OPENCOURSEWARE

NUMERICAL METHODS
BEKG 2452
SOLUTION OF PARTIAL DIFFERENTIAL EQUATIONS (ELLIPTIC EQUATIONS)

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Lesson Outcome

Upon completion of this lesson, the student should be able to:

1. Categorize the types of partial differential equations.
2. Solve numerically Laplace equations.

Linear Second Order Differential Equation

Analytical Methods

- Separation of variables
- Integral transform
- Characteristic etc.

Exact solution

Numerical Methods: Finite Differences

Parabolic eqn.

Hyperbolic eqn.

Elliptic eqn.

Approximated solution

8.4 Elliptic Equation – Laplace's equation

Consider the **Laplace's Equation**:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \qquad 0 \le x \le a, \qquad 0 \le y \le b$$

with boundary conditions

$$\begin{aligned} u(x,0) &= f_1(x,y), \qquad & u(x,b) &= f_2(x,y), \qquad & 0 \le x \le a \\ u(0,y) &= g_1(x,y), \qquad & u(a,y) &= g_2(x,y), \qquad & 0 \le y \le b. \end{aligned}$$

By central-difference formulas,

$$\left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j} + \left(\frac{\partial^2 u}{\partial y^2}\right)_{i,j} = 0$$

is approximated to

$$\boxed{\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} + \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{k^2} = 0}$$

where $h = \Delta x$ and $k = \Delta y$.

8.4 Elliptic Equation – Laplace's equation

Illustrative Example 1:

Find approximation solutions for Laplace's Equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \qquad 0 \le x \le 1, \qquad 0 \le y \le 1$$

with boundary conditions

$$u(x,0) = x^2, \qquad u(x,1) = x^2 + 1, \qquad 0 \le x \le 1$$

$$u(0,y) = y, \qquad u(1,y) = 1 + y^2, \qquad 0 \le y \le 1$$

by using central-difference formula and Gauss elimination method. Given $h = 1/3$ and $k = 1/2$.

8.4 Elliptic Equation – Laplace's equation

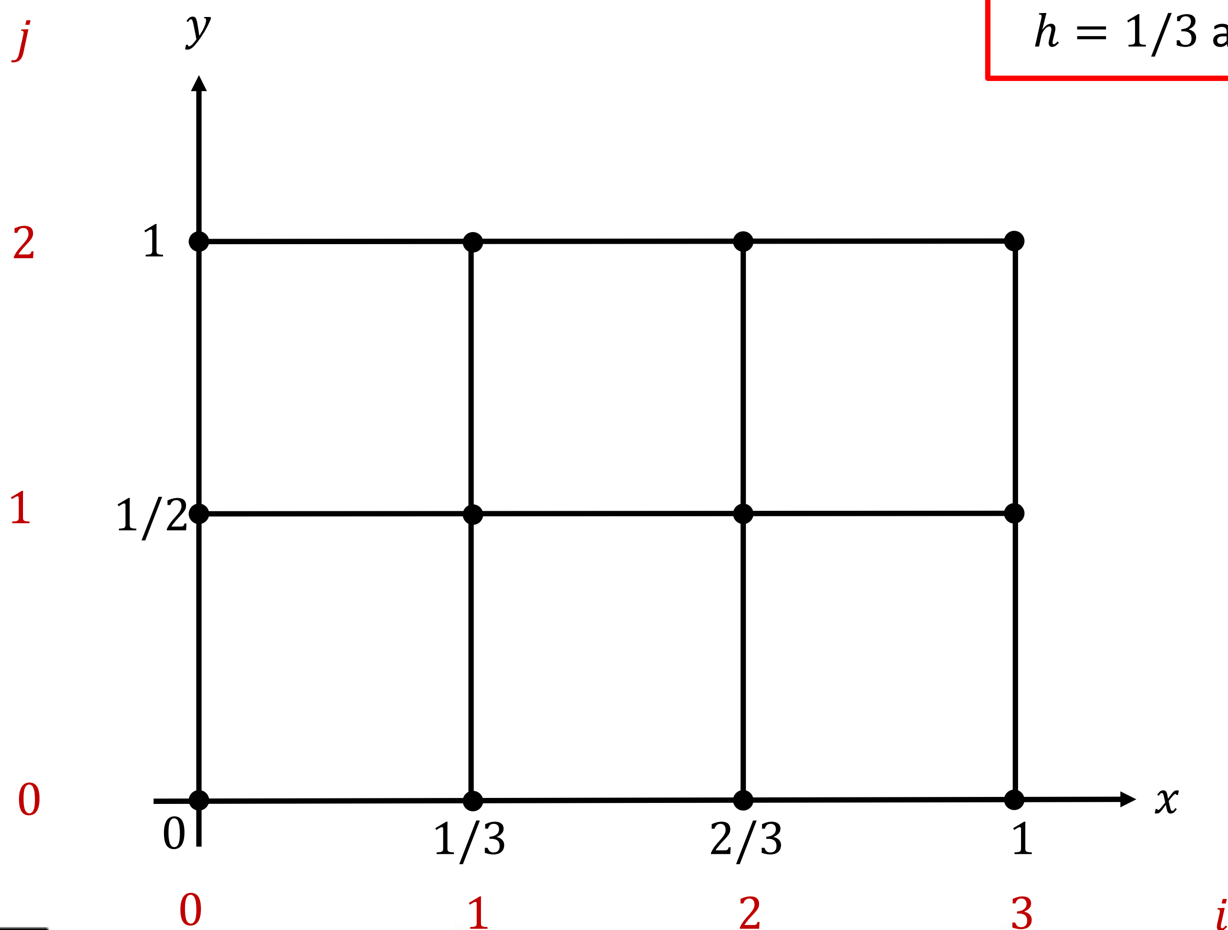
Solution:

Step 1: Sketch the grid points.

Recall:

$0 \le x \le 1, \qquad 0 \le y \le 1$

$h = 1/3$ and $k = 1/2$



8.4 Elliptic Equation – Laplace's equation

Solution:

Step 2: Compute boundary values.

Recall:
$h = 1/3$ and $k = 1/2$

$\boldsymbol{u(x,0) = x^2}$, $\boldsymbol{u(x,1) = x^2 + 1}$, $\boldsymbol{0 \leq x \leq 1}$

$u(0,0) = 0$ $u(0,1) = 1$

$u(1/3,0) = 1/9$ $u(1/3,1) = 10/9$

$u(2/3,0) = 4/9$ $u(2/3,1) = 13/9$

$u(1,0) = 1$ $u(1,1) = 2$

$\boldsymbol{u(0,y) = y}$, $\boldsymbol{u(1,y) = 1 + y^2}$, $\boldsymbol{0 \leq y \leq 1}$

$u(0,0) = 0$ $u(1,0) = 1$

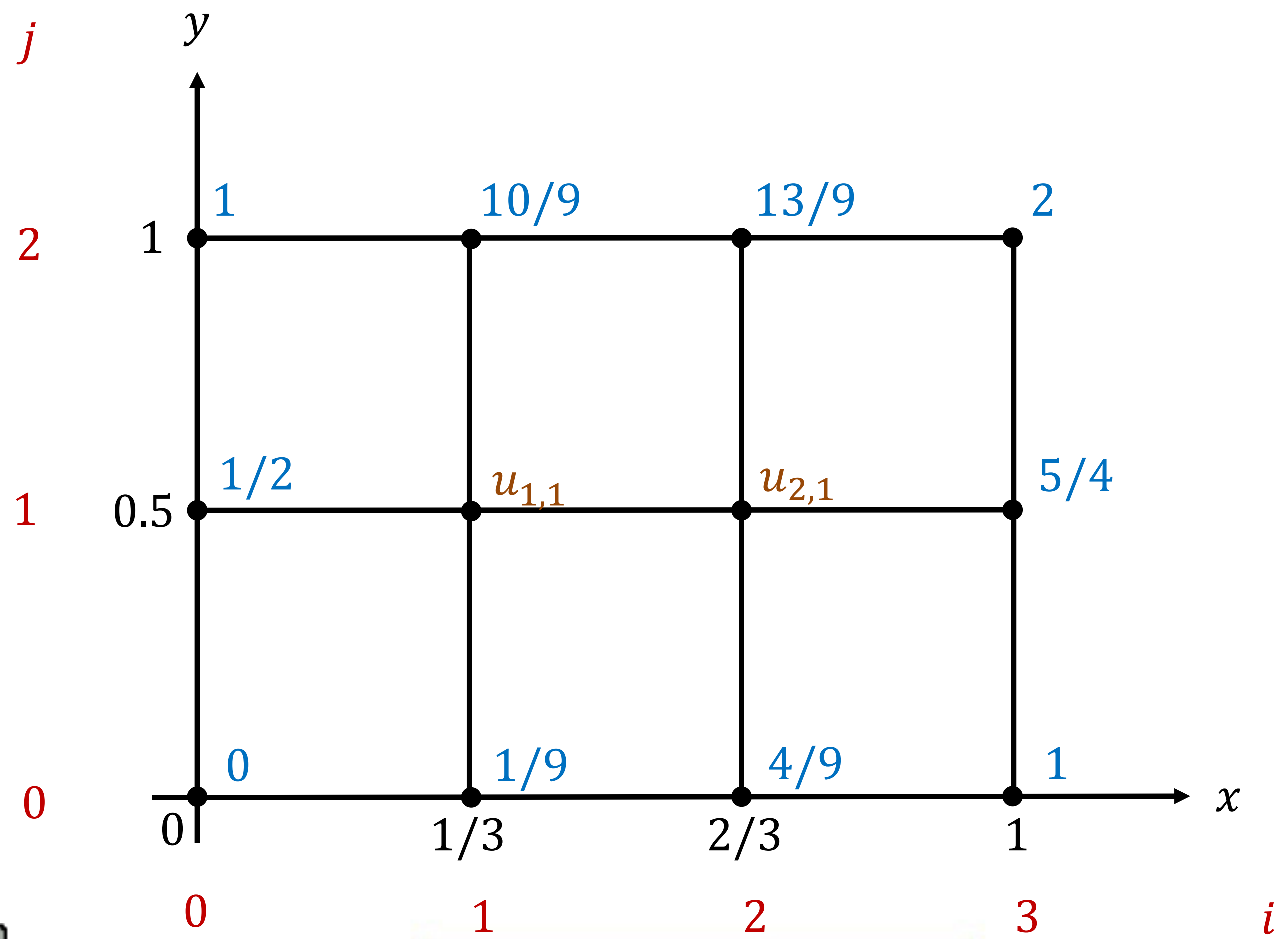
$u(0,1/2) = 1/2$ $u(1,1/2) = 5/4$

$u(0,1) = 1$ $u(1,1) = 2$

8.4 Elliptic Equation – Laplace's equation

Solution:

Step 3: Fill in the values into the grid.



8.4 Elliptic Equation – Laplace's equation

Solution:

Step 4: Obtain formula $u_{i,j}$ from difference formula.

By central-difference formulas,

$$\left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j} + \left(\frac{\partial^2 u}{\partial y^2}\right)_{i,j} = 0$$

Recall:
$h = 1/3$ and $k = 1/2$

is approximated to

$$\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} + \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{k^2} = 0$$

$$\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{(1/3)^2} + \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{(1/2)^2} = 0$$

$$9u_{i+1,j} - 18u_{i,j} + 9u_{i-1,j} + 4u_{i,j+1} - 8u_{i,j} + 4u_{i,j-1} = 0$$

$$u_{i,j} = \frac{9}{26}u_{i+1,j} + \frac{9}{26}u_{i-1,j} + \frac{2}{13}u_{i,j+1} + \frac{2}{13}u_{i,j-1}$$

8.4 Elliptic Equation – Laplace's equation

Solution:

Step 5: Compute equations for internal points $u_{i,j}$

(Refer to grid in Step 3).

$$u_{i,j} = \frac{9}{26}u_{i+1,j} + \frac{9}{26}u_{i-1,j} + \frac{2}{13}u_{i,j+1} + \frac{2}{13}u_{i,j-1}$$

$$u_{1,1} = \frac{9}{26}u_{2,1} + \frac{9}{26}u_{0,1} + \frac{2}{13}u_{1,2} + \frac{2}{13}u_{1,0}$$

$$u_{1,1} = \frac{9}{26}u_{2,1} + \frac{9}{26}(\frac{1}{2}) + \frac{2}{13}(\frac{10}{9}) + \frac{2}{13}(\frac{1}{9})$$

$$\therefore u_{1,1} - \frac{9}{26}u_{2,1} = \frac{13}{36}$$

$$u_{2,1} = \frac{9}{26}u_{3,1} + \frac{9}{26}u_{1,1} + \frac{2}{13}u_{2,2} + \frac{2}{13}u_{2,0}$$

$$u_{2,1} = \frac{9}{26}(\frac{5}{4}) + \frac{9}{26}u_{1,1} + \frac{2}{13}(\frac{13}{9}) + \frac{2}{13}(\frac{4}{9})$$

$$\therefore -\frac{9}{26}u_{1,1} + u_{2,1} = \frac{677}{936}$$

8.4 Elliptic Equation – Laplace's equation

Solution:

Step 6: Form a linear system for internal points and solve it using Gauss-Seidel or Gauss Elimination.

$$u_{1,1} - \frac{9}{26}u_{2,1} = \frac{13}{36}$$
$$-\frac{9}{26}u_{1,1} + u_{2,1} = \frac{677}{936}$$

$$\begin{bmatrix} 1 & -\frac{9}{26} \\ -\frac{9}{26} & 1 \end{bmatrix}\begin{bmatrix} u_{1,1} \\ u_{2,1} \end{bmatrix} = \begin{bmatrix} \frac{13}{36} \\ \frac{677}{936} \end{bmatrix}$$

Augmented matrix form:

$$\left[\begin{array}{cc|c} 1 & -0.3462 & 0.3611 \\ -0.3462 & 1 & 0.7233 \end{array}\right]$$

8.4 Elliptic Equation – Laplace's equation

Solution:

Step 6: Form a linear system for internal points and solve it using Gauss-Seidel or Gauss Elimination.

By using Gauss elimination,

$$\left[\begin{array}{cc|c} 1 & -0.3462 & 0.3611 \\ -0.3462 & 1 & 0.7233 \end{array}\right] \xrightarrow{0.3462r_1+r_2} \left[\begin{array}{cc|c} 1 & -0.3462 & 0.3611 \\ 0 & 0.8801 & 0.8483 \end{array}\right]$$

By backward substitution:

$$0.8801u_{2,1} = 0.8483$$
$$u_{2,1} = 0.9639$$

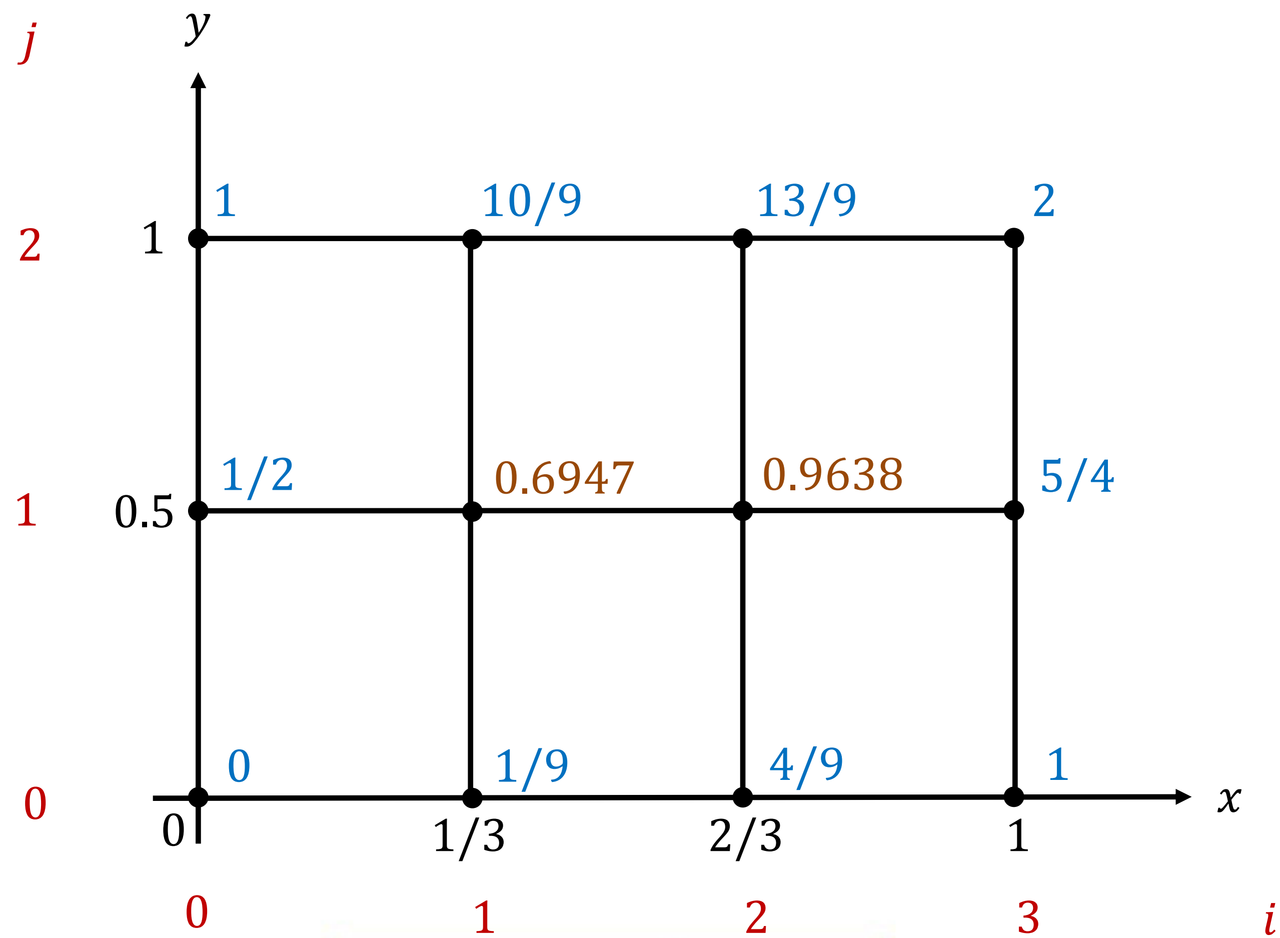
$$u_{1,1} - 0.3462u_{2,1} = 0.3611$$
$$u_{1,1} - 0.3462(0.9639) = 0.3611$$
$$u_{1,1} = 0.6948$$

$$\therefore \begin{bmatrix} u_{1,1} \\ u_{2,1} \end{bmatrix} = \begin{bmatrix} 0.6948 \\ 0.9639 \end{bmatrix}$$

8.4 Elliptic Equation – Laplace's equation

Solution:

Step 7: Complete the grid.



8.4 Elliptic Equation – Laplace's equation

Illustrative Example 2:

Find approximation solutions for Laplace's Equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \qquad 0 \le x \le 1, \qquad 0 \le y \le 1$$

with boundary conditions

$$u(x,0) = x^2, \qquad u(x,1) = x^2 + 1, \qquad 0 \le x \le 1$$

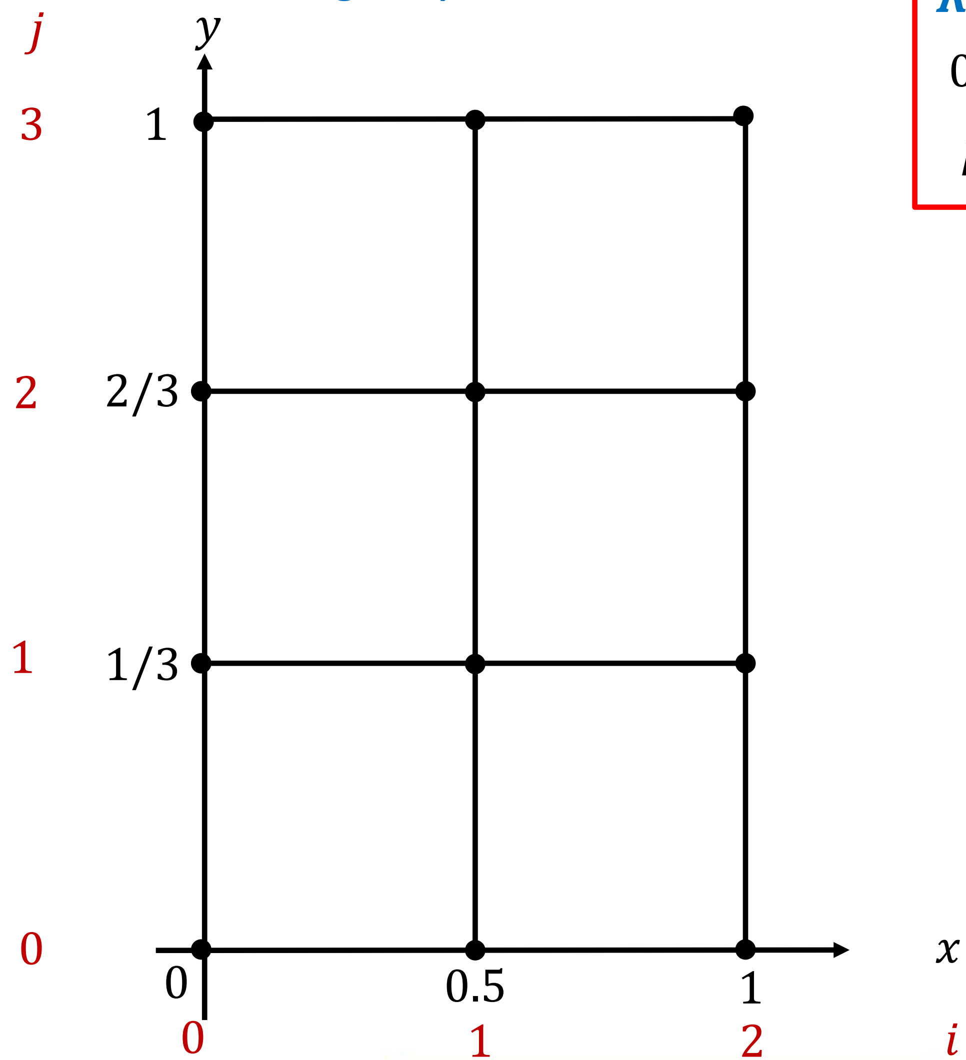
$$u(0,y) = y, \qquad u(1,y) = 1 + y^2, \qquad 0 \le y \le 1$$

by using central-difference formula and Gauss elimination method. Given $h = 1/2$ and $k = 1/3$.

8.4 Elliptic Equation – Laplace's equation

Solution:

Step 1: Sketch the grid points.



Recall:

$0 \leq x \leq 1, \quad 0 \leq y \leq 1$

$h = 1/2$ and $k = 1/3$

8.4 Elliptic Equation – Laplace's equation

Solution:

Step 2: Compute boundary values.

Recall:
$h = 1/2$ and $k = 1/3$

$\boldsymbol{u(x,0) = x^2,} \qquad \boldsymbol{u(x,1) = x^2 + 1,} \qquad \boldsymbol{0 \le x \le 1}$

$u(0,0) = 0 \qquad u(0,1) = 1$

$u(0.5,0) = 1/4 \qquad u(0.5,1) = 5/4$

$u(1,0) = 1 \qquad u(1,1) = 2$

$\boldsymbol{u(0,y) = y,} \qquad \boldsymbol{u(1,y) = 1 + y^2,} \qquad \boldsymbol{0 \le y \le 1}$

$u(0,0) = 0 \qquad u(1,0) = 1$

$u(0,1/3) = 1/3 \qquad u(1,1/3) = 10/9$

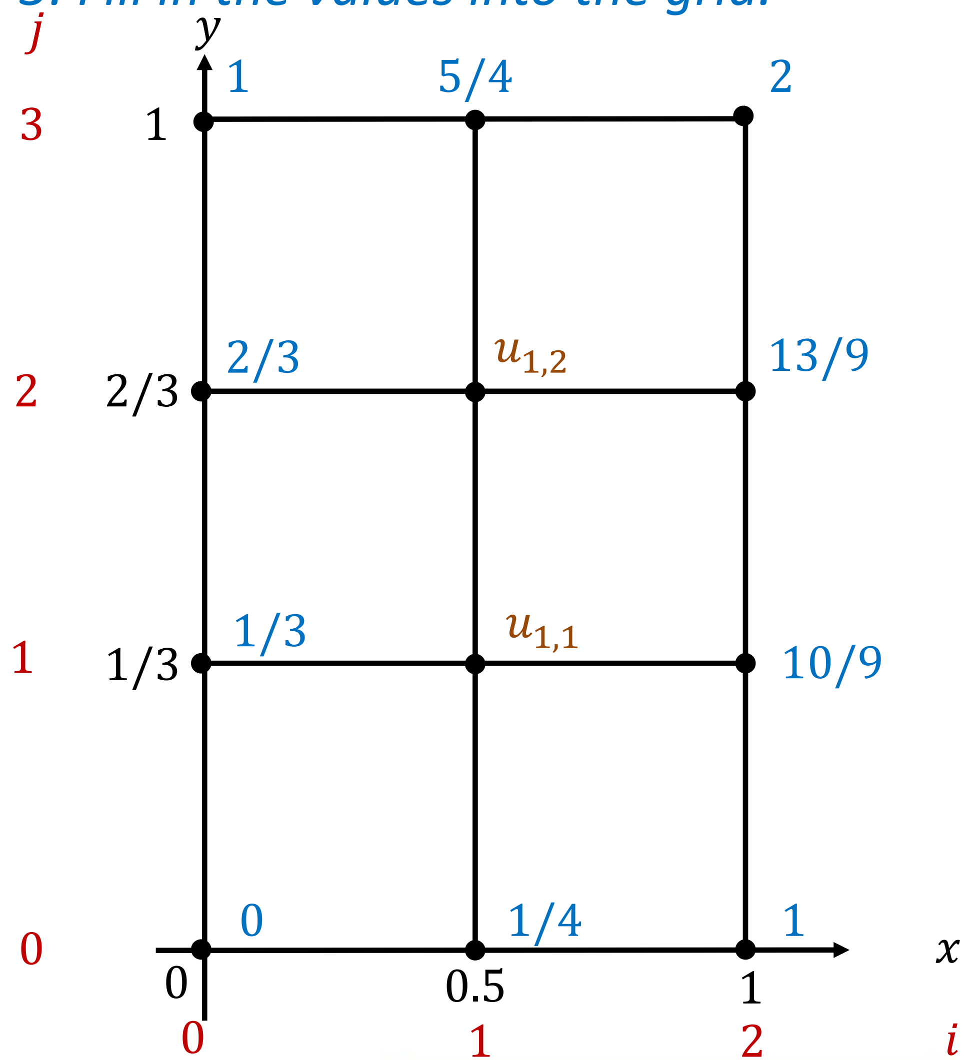
$u(0,2/3) = 2/3 \qquad u(1,2/3) = 13/9$

$u(0,1) = 1 \qquad u(1,1) = 2$

8.4 Elliptic Equation – Laplace's equation

Solution:

Step 3: Fill in the values into the grid.



8.4 Elliptic Equation – Laplace's equation

Solution:

Step 4: Obtain formula $u_{i,j}$ from difference formula.

By central-difference formulas,

$$\left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j} + \left(\frac{\partial^2 u}{\partial y^2}\right)_{i,j} = 0$$

Recall:
$h = 1/2$ and $k = 1/3$

is approximated to

$$\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} + \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{k^2} = 0$$

$$\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{(1/2)^2} + \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{(1/3)^2} = 0$$

$$4u_{i+1,j} - 8u_{i,j} + 4u_{i-1,j} + 9u_{i,j+1} - 18u_{i,j} + 9u_{i,j-1} = 0$$

$$u_{i,j} = \frac{2}{13}u_{i+1,j} + \frac{2}{13}u_{i-1,j} + \frac{9}{26}u_{i,j+1} + \frac{9}{26}u_{i,j-1}$$

8.4 Elliptic Equation – Laplace's equation

Solution:

Step 5: Compute equations for internal points $u_{i,j}$
(Refer to grid in Step 3).

$$u_{i,j} = \frac{2}{13}u_{i+1,j} + \frac{2}{13}u_{i-1,j} + \frac{9}{26}u_{i,j+1} + \frac{9}{26}u_{i,j-1}$$

$$u_{1,1} = \frac{2}{13}u_{2,1} + \frac{2}{13}u_{0,1} + \frac{9}{26}u_{1,2} + \frac{9}{26}u_{1,0}$$

$$u_{1,1} = \frac{2}{13}(\frac{10}{9}) + \frac{2}{13}(\frac{1}{3}) + \frac{9}{26}u_{1,2} + \frac{9}{26}(\frac{1}{4})$$

$$\therefore u_{1,1} - \frac{9}{26}u_{1,2} = \frac{289}{936}$$

$$u_{1,2} = \frac{2}{13}u_{2,2} + \frac{2}{13}u_{0,2} + \frac{9}{26}u_{1,3} + \frac{9}{26}u_{1,1}$$

$$u_{1,2} = \frac{2}{13}(\frac{13}{9}) + \frac{2}{13}(\frac{2}{3}) + \frac{9}{26}(\frac{5}{4}) + \frac{9}{26}u_{1,1}$$

$$\therefore -\frac{9}{26}u_{1,1} + u_{1,2} = \frac{709}{936}$$

8.4 Elliptic Equation – Laplace's equation

Solution:

Step 6: Form a linear system for internal points and solve it using Gauss-Seidel or Gauss Elimination.

$$u_{1,1} - \frac{9}{26}u_{1,2} = \frac{289}{936}$$
$$-\frac{9}{26}u_{1,1} + u_{1,2} = \frac{709}{936}$$

$$\begin{bmatrix} 1 & -\frac{9}{26} \\ -\frac{9}{26} & 1 \end{bmatrix}\begin{bmatrix} u_{1,1} \\ u_{2,1} \end{bmatrix} = \begin{bmatrix} \frac{289}{936} \\ \frac{709}{936} \end{bmatrix}$$

Augmented matrix form:

$$\left[\begin{array}{cc|c} 1 & -0.3462 & 0.3611 \\ -0.3462 & 1 & 0.7233 \end{array}\right]$$

8.4 Elliptic Equation – Laplace's equation

Solution:

Step 6: Form a linear system for internal points and solve it using Gauss-Seidel or Gauss Elimination.

By using Gauss elimination,

$$\left[\begin{array}{cc|c} 1 & -0.3462 & 0.3088 \\ -0.3462 & 1 & 0.7575 \end{array}\right] \xrightarrow{0.3462r_1+r_2} \left[\begin{array}{cc|c} 1 & -0.3462 & 0.3088 \\ 0 & 0.8801 & 0.8644 \end{array}\right]$$

By backward substitution:

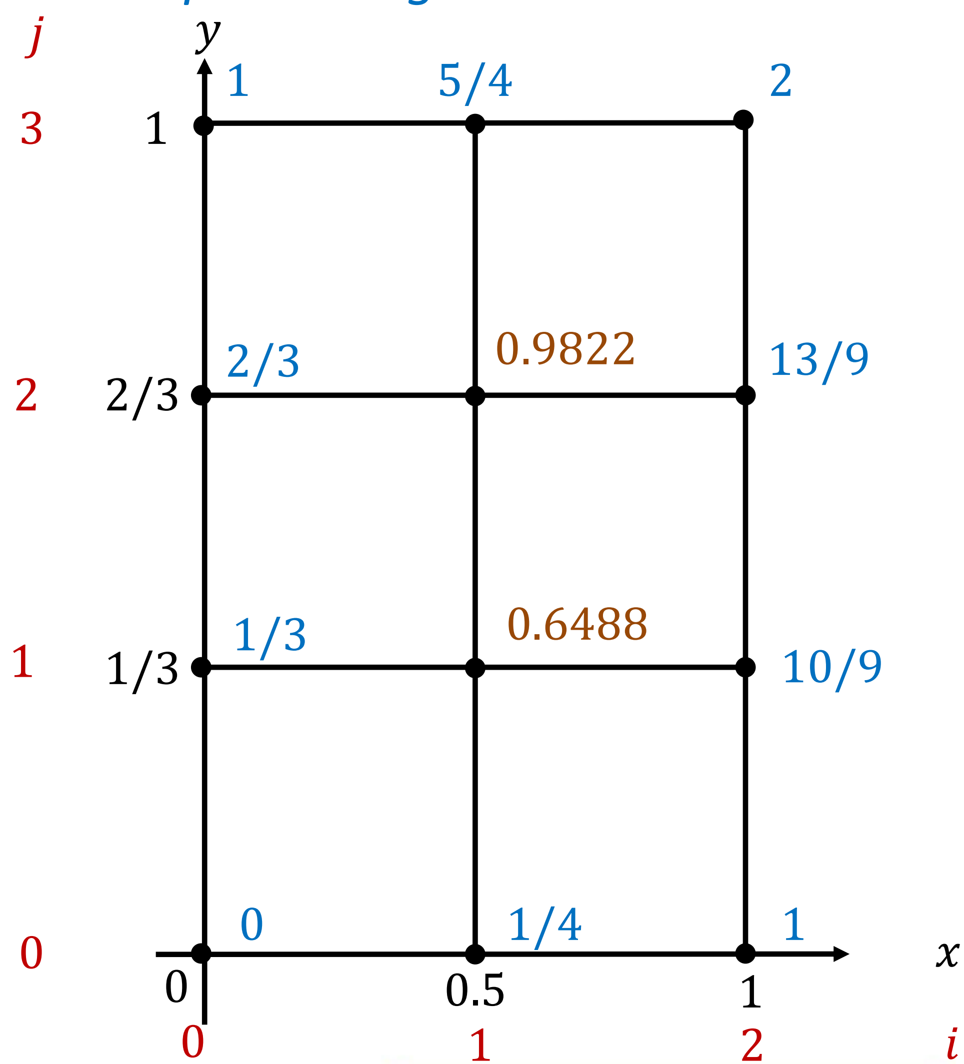
$$0.8801u_{1,2} = 0.8644$$
$$u_{1,2} = 0.9822$$

$$u_{1,1} - 0.3462u_{1,2} = 0.3088$$
$$u_{1,1} - 0.3462(0.9822) = 0.3088$$
$$u_{1,1} = 0.6488$$

$$\therefore \begin{bmatrix} u_{1,1} \\ u_{1,2} \end{bmatrix} = \begin{bmatrix} 0.6488 \\ 0.9822 \end{bmatrix}$$

Solution:

Step 7: Complete the grid.



8.4 Elliptic Equation – Laplace's equation

Illustrative Example 3:

Find approximation solutions for Laplace's Equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \qquad 0 \le x \le 1, \qquad 0 \le y \le 1$$

with boundary conditions

$$u(x,0) = 2x, \qquad u(x,1) = 2, \qquad 0 \le x \le 1$$

$$u(0,y) = 2y, \qquad u(1,y) = 2, \qquad 0 \le y \le 1$$

by using central-difference formula and Gauss elimination method. Given $h = 0.25$ and $k = 0.5$.

8.4 Elliptic Equation – Laplace's equation

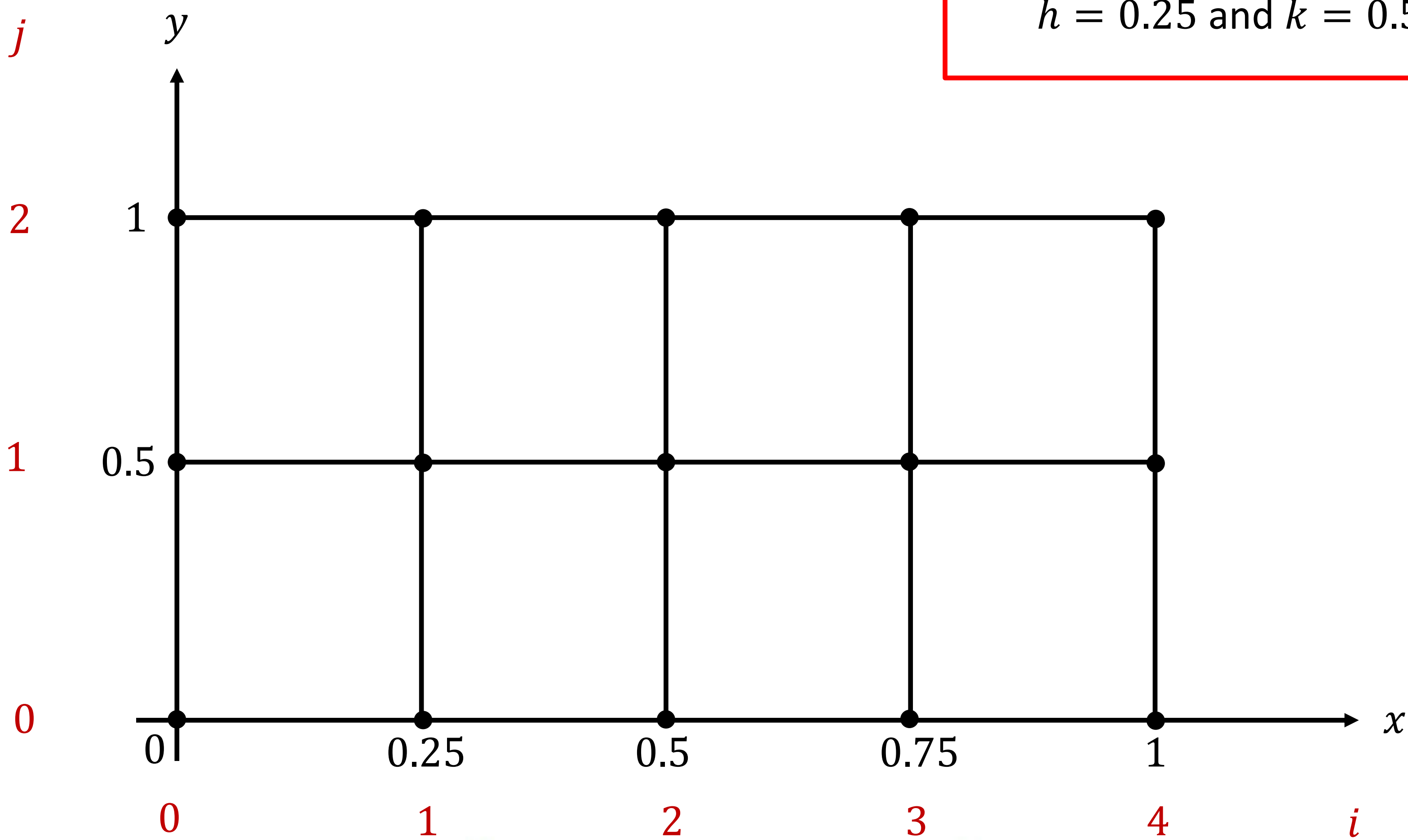
Solution:

Step 1: Sketch the grid points.

Recall:

$0 \leq x \leq 1, \qquad 0 \leq y \leq 1$

$h = 0.25$ and $k = 0.5$



8.4 Elliptic Equation – Laplace's equation

Solution:

Step 2: Compute boundary values.

> ***Recall:***
> $h = 0.25$ and $k = 0.5$

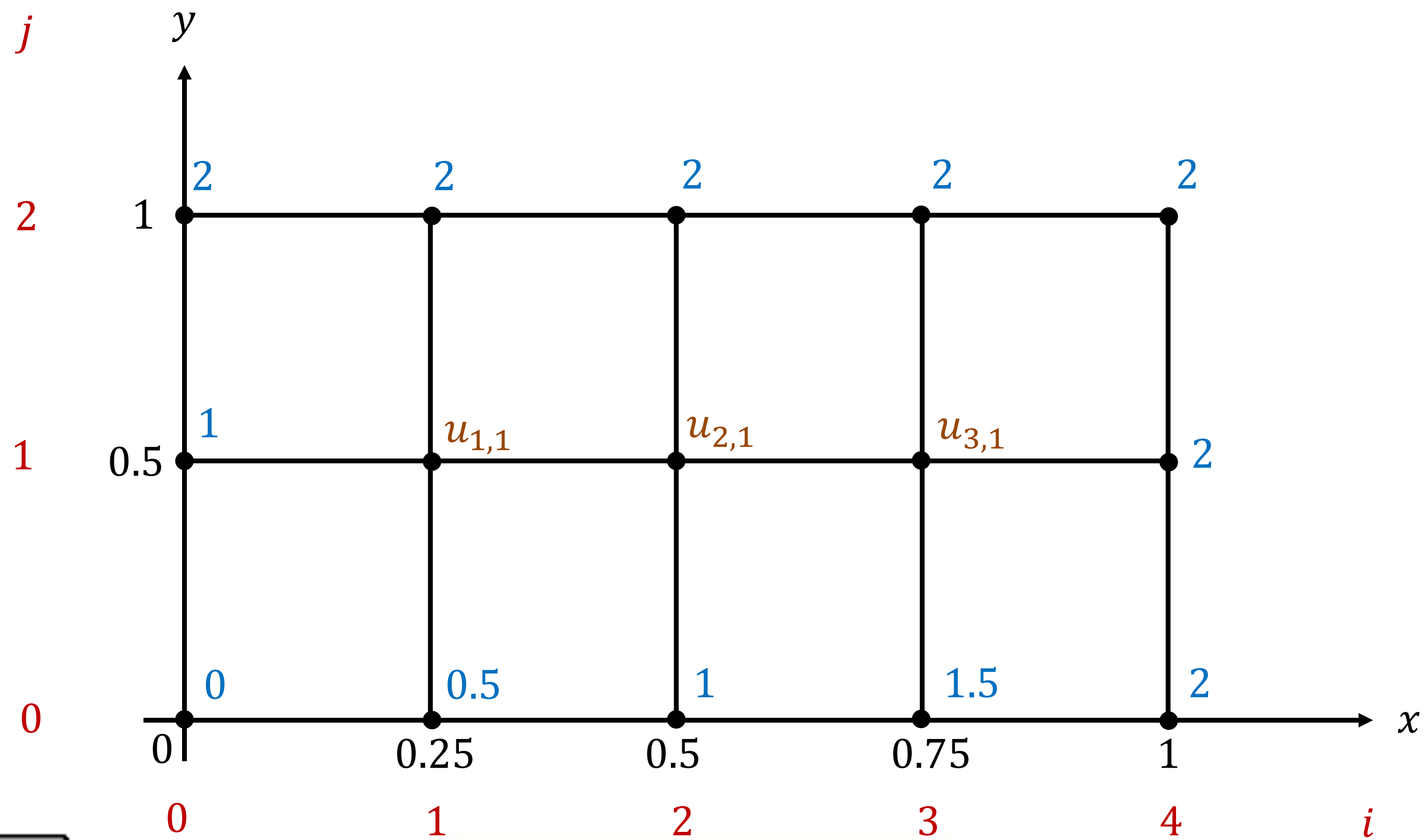
$\boldsymbol{u(x,0) = 2x,}$	$\boldsymbol{u(x,1) = 2,}$	$\boldsymbol{0 \le x \le 1}$
$u(0,0) = 0$	$u(0,1) = 2$	
$u(0.25,0) = 0.5$	$u(0.25,1) = 2$	
$u(0.5,0) = 1$	$u(0.5,1) = 2$	
$u(0.75,0) = 1.5$	$u(0.75,1) = 2$	
$u(1,0) = 2$	$u(1,1) = 2$	

$\boldsymbol{u(0,y) = 2y,}$	$\boldsymbol{u(1,y) = 2,}$	$\boldsymbol{0 \le y \le 1}$
$u(0,0) = 0$	$u(1,0) = 2$	
$u(0,0.5) = 1$	$u(1,0.5) = 2$	
$u(0,1) = 2$	$u(1,1) = 2$	

8.4 Elliptic Equation – Laplace's equation

Solution:

Step 3: Fill in the values into the grid.



8.4 Elliptic Equation – Laplace's equation

Solution:

Step 4: Obtain formula $u_{i,j}$ from difference formula.

By central-difference formulas,

$$\left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j} + \left(\frac{\partial^2 u}{\partial y^2}\right)_{i,j} = 0$$

Recall:
$h = 0.25$ and $k = 0.5$

is approximated to

$$\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} + \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{k^2} = 0$$

$$\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{(0.25)^2} + \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{(0.5)^2} = 0$$

$$16u_{i+1,j} - 32u_{i,j} + 16u_{i-1,j} + 4u_{i,j+1} - 8u_{i,j} + 4u_{i,j-1} = 0$$

$$u_{i,j} = \frac{2}{5}u_{i+1,j} + \frac{2}{5}u_{i-1,j} + \frac{1}{10}u_{i,j+1} + \frac{1}{10}u_{i,j-1}$$

8.4 Elliptic Equation – Laplace's equation

Solution:

Step 5: Compute equations for internal points $u_{i,j}$
(Refer to grid in Step 3).

$$u_{i,j} = \frac{2}{5}u_{i+1,j} + \frac{2}{5}u_{i-1,j} + \frac{1}{10}u_{i,j+1} + \frac{1}{10}u_{i,j-1}$$

$$u_{1,1} = \frac{2}{5}u_{2,1} + \frac{2}{5}u_{0,1} + \frac{1}{10}u_{1,2} + \frac{1}{10}u_{1,0}$$

$$u_{1,1} = \frac{2}{5}u_{2,1} + \frac{2}{5}(1) + \frac{1}{10}(2) + \frac{1}{10}(0.5)$$

$$\therefore u_{1,1} - \frac{2}{5}u_{2,1} = 0.65$$

$$u_{2,1} = \frac{2}{5}u_{3,1} + \frac{2}{5}u_{1,1} + \frac{1}{10}u_{2,2} + \frac{1}{10}u_{2,0}$$

$$u_{2,1} = \frac{2}{5}u_{3,1} + \frac{2}{5}u_{1,1} + \frac{1}{10}(2) + \frac{1}{10}(1)$$

$$\therefore -\frac{2}{5}u_{1,1} + u_{2,1} - \frac{2}{5}u_{3,1} = 0.3$$

$$u_{3,1} = \frac{2}{5}u_{4,1} + \frac{2}{5}u_{2,1} + \frac{1}{10}u_{3,2} + \frac{1}{10}u_{3,0}$$

$$u_{3,1} = \frac{2}{5}(2) + \frac{2}{5}u_{2,1} + \frac{1}{10}(2) + \frac{1}{10}(1.5)$$

$$\therefore -\frac{2}{5}u_{2,1} + u_{3,1} = 1.15$$

8.4 Elliptic Equation – Laplace's equation

Solution:

Step 6: Form a linear system for internal points and solve it using Gauss-Seidel or Gauss Elimination.

$$u_{1,1} - \frac{2}{5}u_{2,1} = 0.65$$

$$-\frac{2}{5}u_{1,1} + u_{2,1} - \frac{2}{5}u_{3,1} = 0.3$$

$$-\frac{2}{5}u_{2,1} + u_{3,1} = 1.15$$

$$\begin{bmatrix} 1 & -0.4 & 0 \\ -0.4 & 1 & -0.4 \\ 0 & -0.4 & 1 \end{bmatrix} \begin{bmatrix} u_{1,1} \\ u_{2,1} \\ u_{3,1} \end{bmatrix} = \begin{bmatrix} 0.65 \\ 0.3 \\ 1.15 \end{bmatrix}$$

By using Gauss elimination,

$$\begin{bmatrix} u_{1,1} \\ u_{2,1} \\ u_{3,1} \end{bmatrix} = \begin{bmatrix} 1.25 \\ 1.5 \\ 1.75 \end{bmatrix}$$

8.4 Elliptic Equation – Laplace's equation

Solution:

Step 6: Form a linear system for internal points and solve it using Gauss-Seidel or Gauss Elimination.

By using Gauss elimination,

$$\left[\begin{array}{ccc|c} 1 & -0.4 & 0 & 0.65 \\ -0.4 & 1 & -0.4 & 0.3 \\ 0 & -0.4 & 1 & 1.15 \end{array}\right] \xrightarrow{0.4r_1+r_2} \left[\begin{array}{ccc|c} 1 & -0.4 & 0 & 0.65 \\ 0 & 0.84 & -0.4 & 0.56 \\ 0 & -0.4 & 1 & 1.15 \end{array}\right]$$

$$\xrightarrow{0.4762r_2+r_3} \left[\begin{array}{ccc|c} 1 & -0.4 & 0 & 0.65 \\ 0 & 0.84 & -0.4 & 0.56 \\ 0 & 0 & 0.8095 & 1.4167 \end{array}\right]$$

$$\therefore \begin{bmatrix} u_{1,1} \\ u_{2,1} \\ u_{3,1} \end{bmatrix} = \begin{bmatrix} 1.2500 \\ 1.5000 \\ 1.7501 \end{bmatrix}$$

8.4 Elliptic Equation – Laplace's equation

Solution:

Step 6: Form a linear system for internal points and solve it using Gauss-Seidel or Gauss Elimination.

Recall:

$$\left[\begin{array}{ccc|c} 1 & -0.4 & 0 & 0.65 \\ 0 & 0.84 & -0.4 & 0.56 \\ 0 & 0 & 0.8095 & 1.4167 \end{array}\right]$$

By backward substitution,

$$0.8095u_{3,1} = 1.4167$$
$$u_{3,1} = 1.7501$$

$$0.84u_{2,1} - 0.4u_{3,1} = 0.56$$
$$0.84u_{2,1} - 0.4(1.7501) = 0.56$$
$$u_{2,1} = 1.5000$$

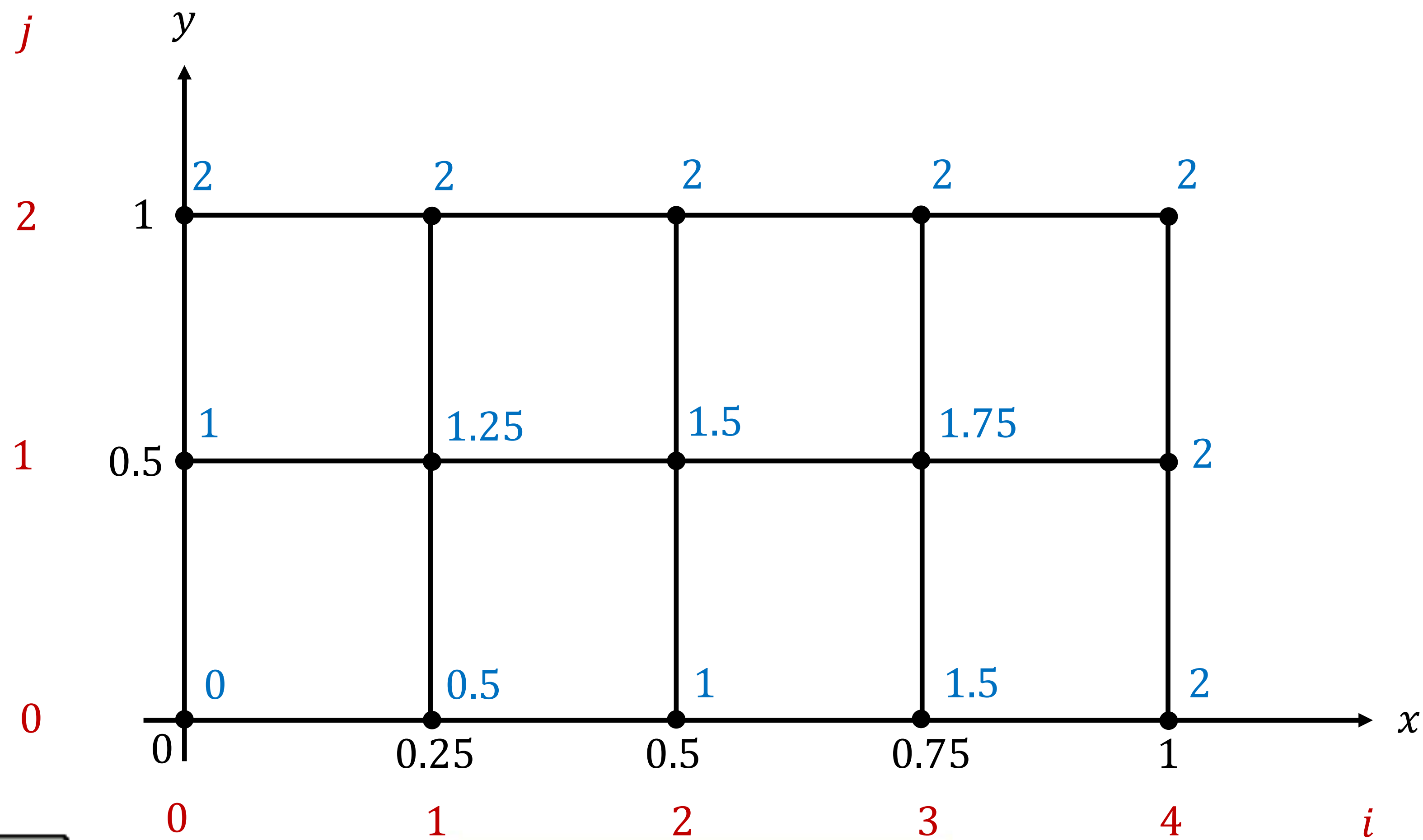
$$u_{1,1} - 0.4u_{2,1} = 0.65$$
$$u_{1,1} - 0.4(1.5) = 0.65$$
$$u_{1,1} = 1.2500$$

$$\therefore \begin{bmatrix} u_{1,1} \\ u_{2,1} \\ u_{3,1} \end{bmatrix} = \begin{bmatrix} 1.2500 \\ 1.5000 \\ 1.7501 \end{bmatrix}$$

8.4 Elliptic Equation – Laplace's equation

Solution:

Step 7: Complete the grid.



8.4 Elliptic Equation – Laplace's equation

Exercise 8.3:

Find approximation solutions for Laplace's Equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \qquad 0 \leq x \leq 1, \qquad 0 \leq y \leq 1$$

with boundary conditions

$$u(x,0) = 0, \qquad u(x,1) = x^2, \qquad 0 \leq x \leq 1$$

$$u(0,y) = 0, \qquad u(1,y) = y^2, \qquad 0 \leq y \leq 1$$

by using central-difference formula and Gauss elimination method. Given $h = 0.5$ and $k = 0.25$.