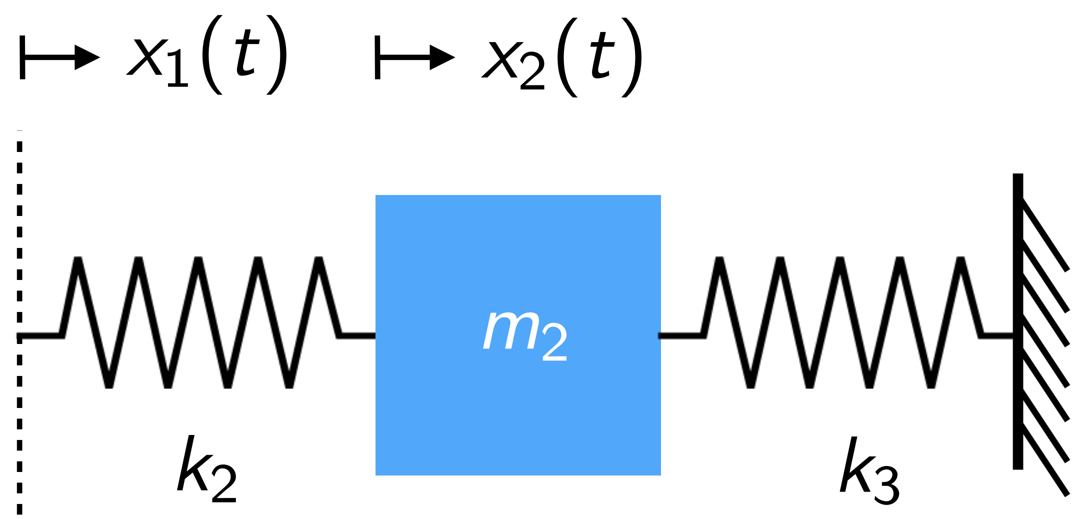
$$-k_1x_1 + k_2(x_2 - x_1) = m_1\ddot{x}_1$$

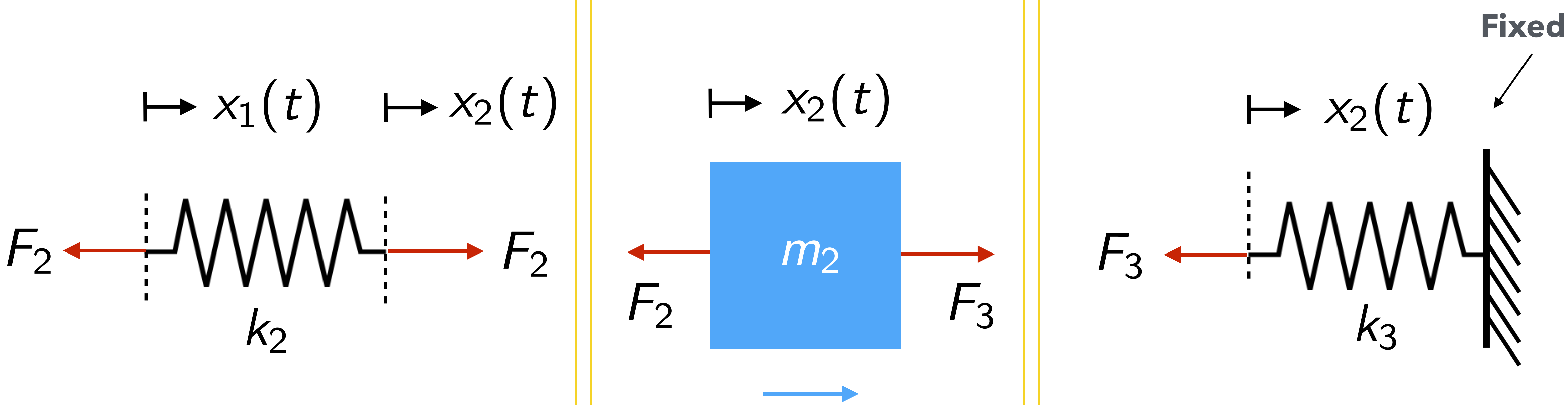



**After rearrangement, we obtain the 1st EOM:**

$$\boxed{m_1\ddot{x}_1 + (k_1 + k_2)x_1 - k_2x_2 = 0}$$

## 2. For mass $m_2$




$$F_2 = k_2(x_2 - x_1)$$

$$m_2\ddot{x}_2$$

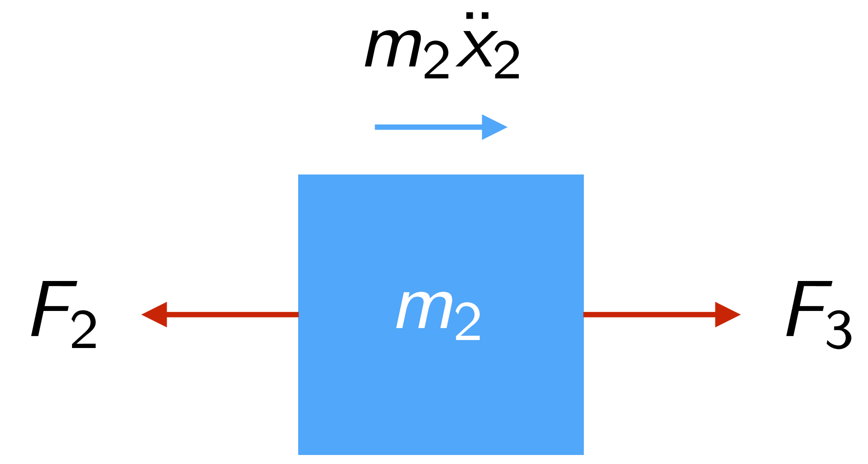
$$F_3 = k_3(0 - x_2) = -k_3x_2$$

## The Newton 2nd Law (at mass $m_2$ )

$$\sum F = m_2 \ddot{x}_2$$

$$-F_2 + F_3 = m_2 \ddot{x}_2$$

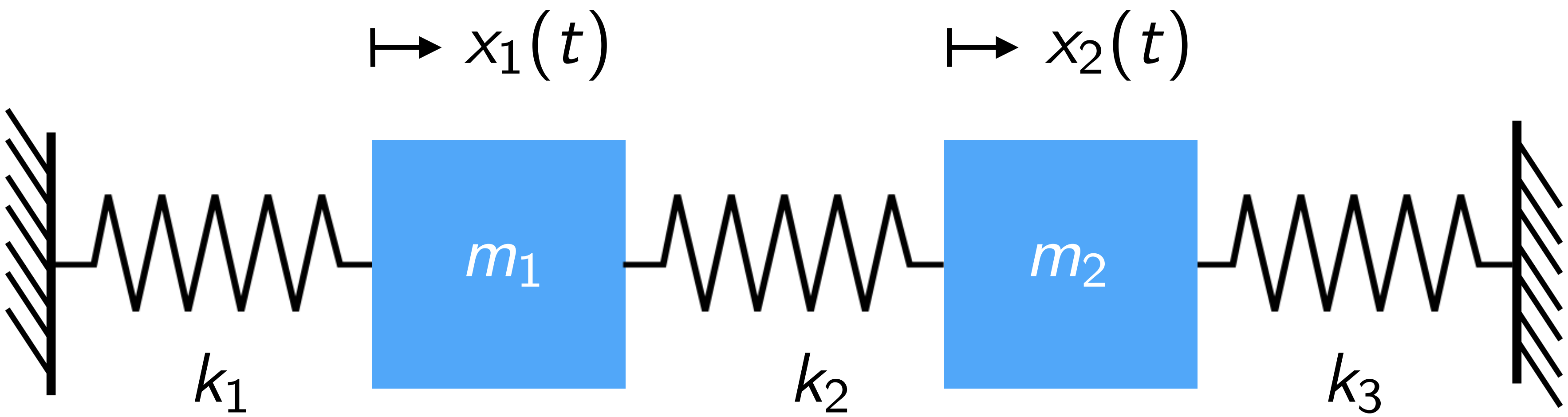
$$-k_2(x_2 - x_1) - k_3 x_2 = m_2 \ddot{x}_2$$




**After rearrangement, we obtain the 2nd EOM:**

$$\boxed{m_2 \ddot{x}_2 - k_2 x_1 + (k_2 + k_3) x_2 = 0}$$

## 2-DOF system:




## Two equations of motion:

$$m_1\ddot{x}_1 + (k_1 + k_2)x_1 - k_2x_2 = 0$$

$$m_2\ddot{x}_2 - k_2x_1 + (k_2 + k_3)x_2 = 0$$

## Watch the video: “MDOF Deriving the Equation of Motion (A quick way)”

Scan this QR code

Or click/tap here.

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{x}_i}\right) - \frac{\partial T}{\partial x_i} + \frac{\partial V}{\partial x_i} = 0\,; \quad i = 1, 2, \cdots$$

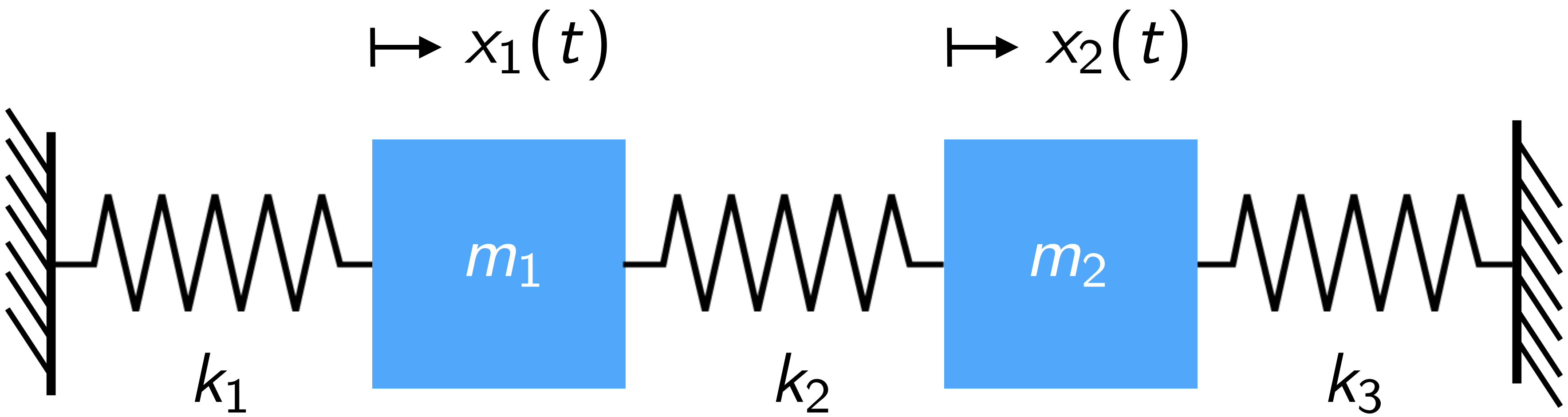
$x$ : **Independent coordinates**

$T$ : **Kinetic energy**

$V$ : **Potential energy**

Joseph-Louis de Lagrange
(1736-1813)

## Derive the equations of motion using the **Lagrange's equation**.




**Total Kinetic Energy:**

$$\boxed{T = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2}$$

**Total Potential Energy:**

$$\boxed{V = \frac{1}{2}k_1x_1^2 + \frac{1}{2}k_2(x_1 - x_2)^2 + \frac{1}{2}k_3x_2^2}$$

**For coordinate $x_1$ :**

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{x}_1}\right) = \frac{\mathrm{d}}{\mathrm{d}t}(m_1\dot{x}_1) = m_1\ddot{x}_1$$

$$\frac{\partial T}{\partial x_1} = 0$$

$$\frac{\partial V}{\partial x_1} = k_1 x_1 + k_2(x_1 - x_2)$$

**From Lagrange's equation:**

$$\boxed{m_1\ddot{x}_1 + (k_1 + k_2)x_1 - k_2 x_2 = 0}$$

**For coordinate $x_2$ :**

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial T}{\partial \dot{x}_1}\right) = \frac{\mathrm{d}}{\mathrm{d}t}(m_2\dot{x}_2) = m_2\ddot{x}_2$$

$$\frac{\partial T}{\partial x_2} = 0$$

$$\frac{\partial V}{\partial x_2} = k_2(x_1 - x_2)(-1) + k_3 x_2$$

**From Lagrange's equation:**

$$\boxed{m_2\ddot{x}_2 + (k_2 + k_3)x_2 - k_2 x_1 = 0}$$

# 5.2

# MATRIX OPERATION

**Two equations of motion:**

$$m_1\ddot{x}_1 + (k_1 + k_2)x_1 - k_2x_2 = 0$$

$$m_2\ddot{x}_2 - k_2x_1 + (k_2 + k_3)x_2 = 0$$

**Matrix form**

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = 0$$

$$\overset{\underline{\mathbf{M}}}{\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}} \overset{\ddot{\tilde{x}}}{\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix}} + \overset{\underline{\mathbf{K}}}{\begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 + k_3 \end{bmatrix}} \overset{\tilde{x}}{\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}} = 0$$

**General form: For $n$ degree-of-freedom system**

Stiffness matrix

$n \times n$

Mass matrix

$n \times n$

$$\boxed{\underline{\mathbf{M}}\ddot{\tilde{x}} + \underline{\mathbf{K}}\tilde{x} = 0}$$

Column vector

$n \times 1$

$$\underline{\mathbf{M}}\ddot{\widetilde{x}} + \underline{\mathbf{K}}\widetilde{x} = 0$$

**For harmonic motion:** $\widetilde{x} = \widetilde{X}e^{j\omega t}$

**yields:**

$$\left[\underline{\mathbf{K}} - \omega^2\underline{\mathbf{M}}\right]\widetilde{X} = 0$$

**Cannot be ZERO (vibration exists)**

**To obtain the solution:**

$$\det\left(\left[\underline{\mathbf{K}} - \omega^2\underline{\mathbf{M}}\right]\right) = 0$$

**Determinant**

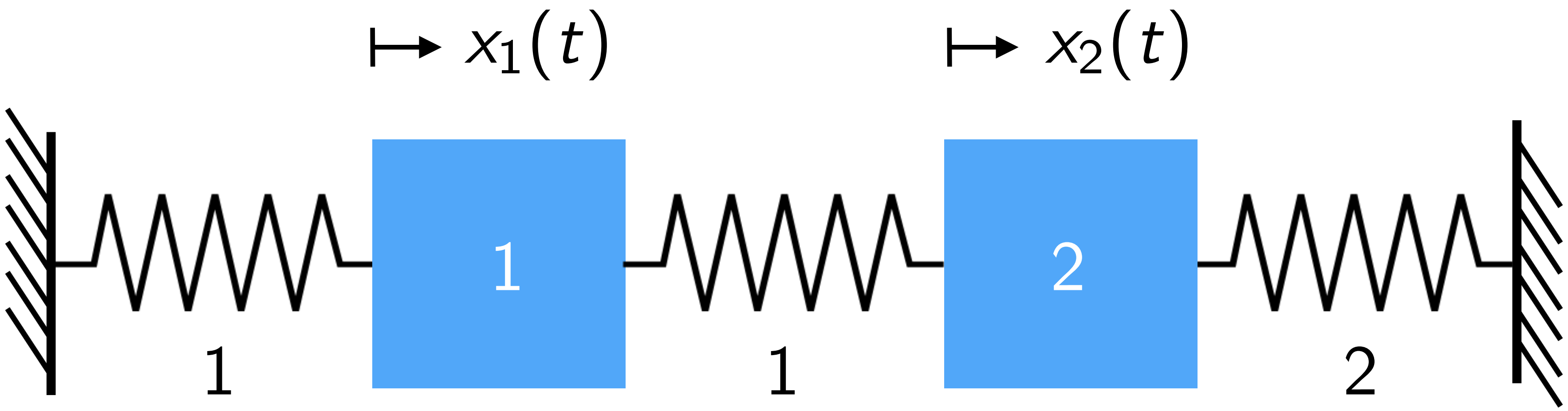
$$\det\left(\left[\underline{\mathbf{K}} - \omega^2 \underline{\mathbf{M}}\right]\right) = 0$$

**Only TRUE for certain values of $\omega$, namely the EIGENVALUES, which are the natural frequencies of the system.**

**For $n$ degrees of freedom**

$$\omega_i \quad \text{for} \quad i = 1, 2, 3, \cdots, n$$

# Example 5.1




**Equations of motion (in matrix):**

$$\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}}_{\underline{\mathbf{M}}} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \underbrace{\begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix}}_{\underline{\mathbf{K}}} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = 0$$

**By substituting** $\widetilde{x} = \widetilde{X} e^{j\omega t}$

$$\left[\underline{\mathbf{K}} - \omega^2 \underline{\mathbf{M}}\right] \widetilde{X} = 0 \qquad (\bigstar)$$

**Solution:**

$$\det\left[\underline{\mathbf{K}} - \omega^2 \underline{\mathbf{M}}\right] = 0$$

$$\det\left(\begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix} - \omega^2 \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\right) = 0$$

$$\det\left(\begin{bmatrix} 2-\omega^2 & -1 \\ -1 & 3-2\omega^2 \end{bmatrix}\right) = 0$$

$$\det\left(\begin{bmatrix} 2-\omega^2 & -1 \\ -1 & 3-2\omega^2 \end{bmatrix}\right) = 0$$

$\ominus$ $\oplus$

$$(2-\omega^2)(3-2\omega^2) - 1 = 0$$

$$2\omega^4 - 7\omega^2 + 5 = 0$$

**If** $\omega_s = \omega^2$ **, then**

$$2\omega_s^2 - 7\omega_s + 5 = 0$$

$$(2\omega_s - 5)(\omega_s - 1) = 0$$

$$\omega_s = \tfrac{5}{2} \quad \textbf{and} \quad \omega_s = 1$$

$$\longrightarrow \quad \omega = \sqrt{\tfrac{5}{2}} \quad \textbf{and} \quad \omega = 1$$

**The system has the 1st natural frequency at** $\omega_1 = 1 \text{ rad/s}$

**and the 2nd natural frequency at** $\omega_2 = \sqrt{\tfrac{5}{2}} \text{ rad/s}$

# 5.3

# MODE SHAPE

# MODE SHAPE

"**HOW** does the system vibrate at each natural frequency?"

**Back to (★) to find relationship between the coordinates:**

$$\begin{bmatrix} 2-\omega^2 & -1 \\ -1 & 3-2\omega^2 \end{bmatrix} \begin{Bmatrix} X_1 \\ X_2 \end{Bmatrix} = 0$$

**We can use any equation of motion from the two EoM:**

$$(2-\omega^2)X_1 - X_2 = 0$$

**For** $\omega_1 = 1 \text{ rad/s}: \rightarrow$

$$(2-\omega^2)X_1 - X_2 = 0$$
$$(2-1)X_1 - X_2 = 0$$
$$X_1 = X_2$$

**For** $\omega_2 = \sqrt{\frac{5}{2}} \text{ rad/s}: \rightarrow$

$$(2-\omega^2)X_1 - X_2 = 0$$
$$(2-\tfrac{5}{2})X_1 - X_2 = 0$$
$$-\tfrac{1}{2}X_1 = X_2$$

**The relative amplitude at each natural frequency:**

**At** $\omega_1 = 1 \text{ rad/s}:$ $\rightarrow$ $X_1 = X_2$

**If** $X_1 = 1$**, then** $X_2 = 1$

**Normalised mode shape function:**

$$\boxed{\tilde{\phi}_1 = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}}$$

**At** $\omega_2 = \sqrt{\frac{5}{2}} \text{ rad/s}:$ $\rightarrow$ $-\frac{1}{2}X_1 = X_2$

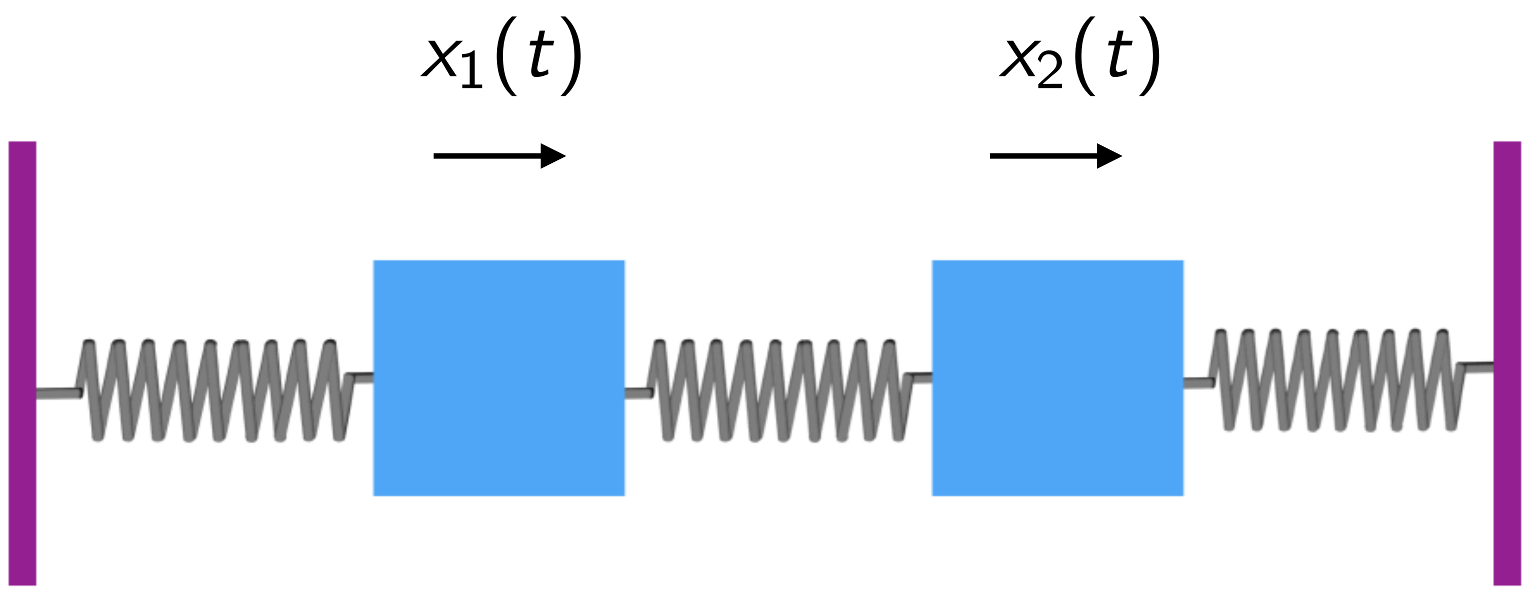
**If** $X_1 = 1$**, then** $X_2 = -\frac{1}{2}$

**Normalised mode shape function:**

$$\boxed{\tilde{\phi}_2 = \begin{Bmatrix} 1 \\ -0.5 \end{Bmatrix}}$$

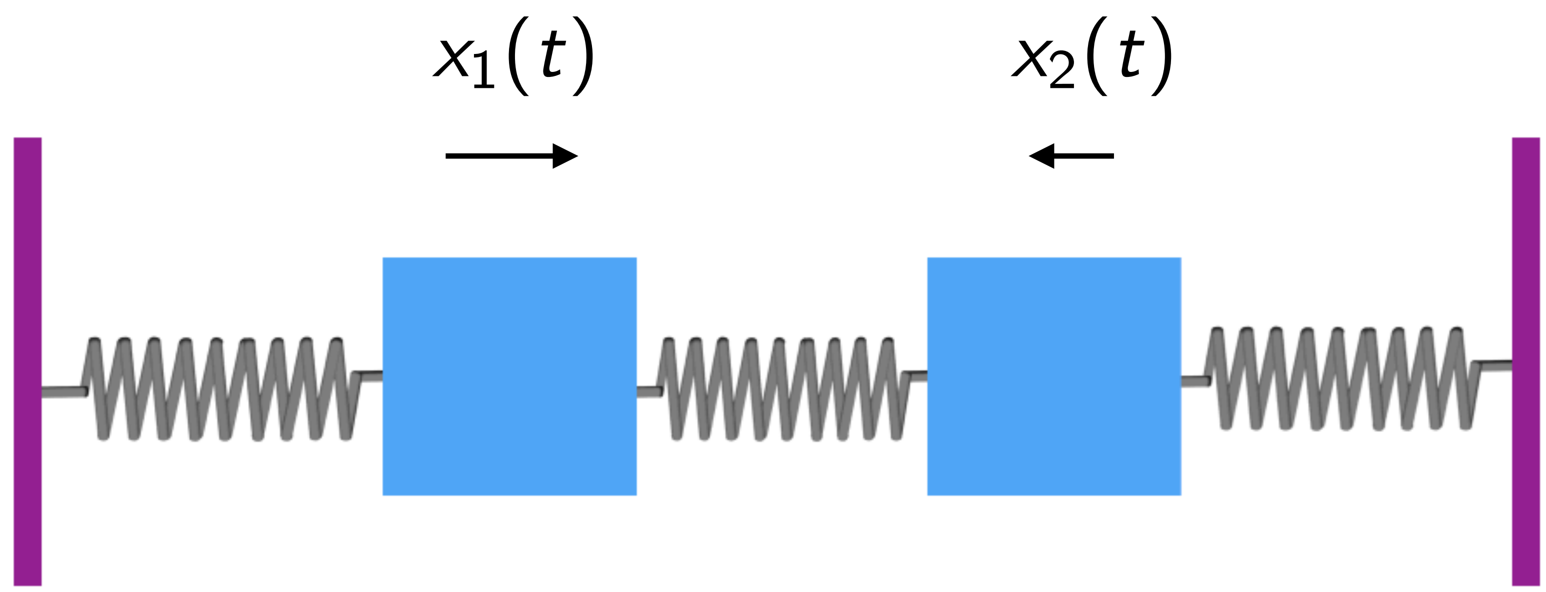
It explains the **behaviour** of vibration of the system

1st mode (at 1st natural frequency)




$$\tilde{\phi}_1 = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$ : $m_1$ and $m_2$ move with the same amplitude and in-phase.

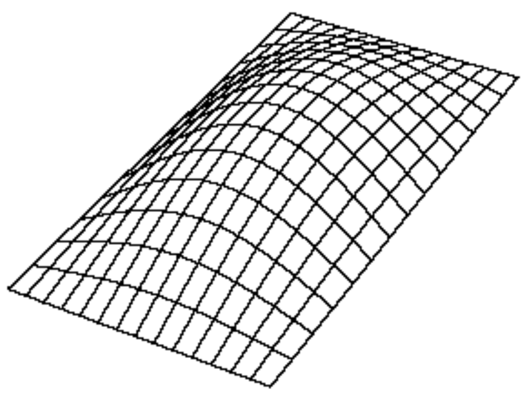
2nd mode(at 2nd natural frequency)



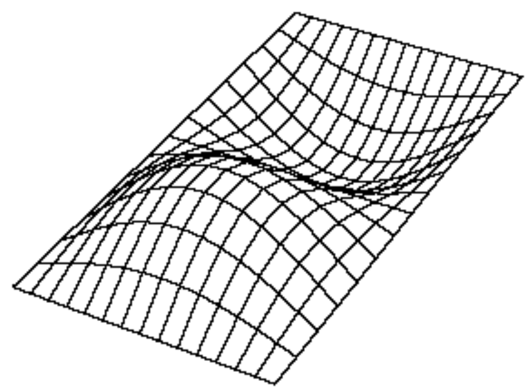

$$\tilde{\phi}_2 = \begin{Bmatrix} 1 \\ -0.5 \end{Bmatrix}$$ : $m_1$ and $m_2$ move in opposite direction (out-of-phase) with amplitude of $m_1$ two times the amplitude of $m_2$

## Continuous structures have infinite number of modes

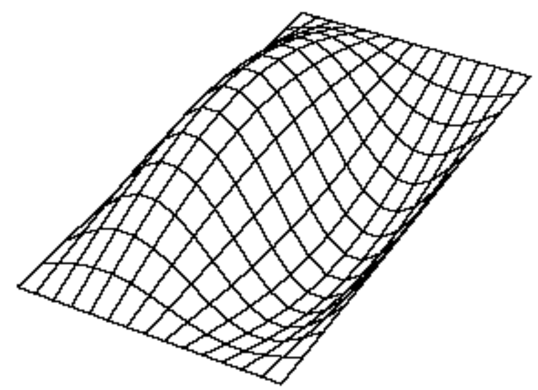
### Rectangular plate with pinned edges



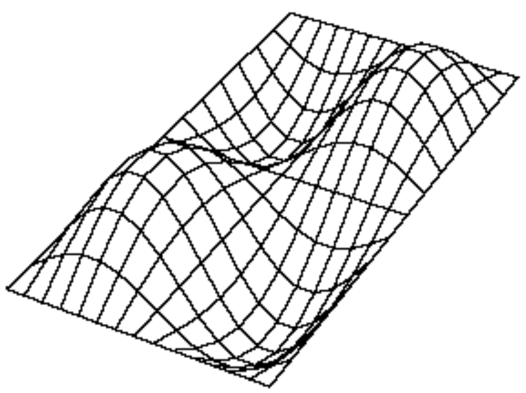
1st mode



2nd mode



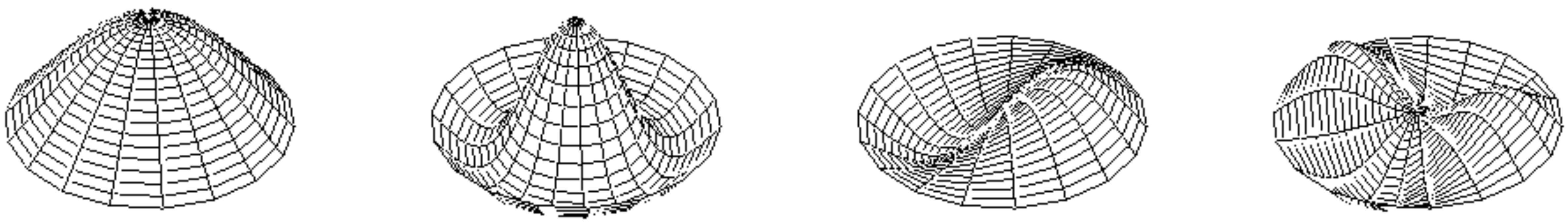
3rd mode



4th mode

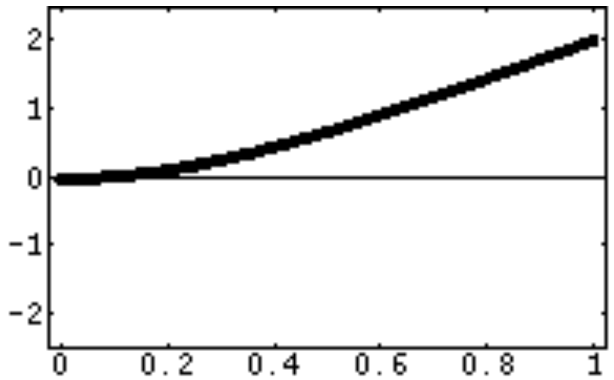
Animation by Dr. Dan Russell, Pen State Univ.

## Membrane of a drum

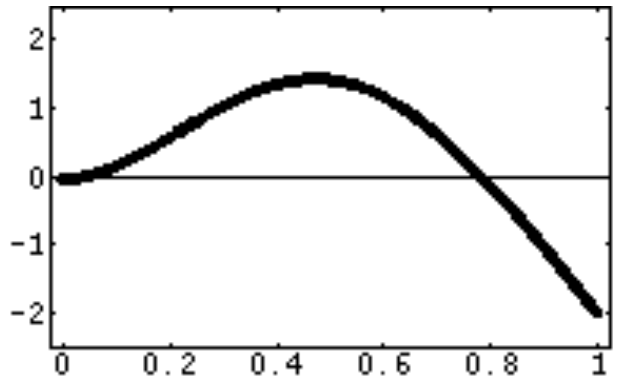


Animation by Dr. Dan Russell, Pen State Univ.

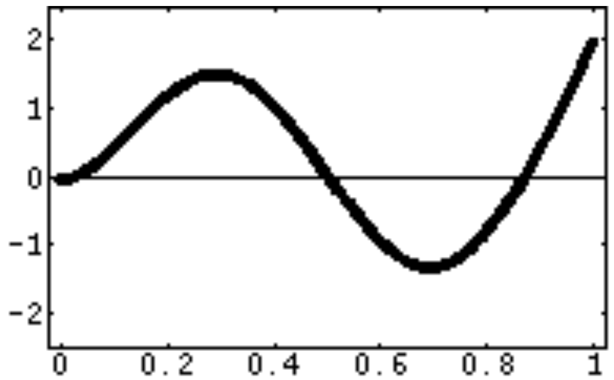
## Fixed-free beam


**1st mode**




**2nd mode**




**3rd mode**

Animation by Dr. Dan Russell, Pen State Univ.

# MODAL MATRIX

$$\underline{\mathbf{B}} = [\tilde{\phi}_1 \quad \tilde{\phi}_2]$$

$\tilde{\phi}_1$ **and** $\tilde{\phi}_2$ **are also called the EIGENVECTORS**

$$\underline{\mathbf{B}} = \begin{bmatrix} 1 & 1 \\ 1 & -0.5 \end{bmatrix}$$

General free vibration is a **superposition of modal components** whose amplitudes are determined by initial conditions.

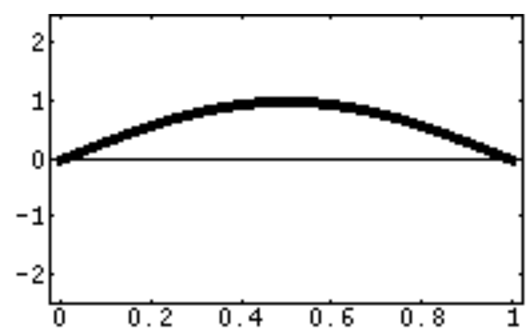
We can express the response in terms of modal matrix:

$$\boxed{\underset{\sim}{x}(t) = \underline{\mathbf{B}}\,\underset{\sim}{u}(t)}$$

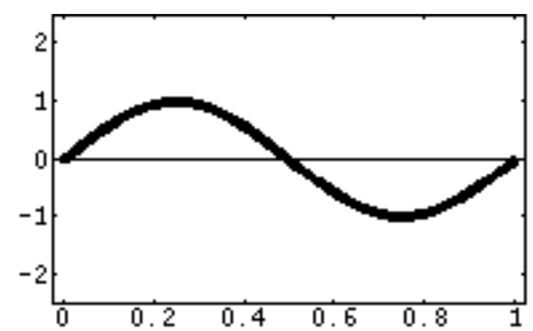
$$\underline{\mathbf{B}} = [\underset{\sim}{\phi}_1 \quad \underset{\sim}{\phi}_2 \quad \underset{\sim}{\phi}_3 \quad \cdots \quad \underset{\sim}{\phi}_N]$$

$$\underset{\sim}{u}(t) = [U_1 \quad U_2 \quad U_3 \quad \cdots \quad U_N]^T e^{j\omega t}$$

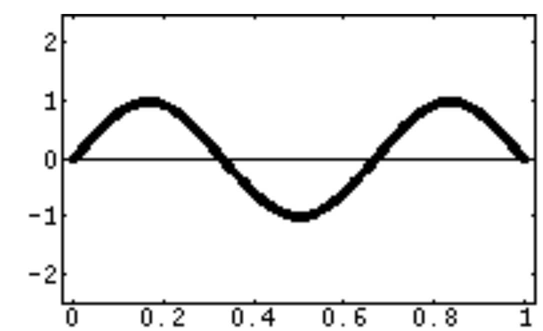
## Example: Simply supported beam


$m = 1$




$m = 2$



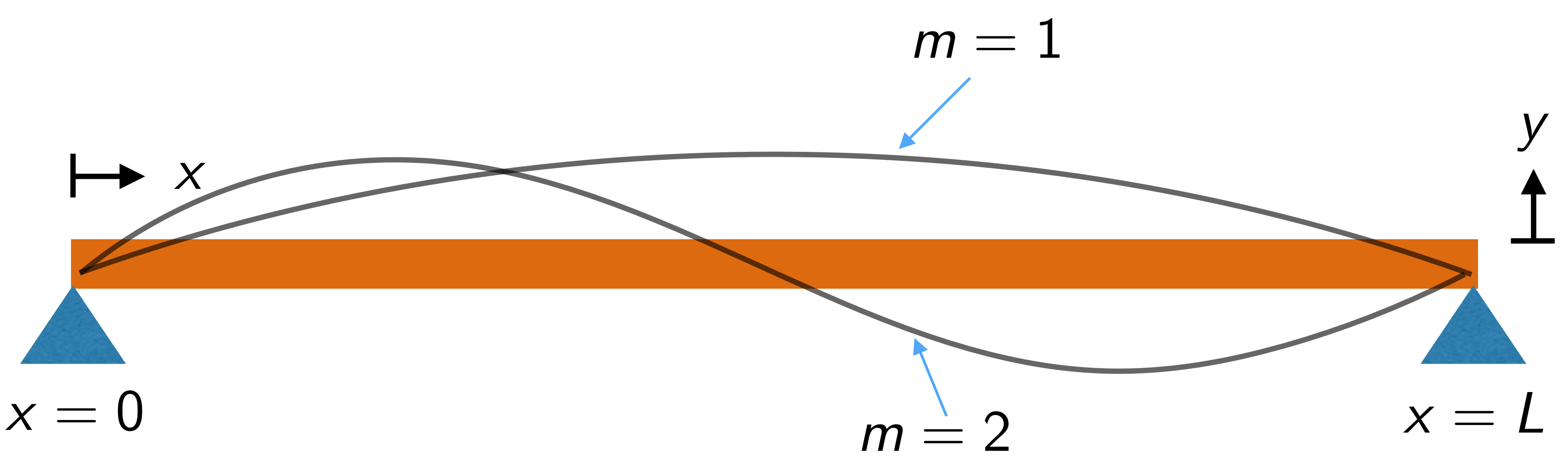

$m = 3$

**Mode shape:**

$$\boxed{\phi(x) = \sin\left(\frac{m\pi x}{L}\right)}$$

**Determine the shape of vibration**

Animation by Dr. Dan Russell, Pen State Univ.




**Mode shape:**

$$\boxed{\phi(x) = \sin\left(\frac{m\pi x}{L}\right)}$$

**Determine the shape of vibration**

**Total vibration** $\longrightarrow$ $y(x) = \phi(x)U$ $\longleftarrow$ **Actual modal amplitude**

**Scaling factor**

# Additional Resources

**My website:**
http://www.azmaputra.com

**My white-board animation videos:**
http://www.youtube.com/c/AzmaPutra-channel

# References

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D. J. Inman, *EngineeringVibrations*, Pearson, 4th Ed. 2014

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