

MECHANICAL VIBRATION

BMCG 3233

CHAPTER 3: HARMONIC FREE VIBRATION (PART 2)

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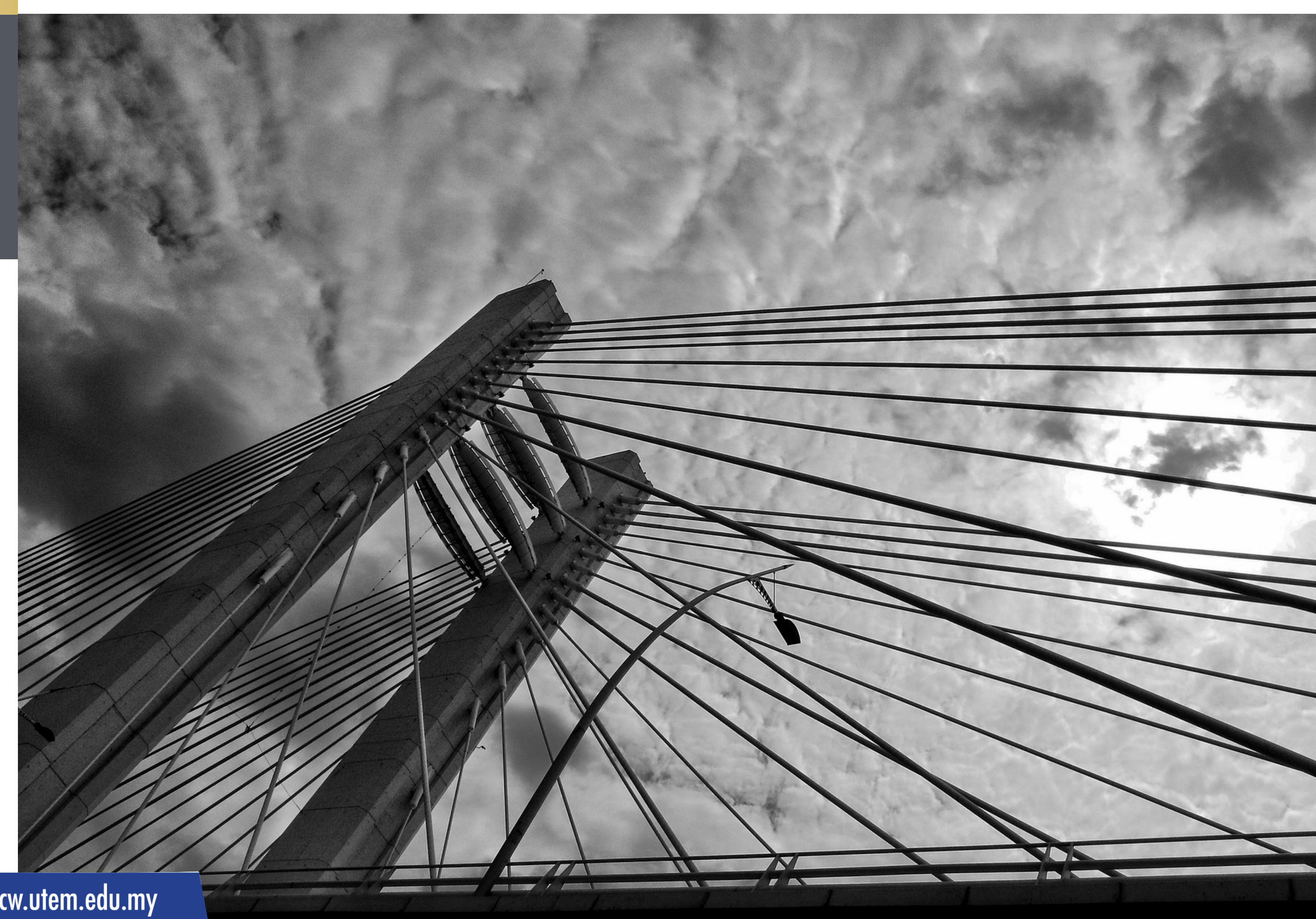
LEARNING OBJECTIVES

1. Derive the equation of motion of a single-degree-of-freedom system
2. Calculate the natural frequency
3. Calculate the damping factor and natural frequency from response of free vibration



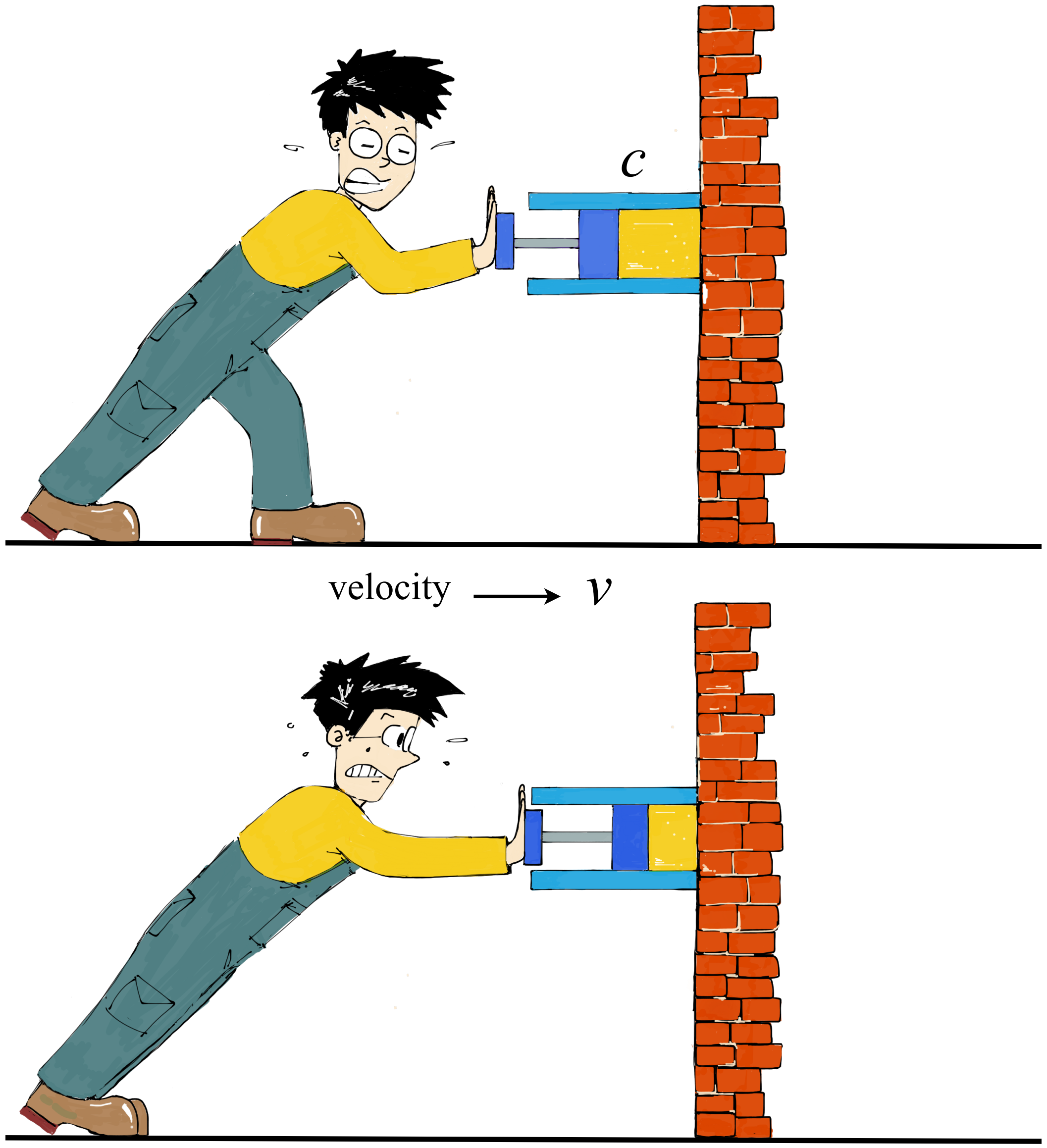
3.4

DAMPING ELEMENT

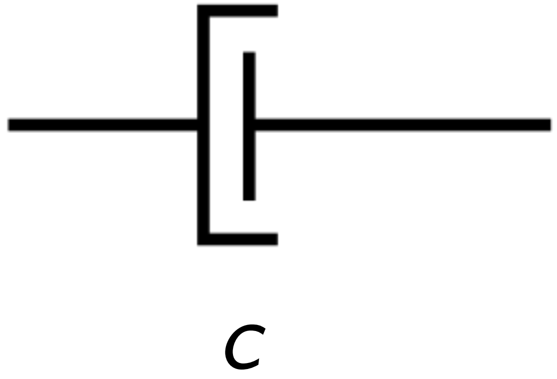
Damper element

Element that provides damping and absorbs energy of vibration.

$$F = c \times v$$

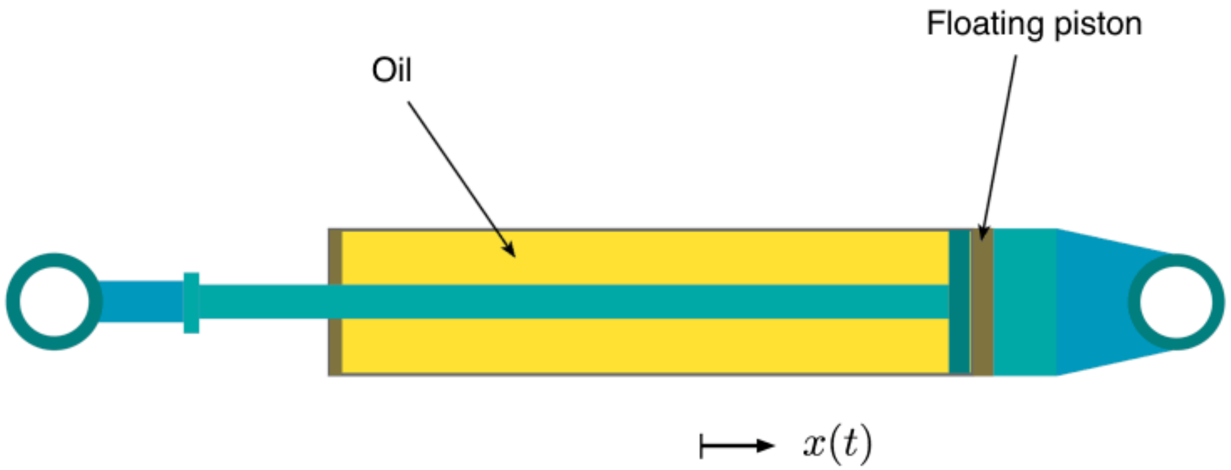


Mathematical symbol:



Damping constant [N.s/m]

Translational Damper



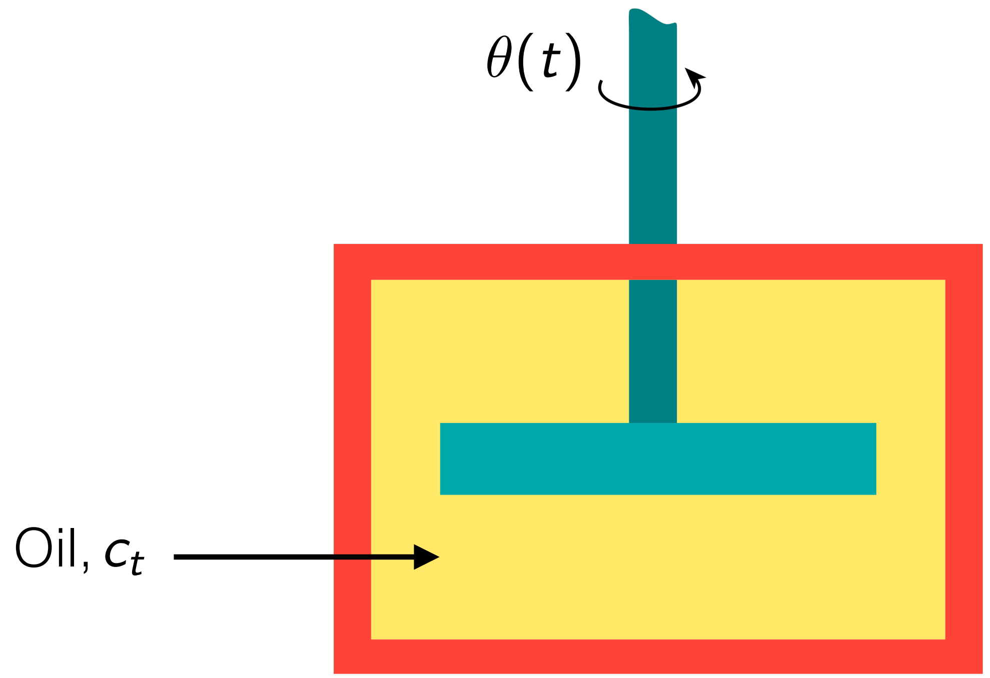
Force:

$$F = c\frac{\mathrm{d}x(t)}{\mathrm{d}t} = c\dot{x}(t)$$

Work done:

$$W = -\int_{x_1}^{x_2} c\dot{x}(t)\mathrm{d}x$$

Rotational Damper

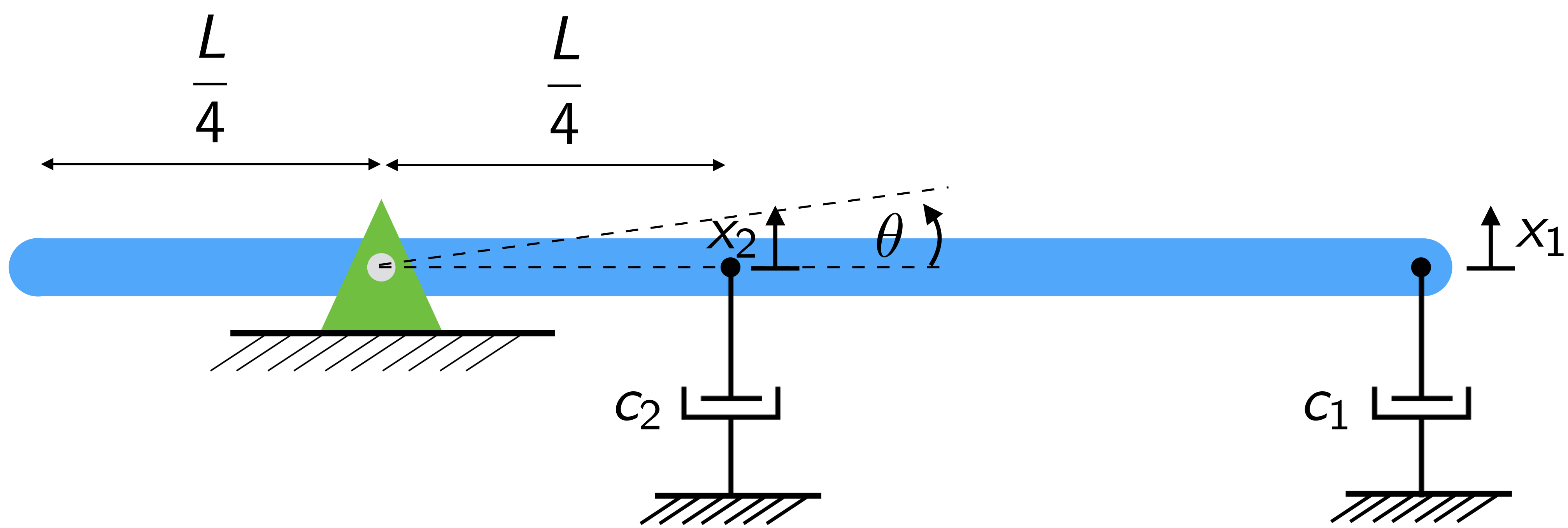


Torsion:

$$T = c_t \frac{\mathrm{d}\theta}{\mathrm{d}t} = c_t \dot{\theta}(t)$$

Work done:

$$W = -\int_{\theta_1}^{\theta_2} c_t \dot{\theta}(t) \mathrm{d}\theta$$



Write down the total work done from the dampers:

$$W = -\int c_1 \dot{x}_1 \mathrm{d}x_1 - \int c_2 \dot{x}_2 \mathrm{d}x_2$$

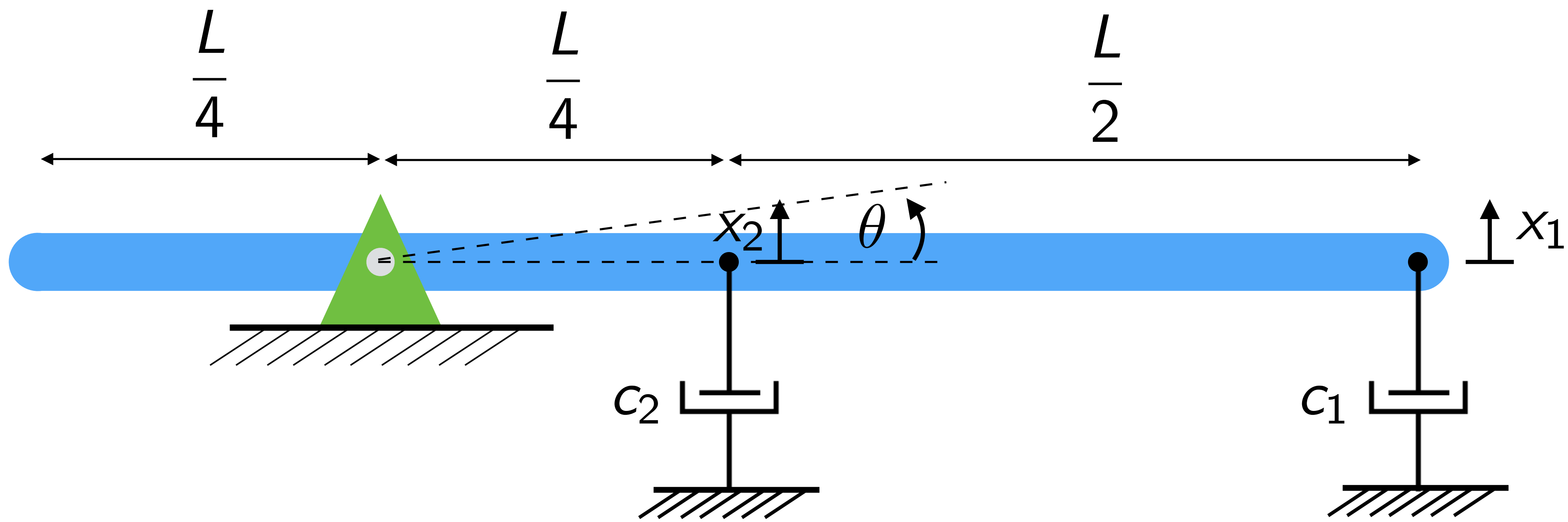
$$x_2 = \frac{1}{3} x_1$$

If x_1 is the generalised coordinate, thus

$$W = -\int c_1 \dot{x}_1 \mathrm{d}x_1 - \int c_2 \left(\tfrac{1}{3}\dot{x}_1\right) \mathrm{d}\left(\tfrac{1}{3}x_1\right) = -\int c_1 \dot{x}_1 \mathrm{d}x_1 - \int \frac{1}{9} c_2 \dot{x}_1 \mathrm{d}x_1$$

$$= -\int \boxed{\left[c_1 + \tfrac{1}{9} c_2\right]} \dot{x}_1 \mathrm{d}x_1$$

Translational equivalent damping constant, c_{eq}



Write down the total work done from the dampers:

$$W = -\int c_1 \dot{x}_1 \mathrm{d}x_1 - \int c_2 \dot{x}_2 \mathrm{d}x_2$$

$$x_1 = (3L/4)\theta$$
$$x_2 = (L/4)\theta$$

If θ is the generalised coordinate, thus

$$W = -\int c_1\left(\tfrac{3L}{4}\dot{\theta}\right)\mathrm{d}\left(\tfrac{3L}{4}\theta\right) - \int c_2\left(\tfrac{L}{4}\dot{\theta}\right)\mathrm{d}\left(\tfrac{L}{4}\theta\right) = -\int c_1\left(\tfrac{9L^2}{16}\dot{\theta}\right)\mathrm{d}\theta - \int c_2\left(\tfrac{L^2}{16}\dot{\theta}\right)\mathrm{d}\theta$$

$$= -\int \boxed{\left[c_1\left(\tfrac{9L^2}{16}\right) + c_2\left(\tfrac{L^2}{16}\right)\right]}\,\dot{\theta}\mathrm{d}\theta$$

Torsional equivalent damping constant, c_{eq}

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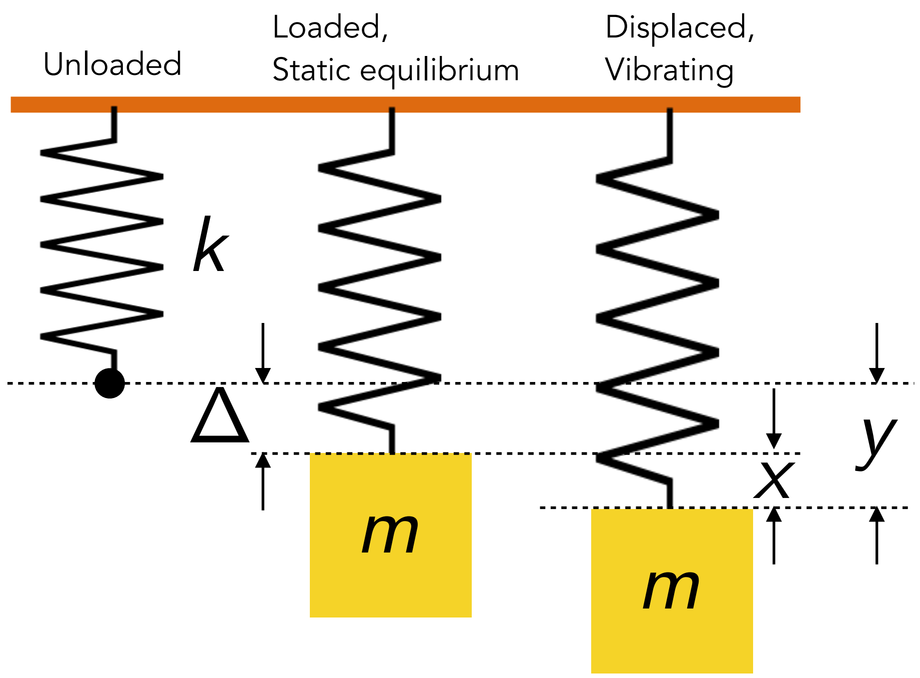
EQUATION OF MOTION



Newton's Law:

$$\sum F = m\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} = m\ddot{y}$$

$$mg - ky = m\ddot{y}$$

$$mg - k(x + \Delta) = m\ddot{x}$$

$$-kx = m\ddot{x}$$

Static deflection: $\Delta = \dfrac{mg}{k}$ **(constant)**

$$y = x + \Delta$$

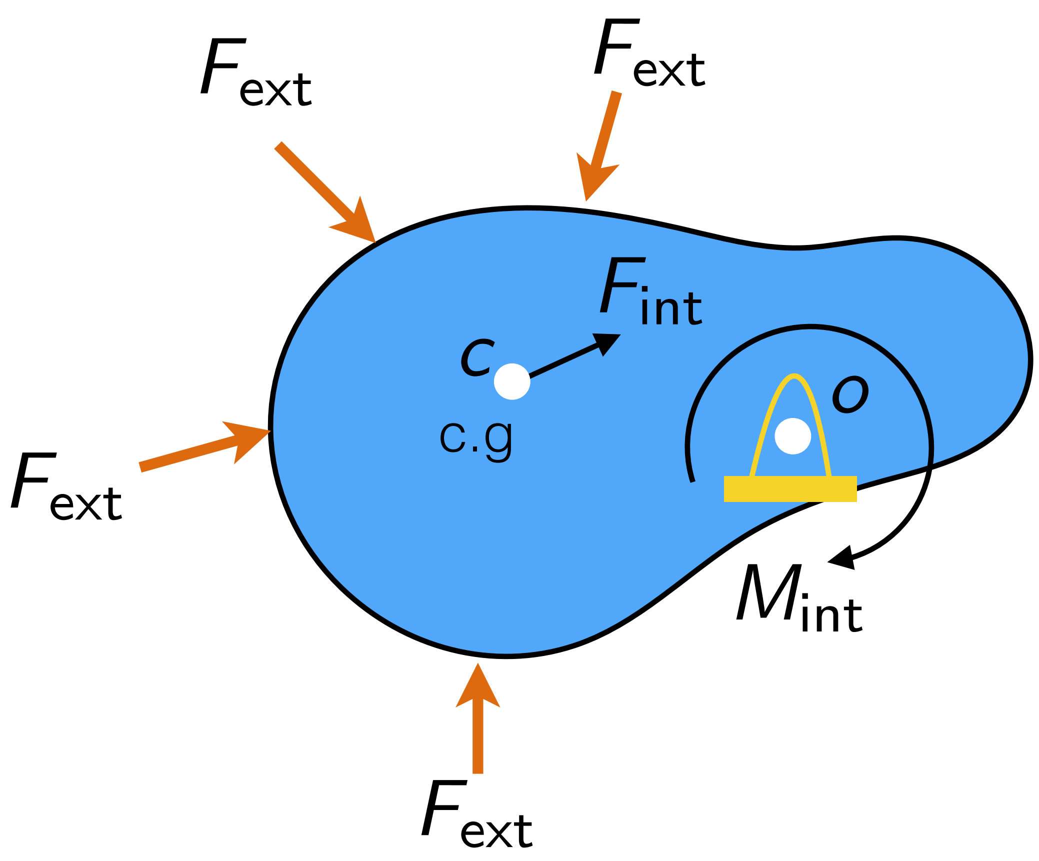
$$\ddot{y} = \ddot{x}$$

Equation of motion:

$$\boxed{m\ddot{x} + kx = 0}$$

(1) The resultant of external forces acting on a rigid body is equal to the resultant of inertia force at the centre of mass.

(2) The resultant of external moment acting on a rigid body is equal to the resultant of inertia moment at the centre of rotation.



(1)

$$\left(\sum F_{ext}\right)_c = \left(\sum F_{int}\right)_c$$

Inertia Force/Inertia Moment

(2)

$$\left(\sum M_{ext}\right)_o = \left(\sum M_{int}\right)_o$$

Watch the video: "D'Alembert Principle"

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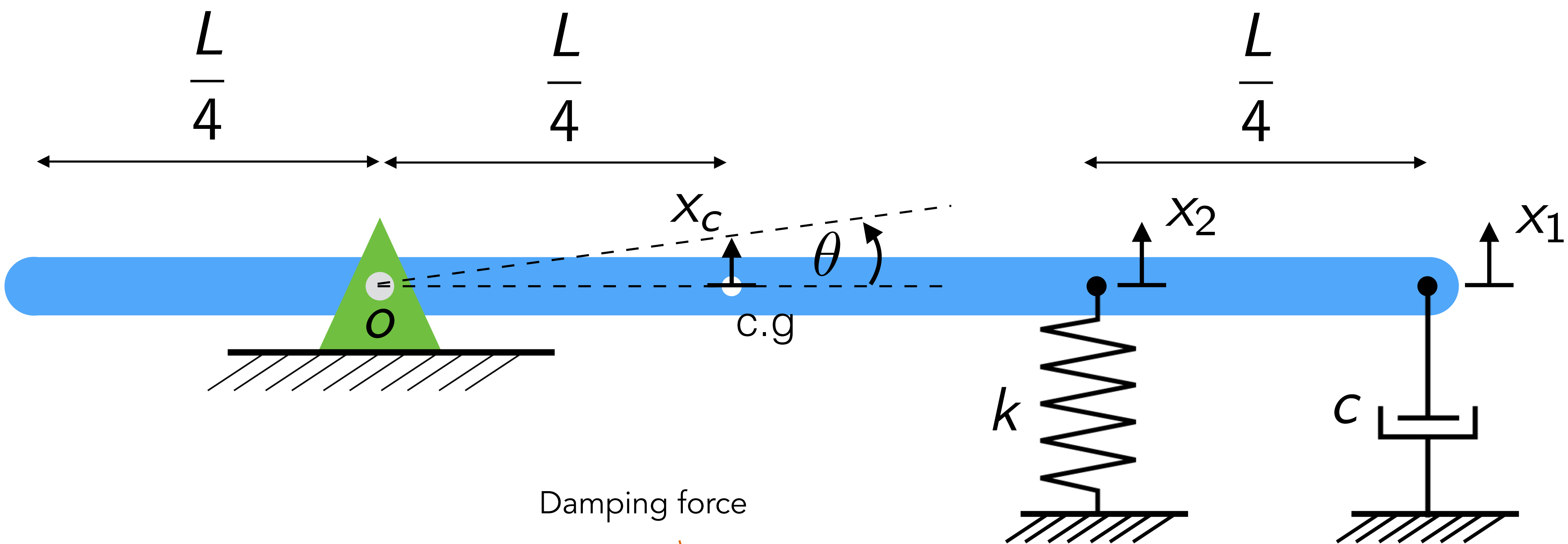
Or click/tap here.

Derive the equation of motion using D'Alembert principle!

Use x_1 as the generalised coordinate.



External moment: $$\left(\sum M_{ext}\right)_o = -c\dot{x}_1\left(\frac{3L}{4}\right) - kx_2\left(\frac{L}{2}\right)$$

Inertia moment: $$\left(\sum M_{int}\right)_o = m\ddot{x}_c\left(\frac{L}{4}\right) + J\ddot{\theta}$$

Damping force

Spring force

Length of arm to the center of rotation, O

Using principle (2):

generalised coordinate

$$\boxed{\left(m + \frac{16J}{L^2}\right)\ddot{x}_1 + 9c\dot{x}_1 + 4kx_1 = 0}$$

Watch the video: "D'Alembert Principle (Example Problem Part 1)"

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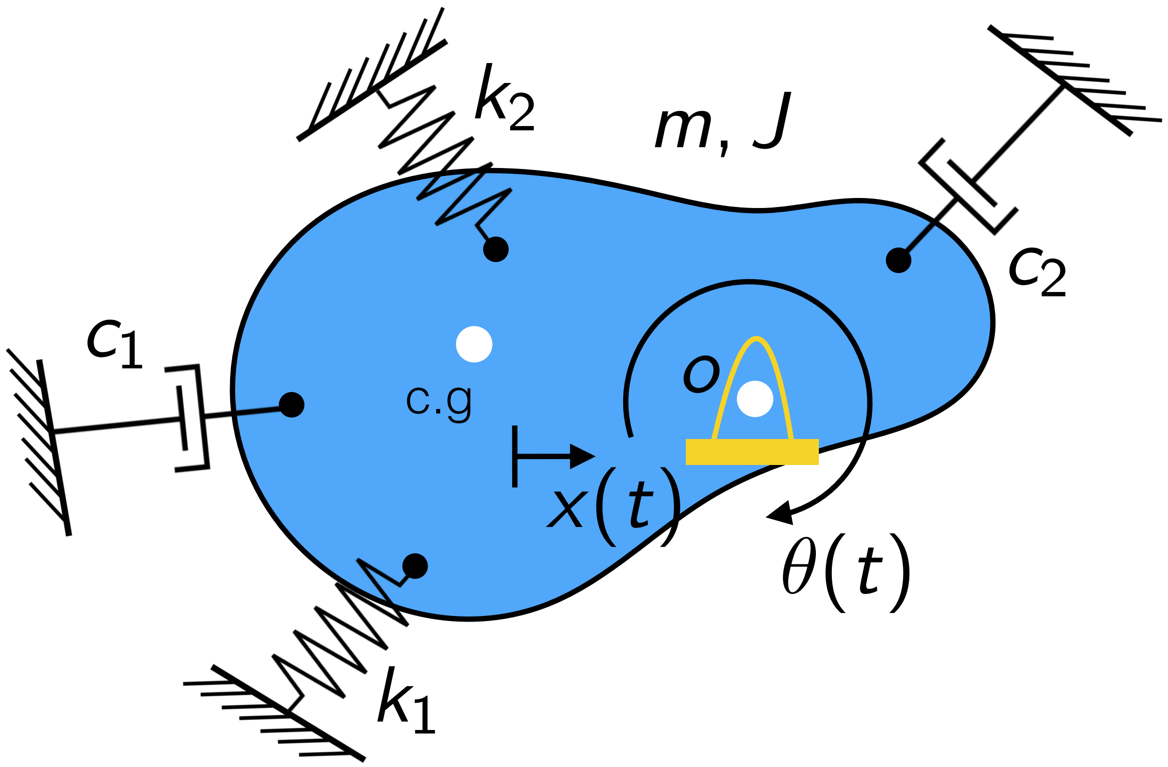
System Equivalent Analysis

Employ energy method to calculate the equivalent parameters.



Equation of Motion:

$$m_{\text{eq}}\ddot{x} + c_{\text{eq}}\dot{x} + k_{\text{eq}}x = 0$$

generalised coordinate

$$J_{\text{eq}}\ddot{\theta} + c_{r,\text{eq}}\dot{\theta} + k_{r,\text{eq}}\theta = 0$$

Watch the video: “System Equivalent Analysis”

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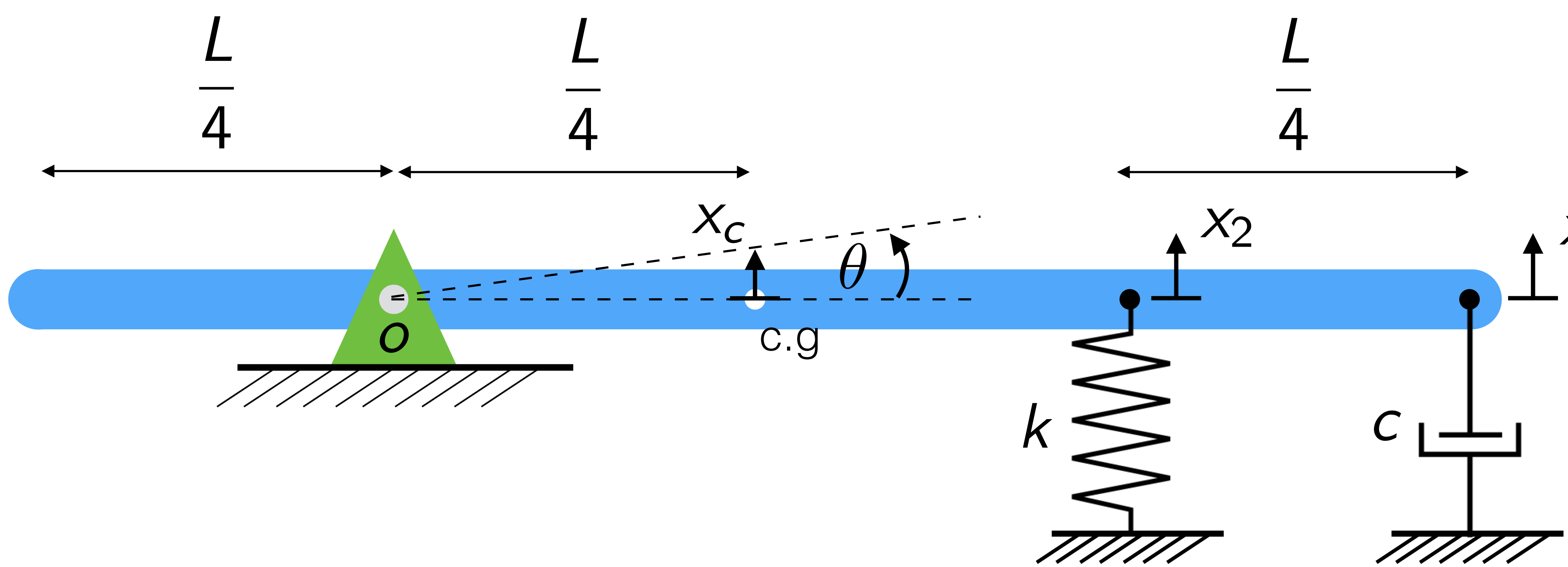
Derive the equation of motion using system equivalent analysis!

Use x_1 as the generalised coordinate.



Kinetic energy: $T = \frac{1}{2} m \dot{x}_c^{\,2} + \frac{1}{2} J \dot{\theta}^2 = \frac{1}{2} \underbrace{\left(\frac{m}{9} + \frac{16J}{9L^2} \right)}_{m_{eq}} \dot{x}_1^2$

Potential energy: $U = \frac{1}{2} k x_2^2 = \frac{1}{2} \underbrace{\left(\frac{4k}{9} \right)}_{k_{eq}} x_1^2$

Work done: $W = -\int \underbrace{c}_{c_{eq}} \dot{x}_1 \mathrm{d} x_1$

Equation of motion:

$$\boxed{\left(\frac{m}{9} + \frac{16J}{9L^2} \right) \ddot{x}_1 + c \dot{x}_1 + \left(\frac{4k}{9} \right) x_1 = 0}$$

Watch the video: "System Equivalent Analysis (Example Problem Part 1)"

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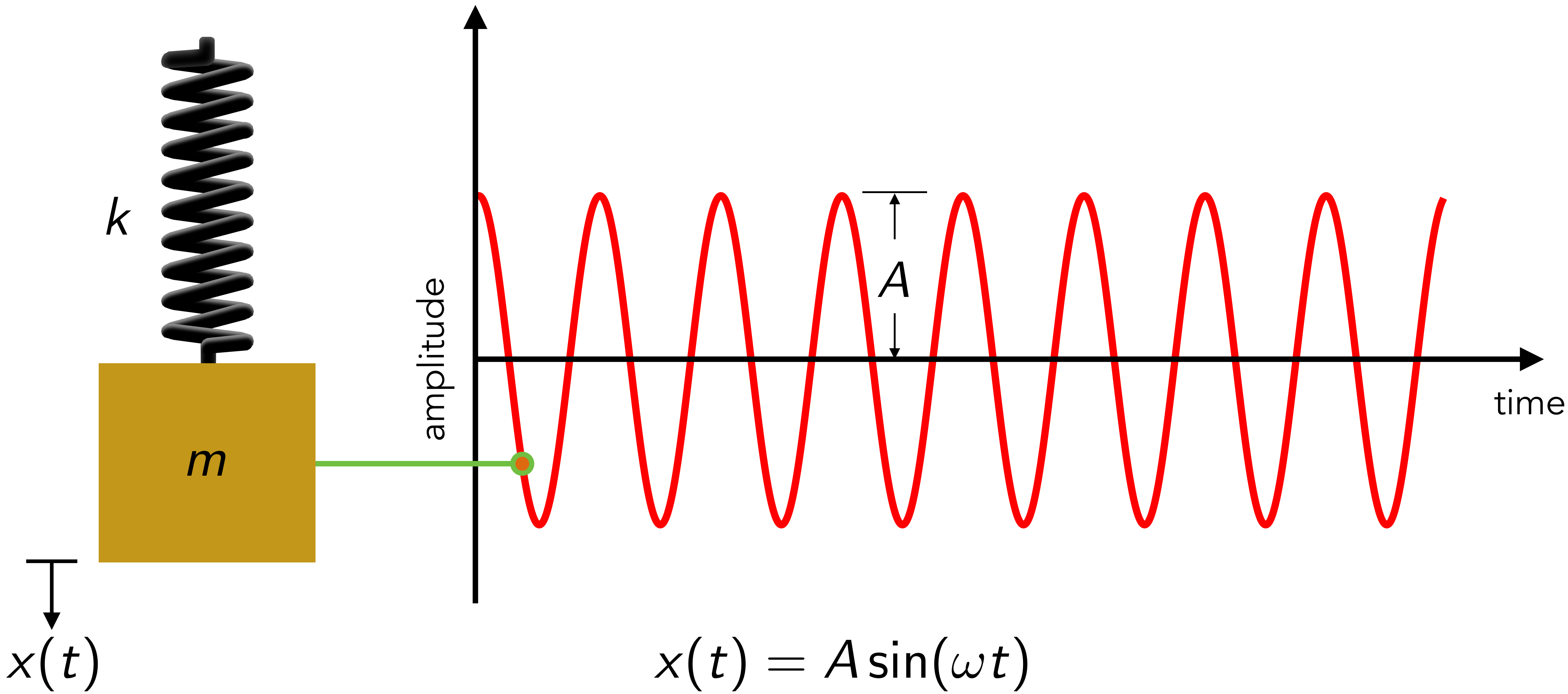
3.6

RESPONSE OF FREE VIBRATION



$$x(t) = A\sin(\omega t)$$

Equation of motion:

$$m\ddot{x} + kx = 0$$

$$-\omega^2 mA\sin(\omega t) + kA\sin(\omega t) = 0$$

Natural frequency:

$$\boxed{\omega = \sqrt{\frac{k}{m}}}$$

Watch the video: "Response of Free Vibration and Natural Frequency"

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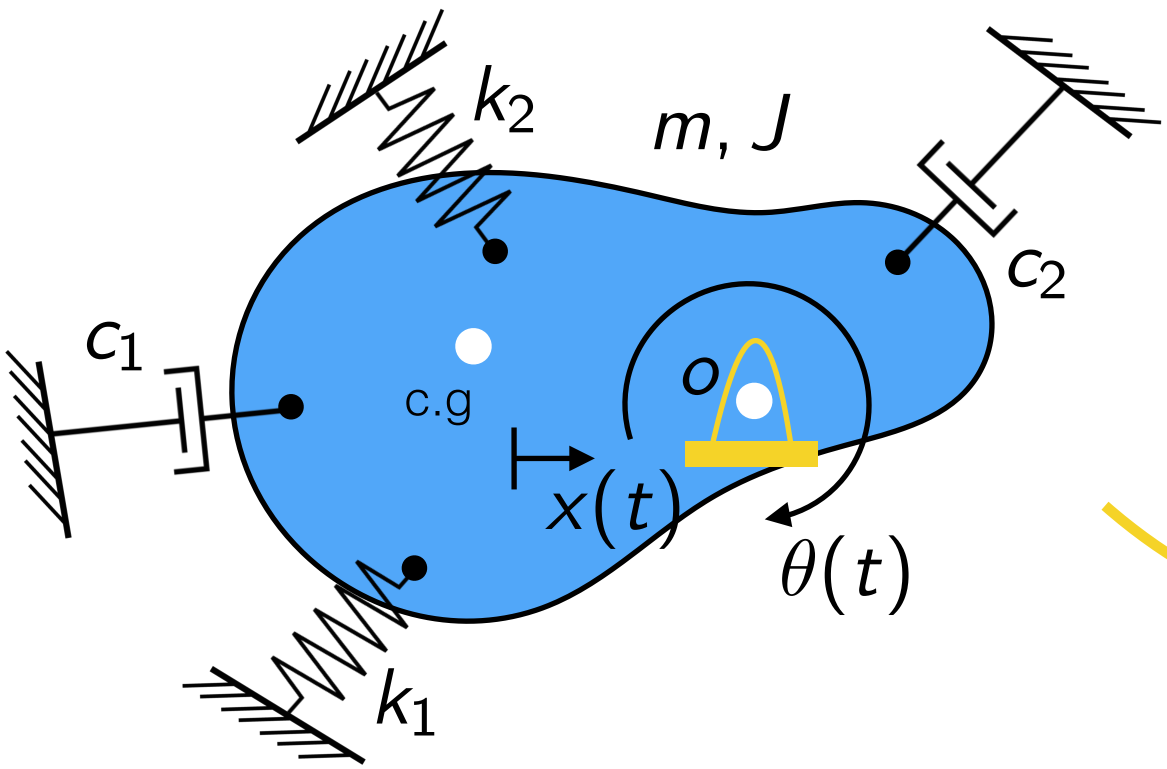
Natural Frequency



$m_{eq},\ k_{eq}$

$$\omega_n = \sqrt{\frac{k_{eq}}{m_{eq}}}$$

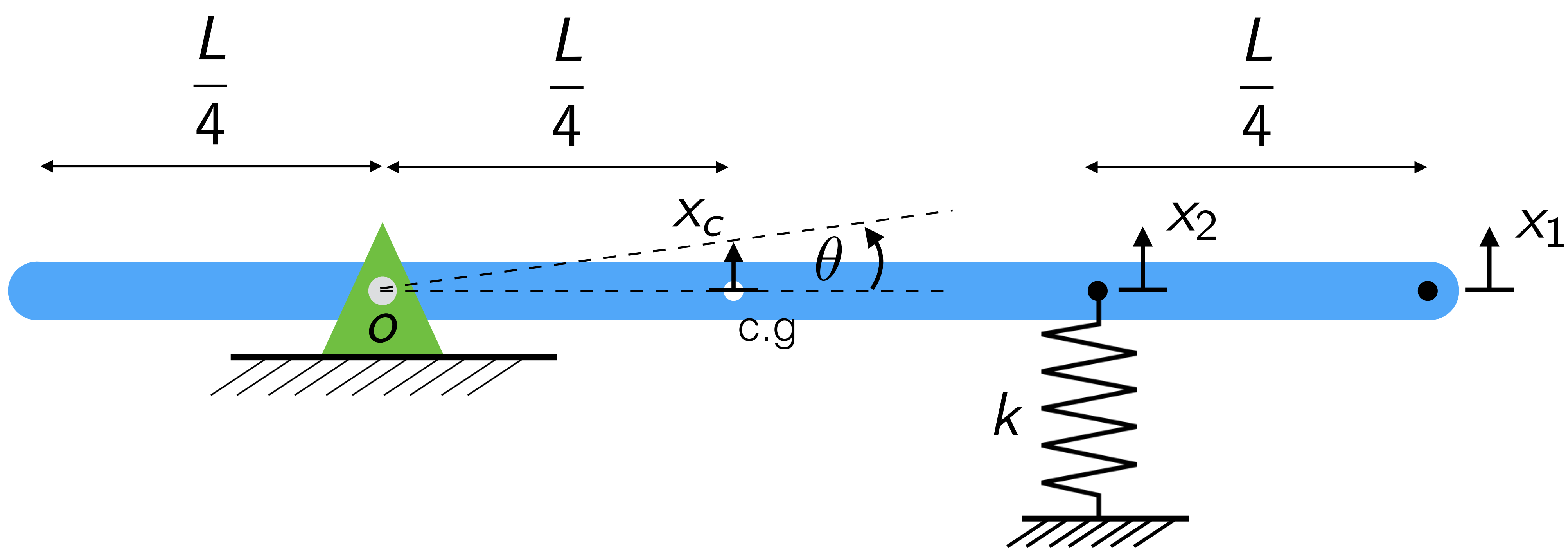
$J_{eq},\ k_{r,eq}$

$$\omega_n = \sqrt{\frac{k_{r,eq}}{J_{eq}}}$$

****Regardless the generalised coordinate**

Calculate the natural frequency!

Use x_1 as the generalised coordinate.



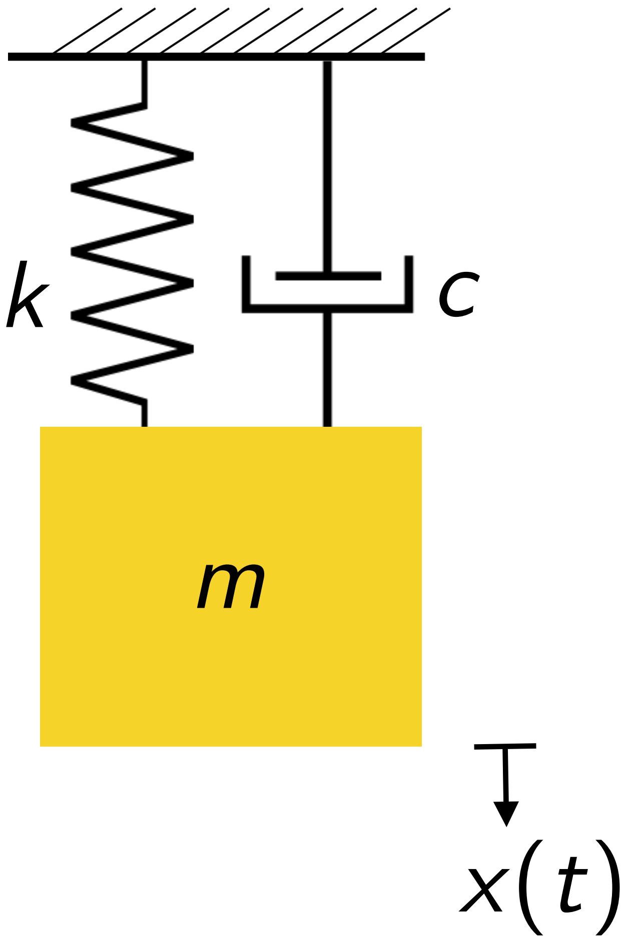
Kinetic energy: $$T = \frac{1}{2}m\dot{x}_c^{\,2} + \frac{1}{2}J\dot{\theta}^2 = \frac{1}{2}\underbrace{\left(\frac{m}{9} + \frac{16J}{9L^2}\right)}_{m_{eq}}\dot{x}_1^2$$

Potential energy: $$U = \frac{1}{2}kx_2^2 = \frac{1}{2}\underbrace{\left(\frac{4k}{9}\right)}_{k_{eq}}x_1^2$$

Natural frequency:

$$\boxed{\omega_n = \sqrt{\frac{k_{eq}}{m_{eq}}} = 2L\sqrt{\frac{k}{mL^2 + 16J}}}$$

Try with generalised coordinate, x_2



Equation of motion:

$$\boxed{m\ddot{x} + c\dot{x} + kx = 0}$$

Can also be written as:

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0$$

Two important parameters in damped free vibration:

Natural frequency

$$\boxed{\omega_n = \sqrt{\frac{k}{m}}}$$

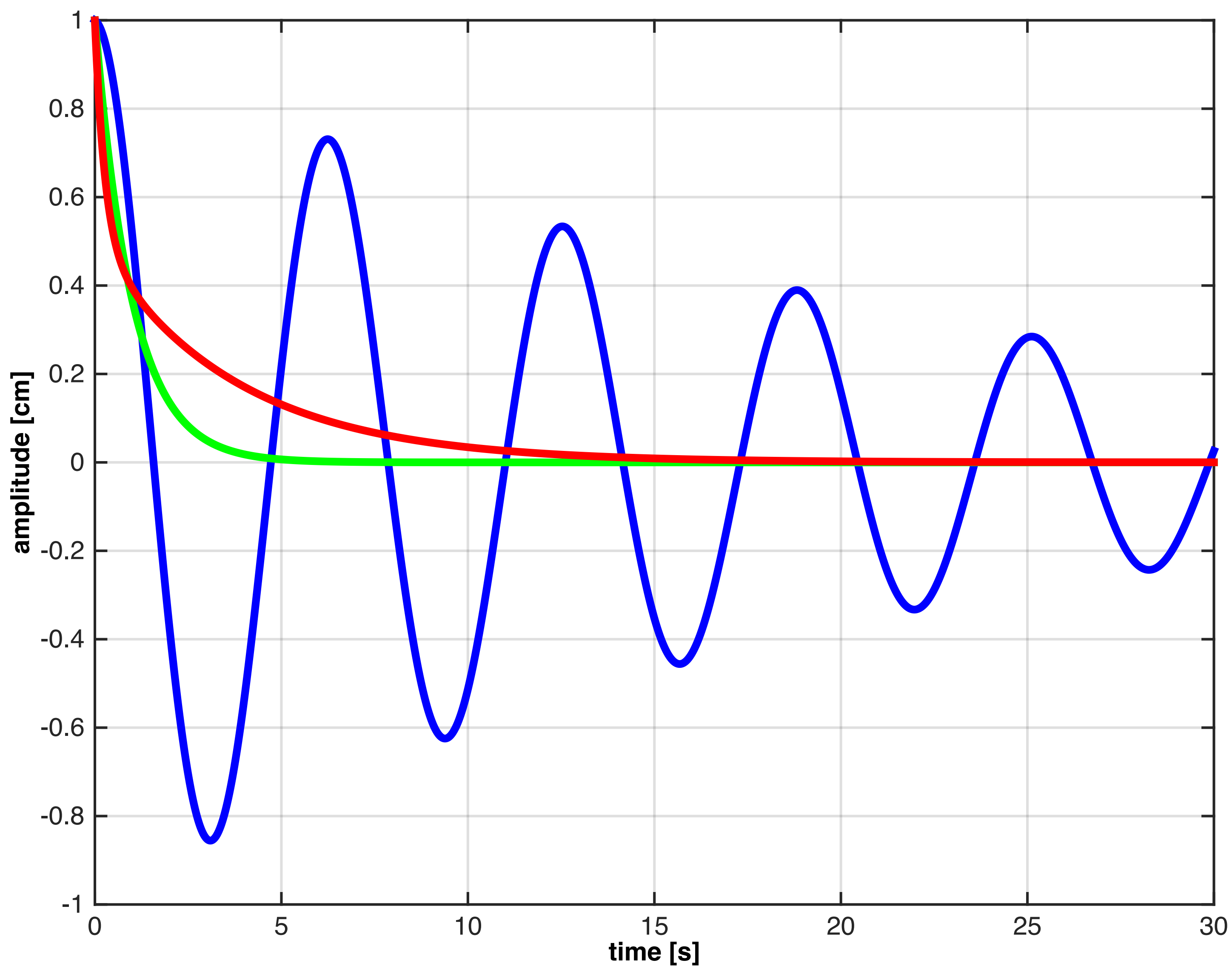
Damping factor

$$\boxed{\zeta = \frac{c}{2m\omega_n} = \frac{c}{2\sqrt{km}}}$$

$0 < \zeta < 1$ **: UNDERDAMPED, decaying oscillatory motion**

$\zeta = 1$ **: CRITICALLY DAMPED**

$\zeta > 1$ **: OVERDAMPED, non-oscillatory decaying motion**

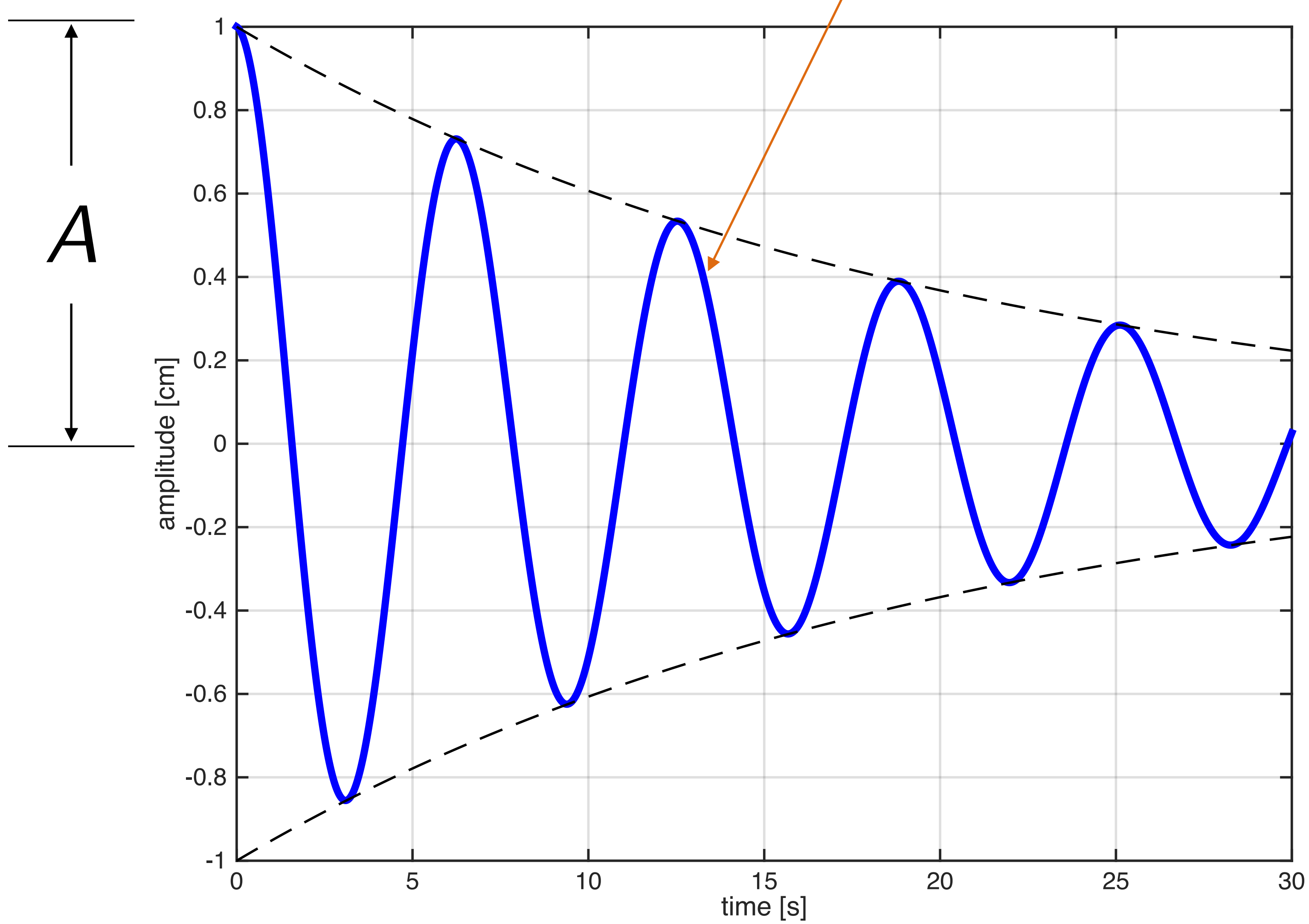


$$x(t) = A\underbrace{e^{-\zeta\omega_n t}}\,\underbrace{\cos(\omega_d t + \phi)}$$

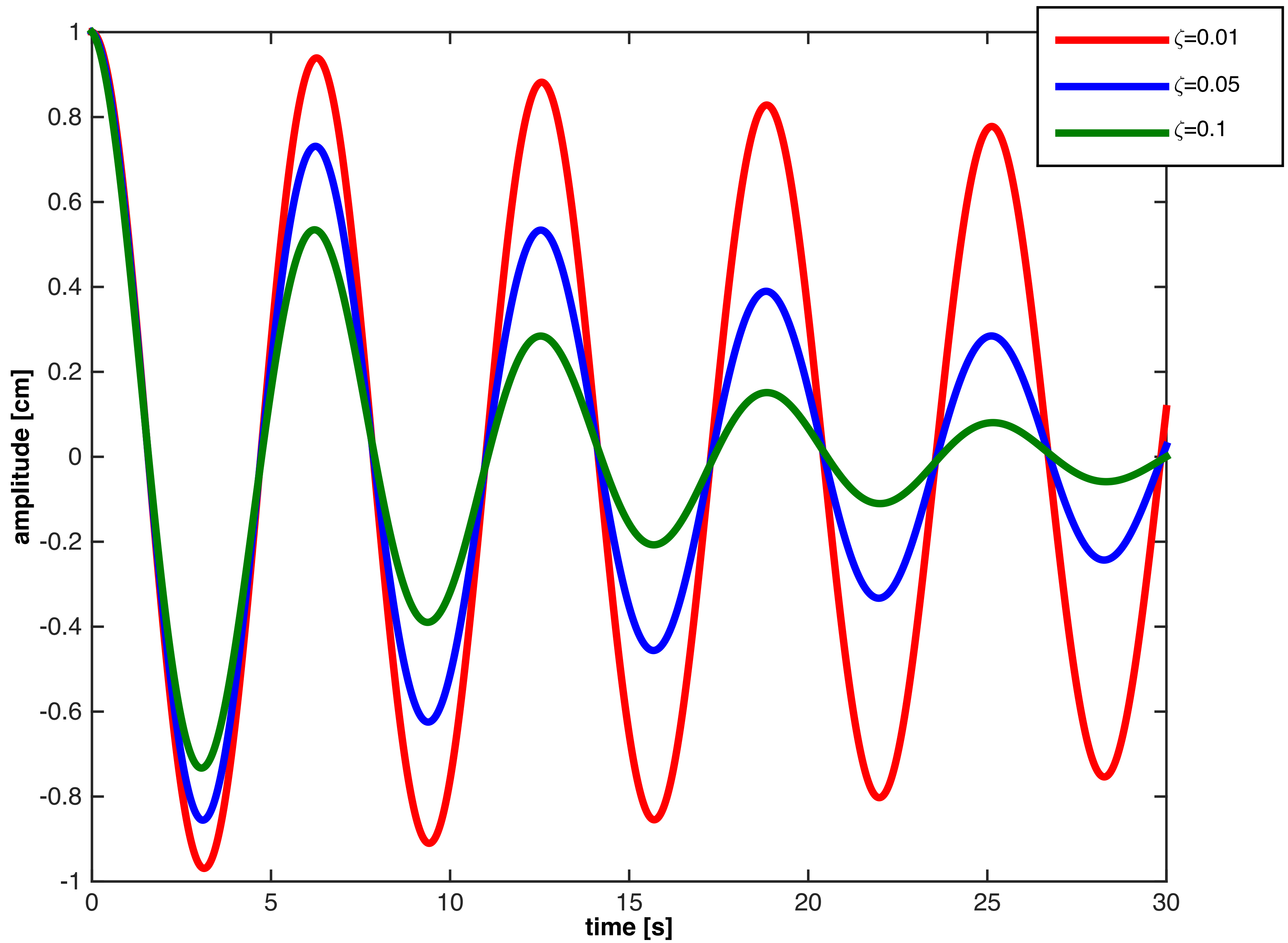
Decay rate, depends on ζ

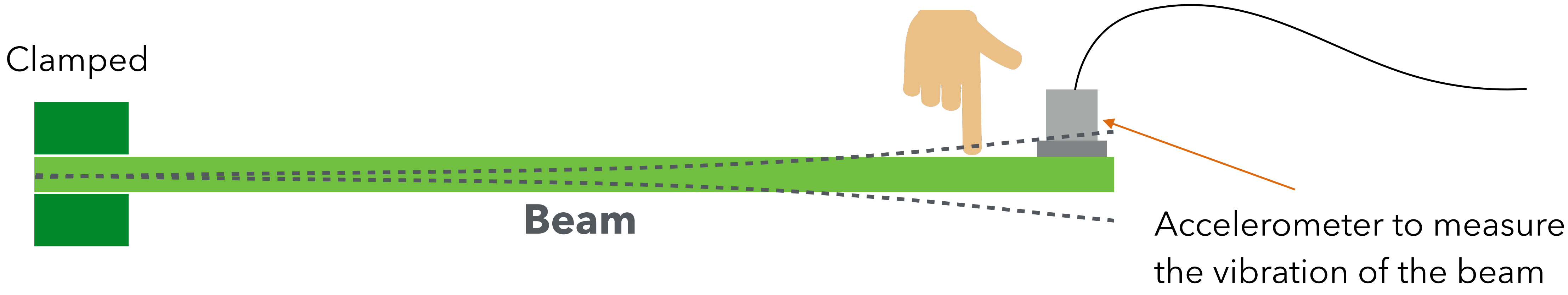
Oscillation at frequency:

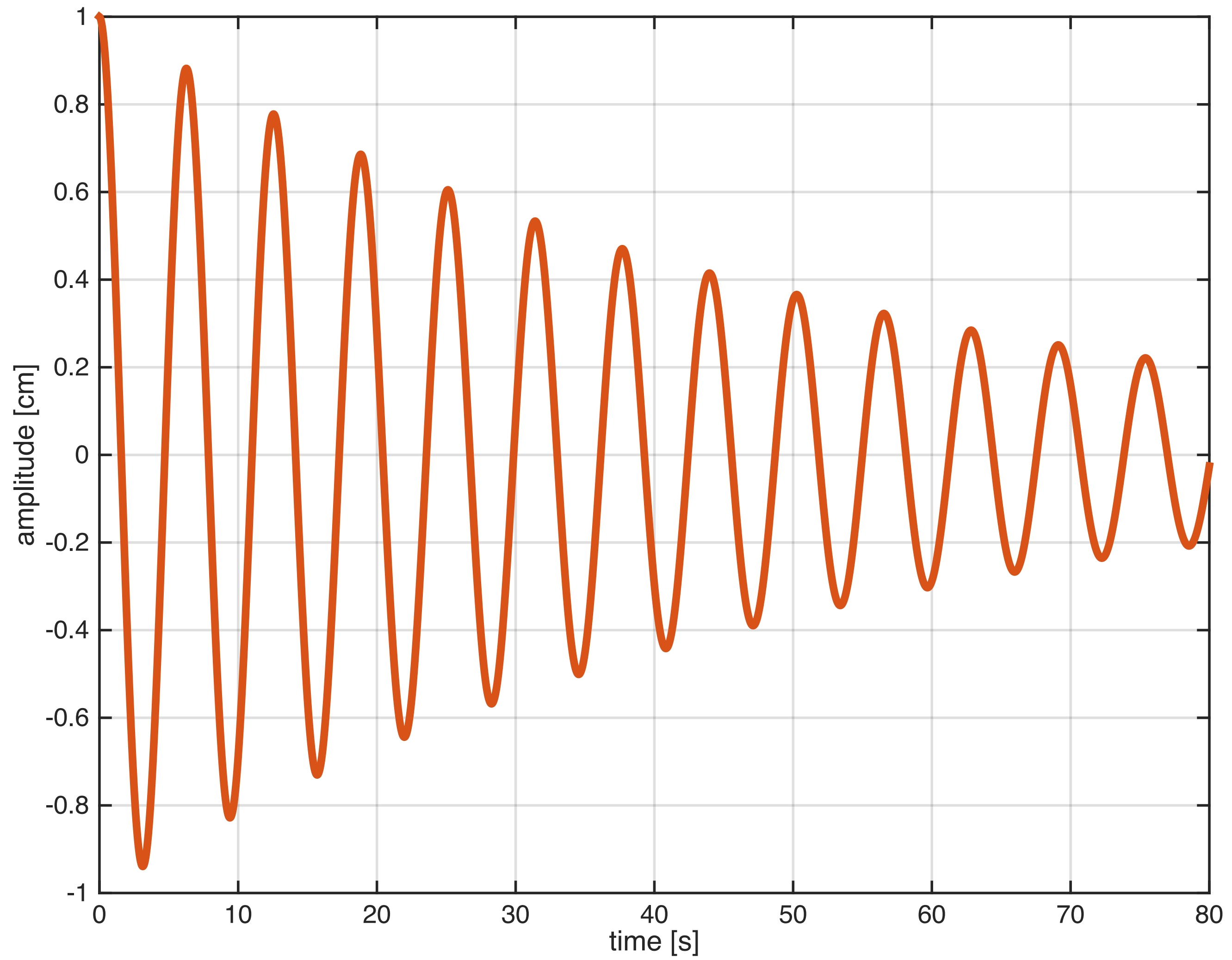
$$\omega_d = \omega_n\sqrt{1-\zeta^2}$$



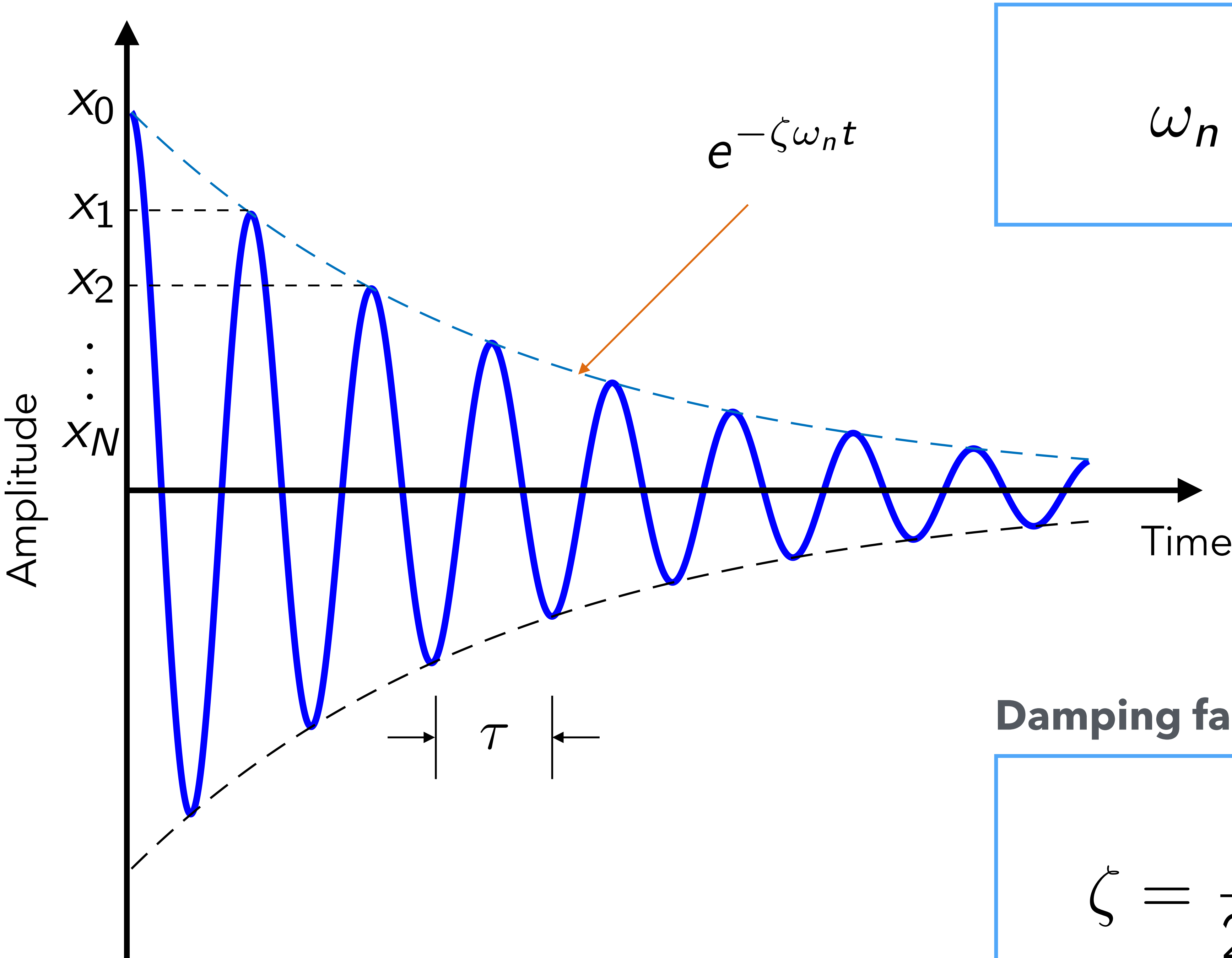
Frequency is the same, but decaying faster with increasing damping factor, ζ







From this measurement data, we can extract information about the damping and natural frequency of the beam.



Natural frequency:

$$\boxed{\omega_n \approx \frac{2\pi}{\tau}}$$

$$\frac{x_N}{x_0} = e^{-\zeta\omega_n t_N} = e^{-\zeta\omega_n (N\tau)} = e^{-\zeta\frac{2\pi}{\tau}(N\tau)}$$

Damping factor:

$$\boxed{\zeta = \frac{1}{2\pi N}\ln\left(\frac{x_0}{x_N}\right)}$$

The amplitude of the *N*-th peak after x_0

Interact with my animations:

http://www.azmaputra.com/animations/

My white-board animation videos:

http://www.youtube.com/c/AzmaPutra-channel

References

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