NUMERICAL METHODS
BEKG 2452
SOLUTION OF PARTIAL DIFFERENTIAL EQUATIONS (PARABOLIC AND HYPERBOLIC EQUATIONS)

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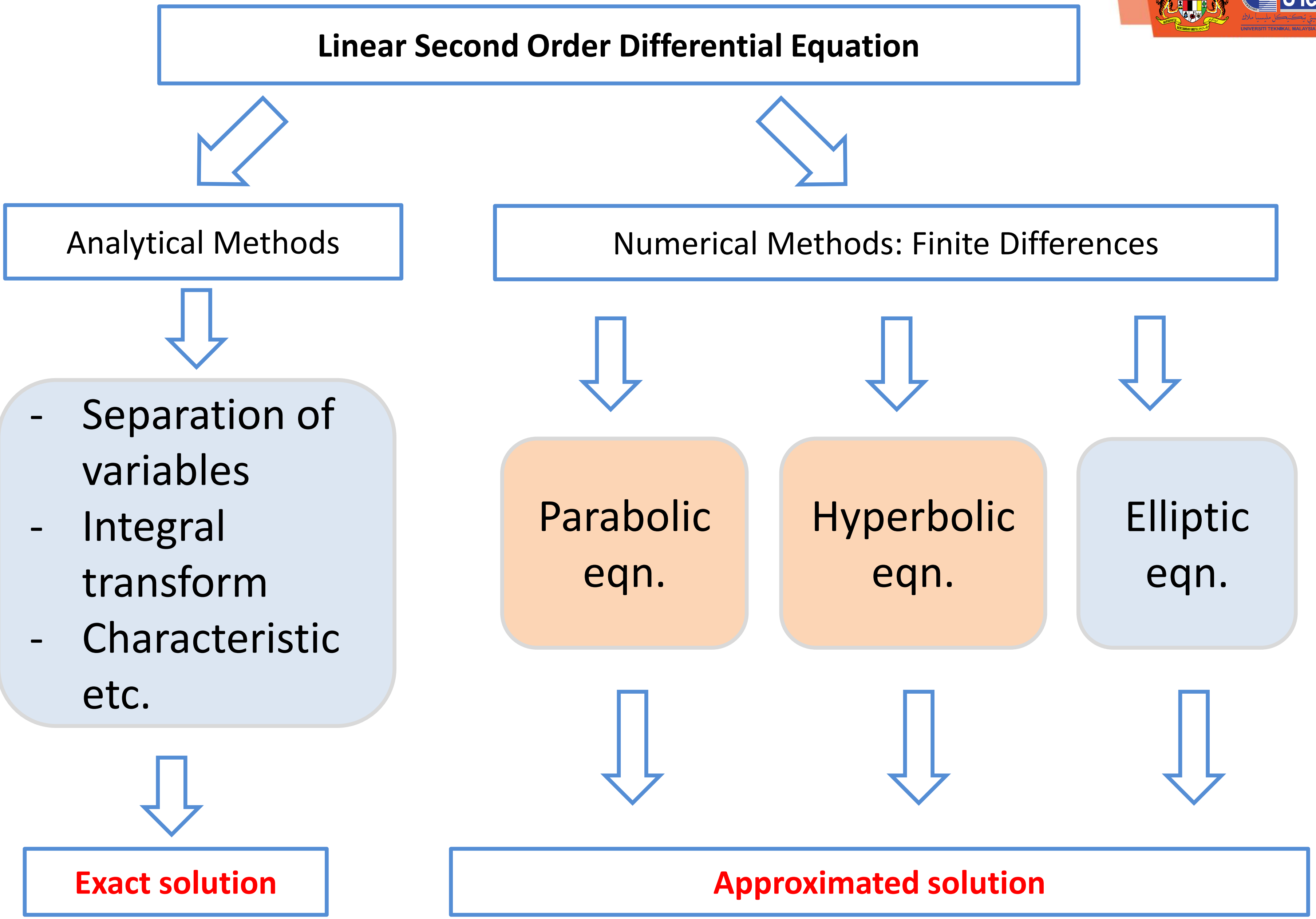
Lesson Outcome

Upon completion of this lesson, the student should be able to:

1. Categorize the types of partial differential equations.
2. Solve numerically heat and wave equations.


Linear Second Order Differential Equation
Analytical Methods
Numerical Methods: Finite Differences
- Separation of variables
- Integral transform
- Characteristic etc.
Parabolic eqn.
Hyperbolic eqn.
Elliptic eqn.
Exact solution
Approximated solution

8.1 Introduction

A partial differential equation involves partial derivatives of an unknown of 2 or more independent variables.

General form of linear second-order PDE:

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial y \partial x} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu + G = 0 \quad (8.1)$$

There are three types of PDE equations for Eqn (8.1):

1) Parabolic if $B^2 - 4AC = 0$
2) Hyperbolic if $B^2 - 4AC > 0$
3) Elliptic if $B^2 - 4AC < 0$

8.1.1 Classification of PDE

Example:

Classify the category of PDE $4\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial y^2} + 3\frac{\partial u}{\partial y} = 0.$

Solution:

$$A = 4, B = 0, C = 2$$
$$B^2 - 4AC = 0 - 4(4)(2) = -32 < 0$$

Hence, it is an elliptic equation.

Example 8.1:

Classify the category of PDE for the following equations.

1) $3u_{xx} - u_{xy} + 4u_{yy} - 6u_x + u_y = 0$

2) $-\frac{\partial^2 u}{\partial y \partial x} + 3\frac{\partial^2 u}{\partial y^2} + 6\frac{\partial u}{\partial x} + 2u = 0$

3) $u_{xx} + 4u_{xy} + 4u_{yy} + 5u = 0$

Solution:

1) $3u_{xx} - u_{xy} + 4u_{yy} - 6u_x + u_y = 0$

$$A = 3, B = -1, C = 4$$

$$B^2 - 4AC = 1 - 4(3)(4) = -47 < 0$$

∴ Elliptic equation

2) $-\frac{\partial^2 u}{\partial y \partial x} + 3\frac{\partial^2 u}{\partial y^2} + 6\frac{\partial u}{\partial x} + 2u = 0$

$$A = 0, B = -1, C = 3$$

$$B^2 - 4AC = 1 - 4(0)(3) = 1 > 0$$

∴ Hyperbolic equation

3) $u_{xx} + 4u_{xy} + 4u_{yy} + 5u = 0$

$$A = 1, B = 4, C = 4$$

$$B^2 - 4AC = 16 - 4(1)(4) = 0$$

∴ Parabolic equation

8.1.2 Explicit Finite Difference Method

Domain of the problem is subdivided into a mesh of discrete points for each of the independent variables

↓

Derivatives are replaced by appropriate difference quotients

↓

For Elliptic equation:
Resulting linear system is solved by Gauss-Seidel or Gauss Elimination

8.1.3 Difference Formulas

Forward-Difference Formula:

$$\left(\frac{\partial u}{\partial t}\right)_{i,j} = \frac{u_{i,j+1} - u_{i,j}}{k} \qquad \left(\frac{\partial u}{\partial x}\right)_{i,j} = \frac{u_{i+1,j} - u_{i,j}}{h}$$

Backward-Difference Formula:

$$\left(\frac{\partial u}{\partial t}\right)_{i,j} = \frac{u_{i,j} - u_{i,j-1}}{k} \qquad \left(\frac{\partial u}{\partial x}\right)_{i,j} = \frac{u_{i,j} - u_{i-1,j}}{h}$$

Central-Difference Formula:

$$\left(\frac{\partial u}{\partial t}\right)_{i,j} = \frac{u_{i,j+1} - u_{i,j-1}}{2k} \qquad \left(\frac{\partial u}{\partial x}\right)_{i,j} = \frac{u_{i+1,j} - u_{i-1,j}}{2h}$$

$$\left(\frac{\partial^2 u}{\partial t^2}\right)_{i,j} = \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{k^2} \qquad \left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2}$$

8.2 Parabolic Equation – Heat Equation

Consider the **Heat Equation**:

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < a, \qquad t > 0$$

with boundary condition

$$u(0,t) = u(a,t) = 0, \qquad t > 0$$

and initial condition

$$u(x,0) = f(x), \qquad 0 \le x \le a.$$

By forward-difference and central-difference formulas,

$$\left(\frac{\partial u}{\partial t}\right)_{i,j} - c^2 \left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j} = 0$$

is approximated to

$$\boxed{\frac{u_{i,j+1} - u_{i,j}}{k} - c^2 \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} = 0}$$

where $h = \Delta x$ and $k = \Delta t$.

8.2 Parabolic Equation – Heat Equation

Illustrative Example 1:

Find approximation solutions for Heat Equation

$$\frac{\partial u}{\partial t} = 4\frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < 1, \qquad t > 0$$

with boundary condition

$$u(0,t) = u(1,t) = 0, \qquad 0 \le t \le 0.1$$

and initial condition

$$u(x,0) = x\cos\left(\frac{\pi x}{2}\right) \quad 0 \le x \le 1.$$

by using forward-difference and central-difference formula.

Given $h = 0.25$ and $k = 0.05$.

8.2 Parabolic Equation – Heat Equation

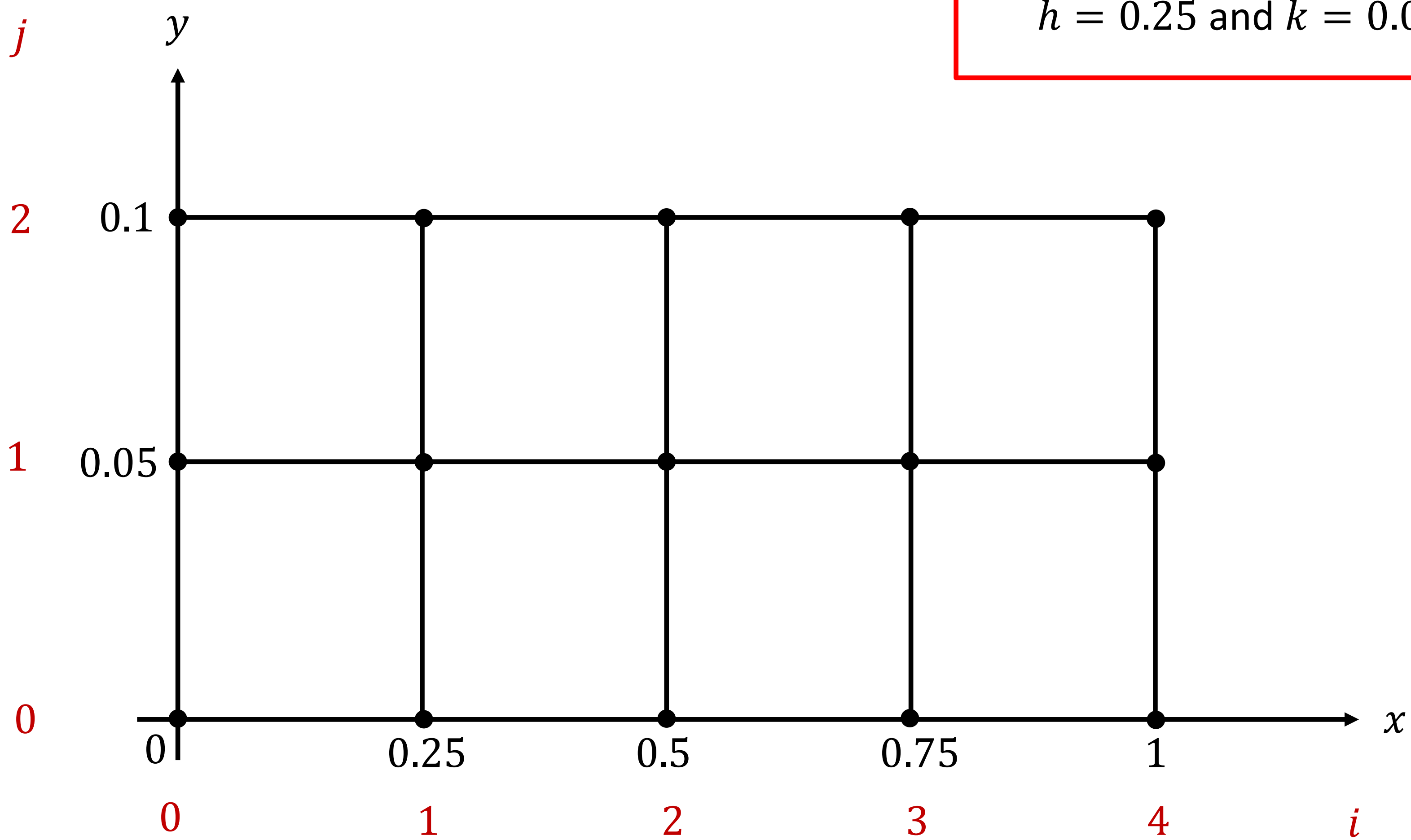
Solution:

Step 1: Sketch the grid points.

Recall:

$0 \leq x \leq 1, \qquad 0 \leq t \leq 0.1$

$h = 0.25$ and $k = 0.05$



8.2 Parabolic Equation – Heat Equation

Solution:

Step 2: Compute boundary and initial values.

Recall:
$h = 0.25$ and $k = 0.05$

$$\boldsymbol{u(x,0) = x\cos\left(\frac{\pi x}{2}\right), \qquad 0 \le x \le 1}$$

$$u(0,0) = 0$$
$$u(0.25,0) = 0.2310$$
$$u(0.5,0) = 0.3536$$
$$u(0.75,0) = 0.2870$$
$$u(1,0) = 0$$

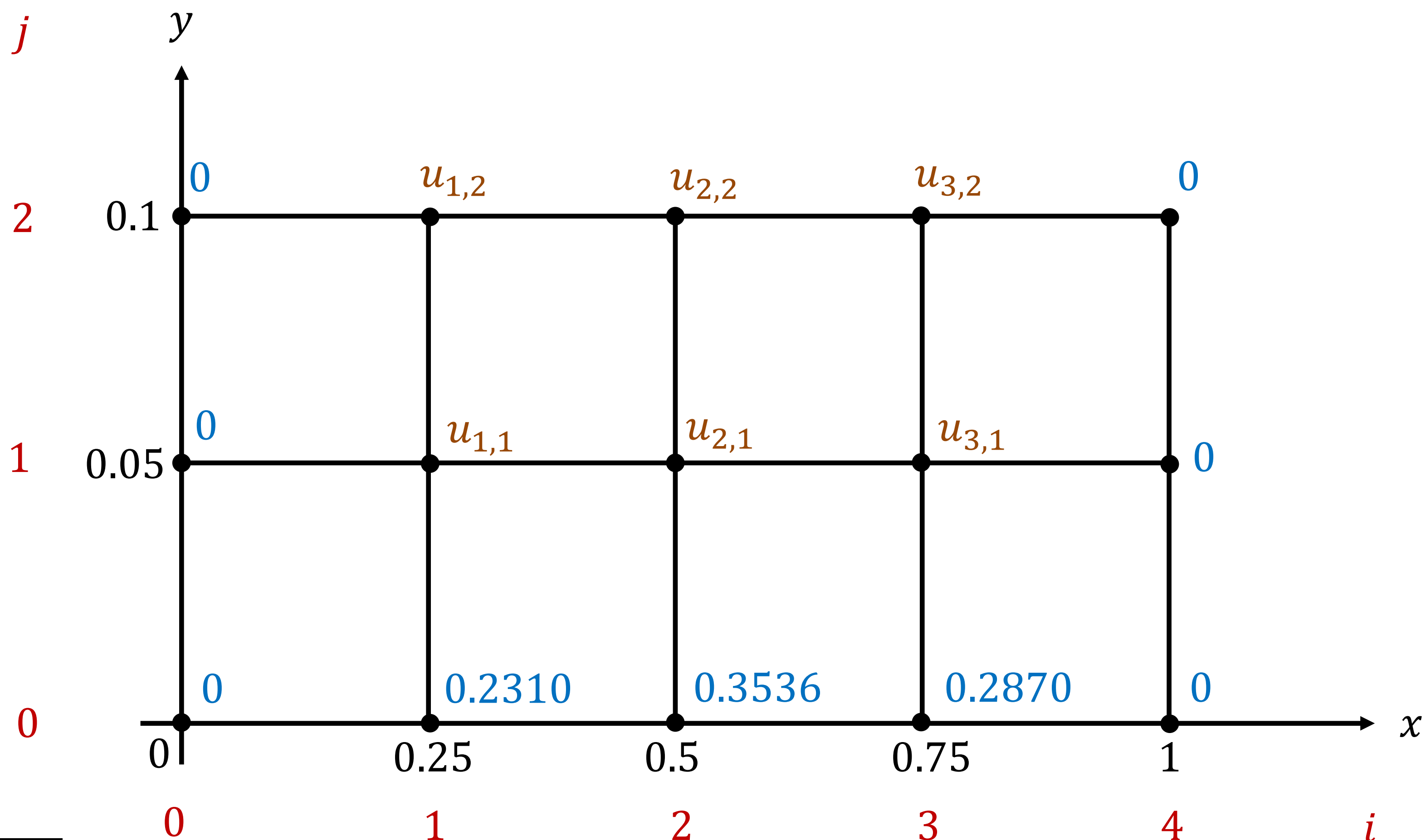
$$\boldsymbol{u(0,t) = 0, \qquad u(1,t) = 0, \qquad 0 \le t \le 0.1}$$

$u(0,0) = 0$	$u(1,0) = 0$
$u(0,0.05) = 0$	$u(1,0.05) = 0$
$u(0,0.1) = 0$	$u(1,0.1) = 0$

8.2 Parabolic Equation – Heat Equation

Solution:

Step 3: Fill in the values into the grid.



8.2 Parabolic Equation – Heat Equation

Solution:

Step 4: Obtain formula $u_{i,j}$ from difference formula.

By forward-difference and central-difference formulas,

$$\left(\frac{\partial u}{\partial t}\right)_{i,j} - 4\left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j} = 0$$

Recall:
$h = 0.25$ and $k = 0.5$

is approximated to

$$\frac{u_{i,j+1} - u_{i,j}}{k} - 4\left(\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2}\right) = 0$$

$$\frac{u_{i,j+1} - u_{i,j}}{0.05} - 4\left(\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{(0.25)^2}\right) = 0$$

$$u_{i,j+1} - u_{i,j} - \frac{4(0.05)}{(0.25)^2}(u_{i+1,j} - 2u_{i,j} + u_{i-1,j}) = 0$$

$$u_{i,j+1} = u_{i,j} + 3.2(u_{i+1,j} - 2u_{i,j} + u_{i-1,j})$$

8.2 Parabolic Equation – Heat Equation

Solution:

Step 5: Compute equations for internal points $u_{i,j}$
(Refer to grid in Step 3).

$$u_{i,j+1} = u_{i,j} + 3.2(u_{i+1,j} - 2u_{i,j} + u_{i-1,j})$$

$u_{1,1} = u_{1,0} + 3.2(u_{2,0} - 2u_{1,0} + u_{0,0})$
$u_{1,1} = 0.2310 + 3.2[0.3536 - 2(0.2310) + 0]$
$u_{1,1} = -0.1159$

$u_{2,1} = u_{2,0} + 3.2(u_{3,0} - 2u_{2,0} + u_{1,0})$
$u_{2,1} = 0.3536 + 3.2[0.2870 - 2(0.3536) + 0.2310]$
$u_{2,1} = -0.2518$

$u_{3,1} = u_{3,0} + 3.2(u_{4,0} - 2u_{3,0} + u_{2,0})$
$u_{3,1} = 0.287 + 3.2[0 - 2(0.2870) + 0.3536]$
$u_{3,1} = -0.4183$

$u_{1,2} = u_{1,1} + 3.2(u_{2,1} - 2u_{1,1} + u_{0,1})$
$u_{1,2} = -0.1159 + 3.2[-0.2518 - 2(-0.1159) + 0]$
$u_{1,2} = -0.1799$

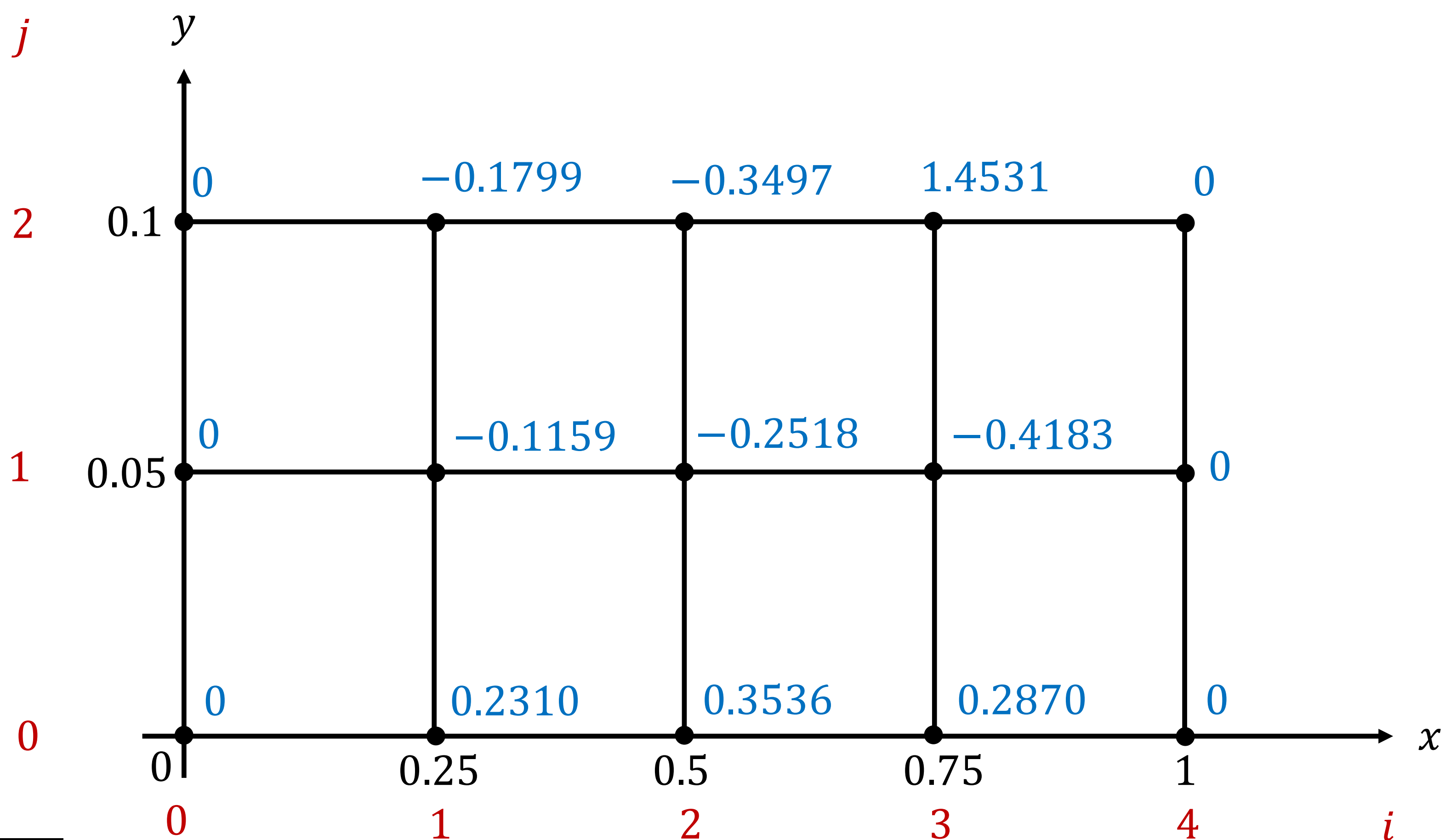
$u_{2,2} = u_{2,1} + 3.2(u_{3,1} - 2u_{2,1} + u_{1,1})$
$u_{2,2} = -0.2518 + 3.2[-0.4183 - 2(-0.2518) - 0.1159]$
$u_{2,2} = -0.3497$

$u_{3,2} = u_{3,1} + 3.2(u_{4,1} - 2u_{3,1} + u_{2,1})$
$u_{3,2} = -0.4183 + 3.2[0 - 2(-0.4183) - 0.2518]$
$u_{3,2} = 1.4531$

8.2 Parabolic Equation – Heat Equation

Solution:

Step 6: Fill in the values into the grid.



8.2 Parabolic Equation – Heat Equation

Illustrative Example 2:

Find approximation solutions for Heat Equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < 1, \qquad t > 0$$

with boundary condition

$$u(0,t) = u(1,t) = 0, \qquad 0 \le t \le 0.4$$

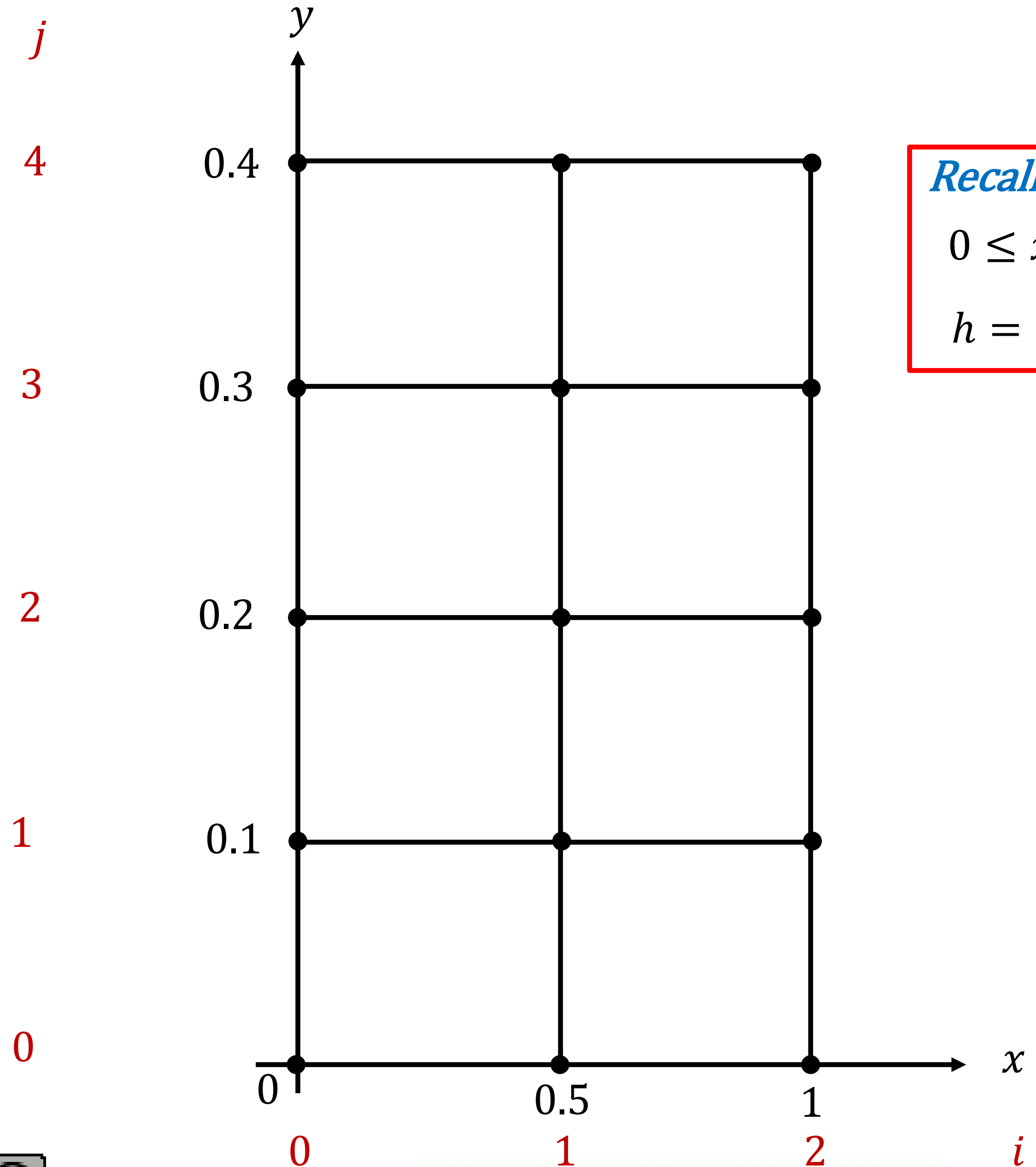
and initial condition

$$u(x,0) = x^2(x-1), \qquad 0 \le x \le 1.$$

by using forward-difference and central-difference formula.

Given $h = 0.5$ and $k = 0.1$.

Solution: *Step 1: Sketch the grid points.*



Recall:

$0 \leq x \leq 1, \qquad 0 \leq t \leq 0.4$

$h = 0.5$ and $k = 0.1$

8.2 Parabolic Equation – Heat Equation

Solution (Cont.):

Step 2: Compute boundary and initial values.

Recall:
$h = 0.5$ and $k = 0.1$

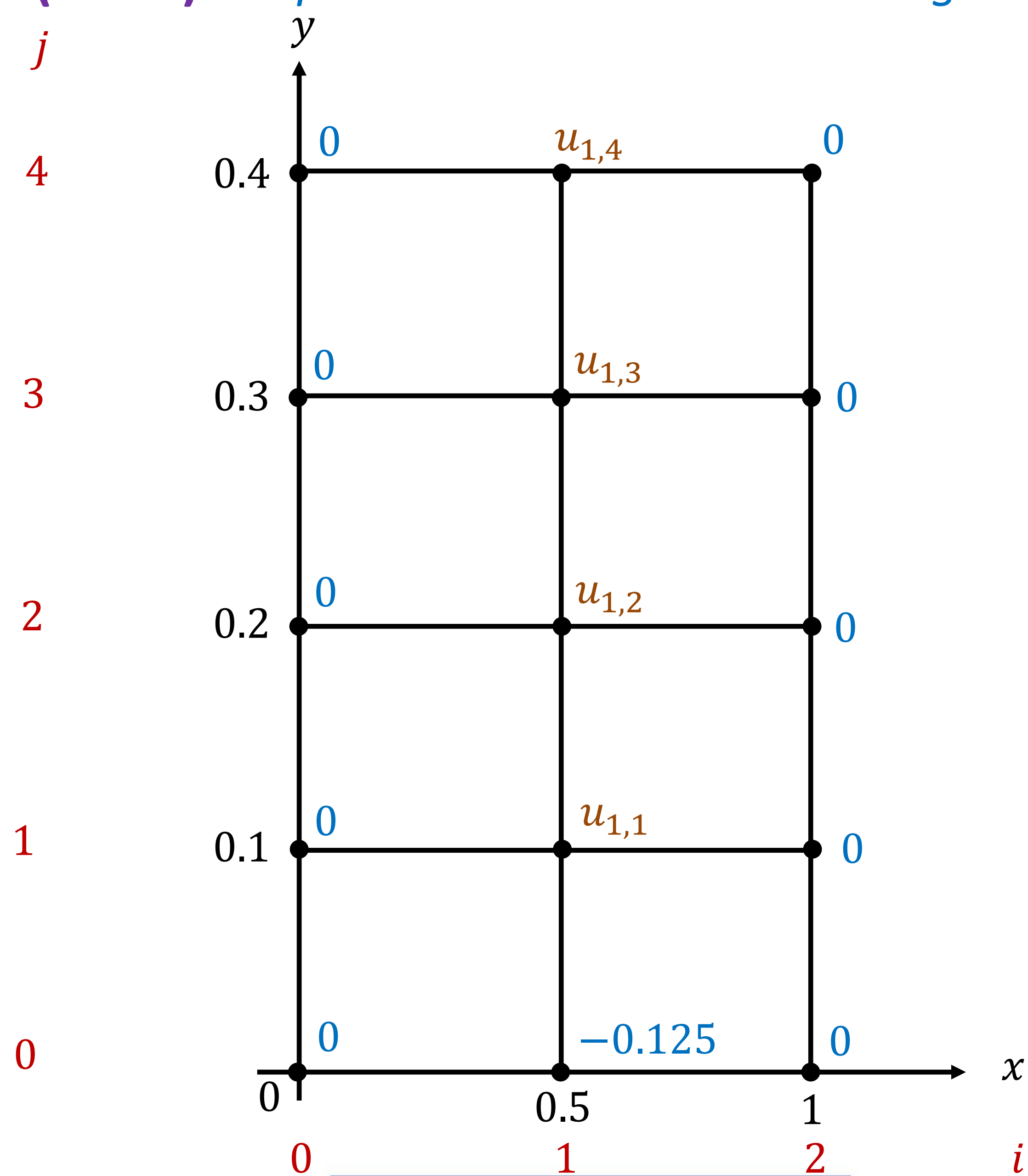
$$\boldsymbol{u(x,0) = x^2(x-1), \qquad 0 \le x \le 1}$$

$$u(0,0) = 0$$
$$u(0.5,0) = -0.125$$
$$u(1,0) = 0$$

$$\boldsymbol{u(0,t) = 0, \qquad u(1,t) = 0, \qquad 0 \le t \le 0.4}$$

$u(0,0) = 0$	$u(1,0) = 0$
$u(0,0.1) = 0$	$u(1,0.1) = 0$
$u(0,0.2) = 0$	$u(1,0.2) = 0$
$u(0,0.3) = 0$	$u(1,0.3) = 0$
$u(0,0.4) = 0$	$u(1,0.4) = 0$

Solution (Cont.): *Step 3: Fill in the values into the grid.*



8.2 Parabolic Equation – Heat Equation

Solution (Cont.):

Step 4: Obtain formula $u_{i,j}$ from difference formula.

By forward-difference and central-difference formulas,

$$\left(\frac{\partial u}{\partial t}\right)_{i,j} - \left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j} = 0$$

Recall:
$h = 0.5$ and $k = 0.1$

is approximated to

$$\frac{u_{i,j+1} - u_{i,j}}{k} - \left(\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2}\right) = 0$$

$$\frac{u_{i,j+1} - u_{i,j}}{0.1} - \left(\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{(0.5)^2}\right) = 0$$

$$u_{i,j+1} - u_{i,j} - \frac{(0.1)}{(0.5)^2}\left(u_{i+1,j} - 2u_{i,j} + u_{i-1,j}\right) = 0$$

$$u_{i,j+1} = u_{i,j} + 0.4\left(u_{i+1,j} - 2u_{i,j} + u_{i-1,j}\right)$$

8.2 Parabolic Equation – Heat Equation

Solution (Cont.):

Step 5: Compute equations for internal points $u_{i,j}$

(Refer to grid in Step 3).

$$u_{i,j+1} = u_{i,j} + 0.4(u_{i+1,j} - 2u_{i,j} + u_{i-1,j})$$

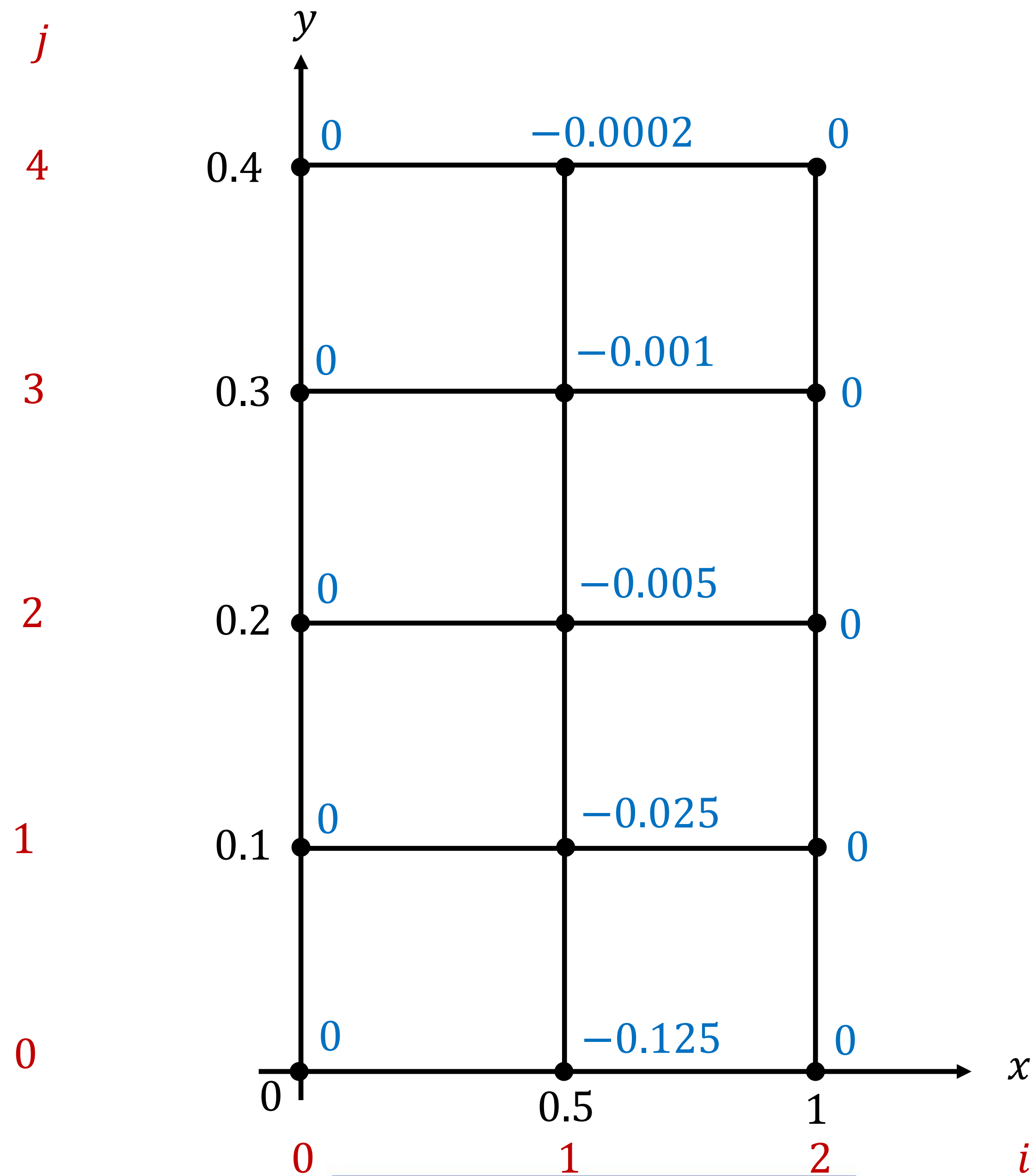
$$u_{1,1} = u_{1,0} + 0.4(u_{2,0} - 2u_{1,0} + u_{0,0})$$
$$u_{1,1} = -0.125 + 0.4[0 - 2(-0.125) + 0]$$
$$u_{1,1} = -0.025$$

$$u_{1,2} = u_{1,1} + 0.4(u_{2,1} - 2u_{1,1} + u_{0,1})$$
$$u_{1,2} = -0.025 + 0.4[0 - 2(-0.025) + 0]$$
$$u_{1,2} = -0.005$$

$$u_{1,3} = u_{1,2} + 0.4(u_{2,2} - 2u_{1,2} + u_{0,2})$$
$$u_{1,3} = -0.005 + 0.4[0 - 2(-0.005) + 0]$$
$$u_{1,3} = -0.001$$

$$u_{1,4} = u_{1,3} + 0.4(u_{2,3} - 2u_{1,3} + u_{0,3})$$
$$u_{1,4} = -0.001 + 0.4[0 - 2(-0.001) + 0]$$
$$u_{1,4} = -0.0002$$

Solution (Cont.): *Step 6: Fill in the values into the grid.*



8.2 Parabolic Equation – Heat Equation

Exercise 8.1:

Find approximation solutions for Heat Equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < 1, \qquad t > 0$$

with boundary condition

$$u(0,t) = u(1,t) = 0, \qquad 0 \le t \le 0.02$$

and initial condition

$$u(x,0) = x(1-x) \;\; 0 \le x \le 1.$$

by using forward-difference and central-difference formula.

Given $h = 1/5$ and $k = 1/100$.

8.3 Hyperbolic Equation – Wave Equation

Consider the **Wave Equation**:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < a, \qquad t > 0$$

with boundary condition

$$u(0,t) = u(a,t) = 0, \qquad t > 0$$

and initial condition

$$u(x,0) = f(x), \qquad 0 \le x \le a$$

$$\frac{\partial u}{\partial t}(x,0) = g(x), \qquad 0 \le x \le a$$

By central-difference formulas,

$$\left(\frac{\partial^2 u}{\partial t^2}\right)_{i,j} - c^2 \left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j} = 0$$

is approximated to

$$\boxed{\frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{k^2} - c^2 \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2} = 0}$$

where $h = \Delta x$ and $k = \Delta t$.

8.3 Hyperbolic Equation – Wave Equation

Illustrative Example 1:

Find approximation solutions for Wave Equation

$$\frac{\partial^2 u}{\partial t^2} = 4\frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < 1, \qquad t > 0$$

with boundary condition

$$u(0,t) = u(1,t) = 0, \qquad 0 \le t \le 1$$

and initial condition

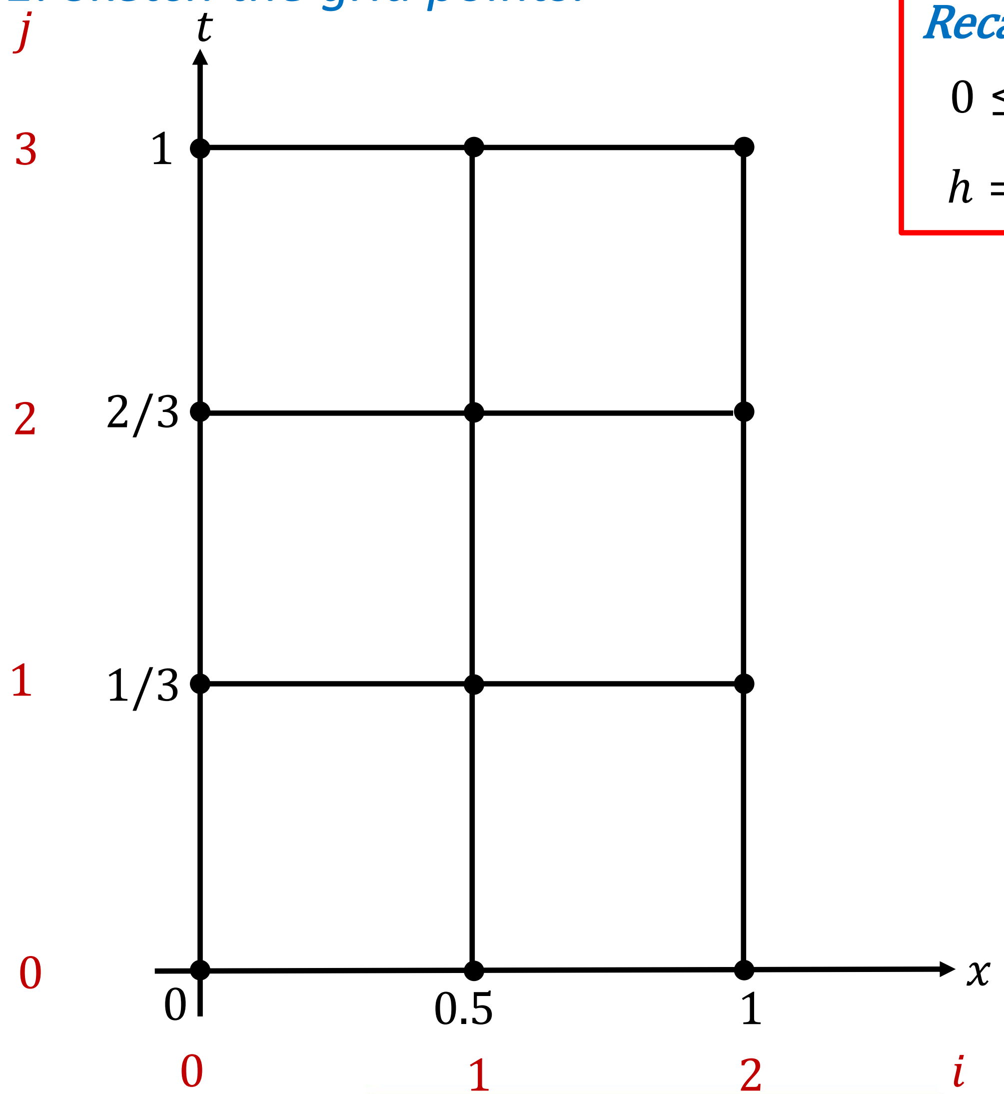
$$u(x,0) = \sin 2\pi x\,, \qquad 0 \le x \le 1$$

$$\frac{\partial u}{\partial t}(x,0) = 4, \qquad 0 \le x \le 1$$

by using central-difference formula. Given $h = 0.5$ and $k = \frac{1}{3}$.

Solution:

Step 1: Sketch the grid points.



Recall:

$0 \leq x \leq 1, \qquad 0 \leq t \leq 1$

$h = 0.5$ and $k = 1/3$

8.3 Hyperbolic Equation – Wave Equation

Solution:

Step 2: Compute boundary and initial values.

Recall:
$h = 0.5$ and $k = 1/3$

$$\boldsymbol{u(x,0) = \sin 2\pi x\,, \qquad 0 \le x \le 1}$$

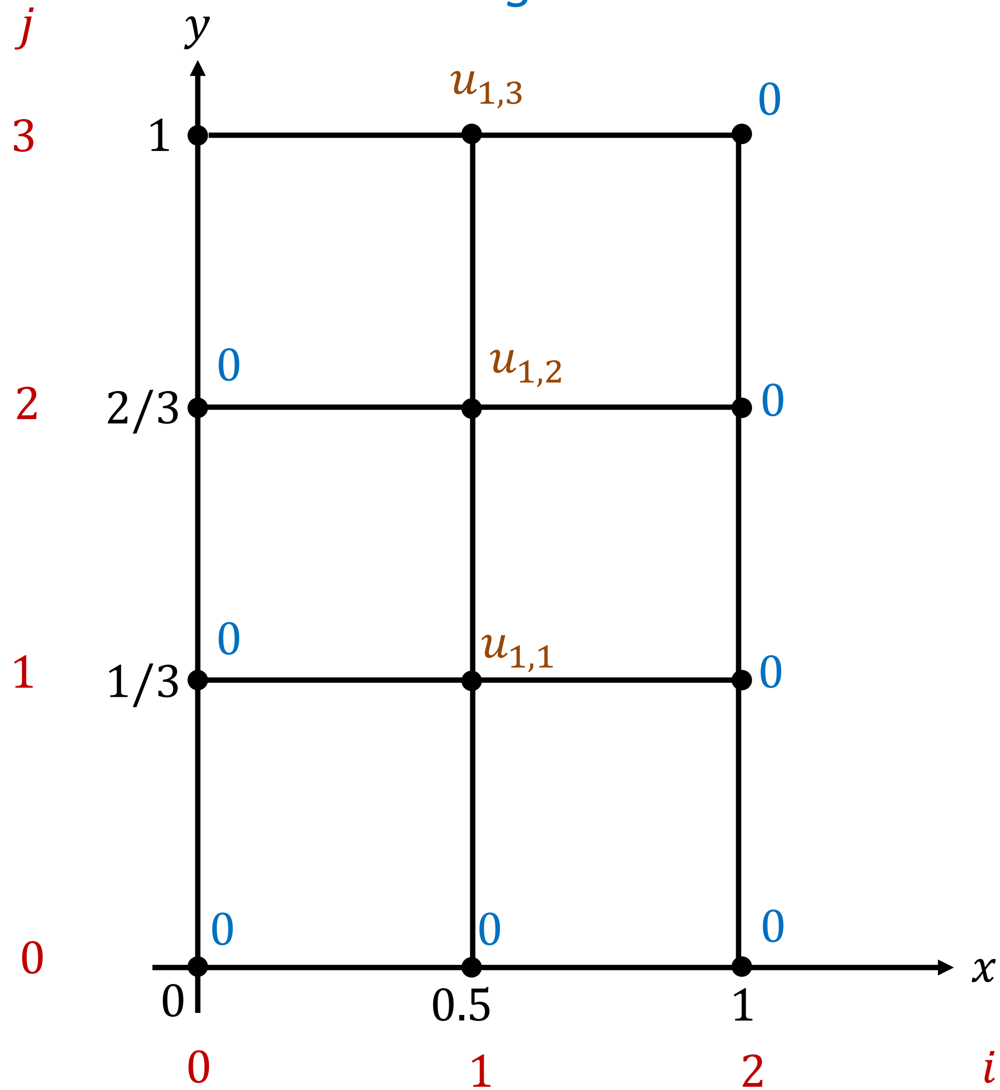
$$u(0,0) = 0$$
$$u(0.5,0) = 0$$
$$u(1,0) = 0$$

$$\boldsymbol{u(0,t) = 0, \qquad u(1,t) = 0, \qquad 0 \le t \le 1}$$

$u(0,0) = 0$	$u(1,0) = 0$
$u(0,0.3333) = 0$	$u(1,0.3333) = 0$
$u(0,0.6667) = 0$	$u(1,0.6667) = 0$
$u(0,1) = 0$	$u(1,1) = 0$

Solution:

Step 3: Fill in the values into the grid.



8.3 Hyperbolic Equation – Wave Equation

Solution:

Step 4: Obtain formula $u_{i,j}$ from difference formula.

By central-difference formulas,

$$\left(\frac{\partial^2 u}{\partial t^2}\right)_{i,j} - 4\left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j} = 0$$

Recall:
$h = 0.5$ and $k = 1/3$

is approximated to

$$\frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{k^2} - 4\left(\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2}\right) = 0$$

$$\frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{(1/3)^2} - 4\left(\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{(0.5)^2}\right) = 0$$

$$u_{i,j+1} - 2u_{i,j} + u_{i,j-1} - \frac{4(1/3)^2}{(0.5)^2}(u_{i+1,j} - 2u_{i,j} + u_{i-1,j}) = 0$$

$$u_{i,j+1} = 2u_{i,j} - u_{i,j-1} + (16/9)(u_{i+1,j} - 2u_{i,j} + u_{i-1,j})$$

8.3 Hyperbolic Equation – Wave Equation

Solution:

Step 4: Obtain formula $u_{i,j}$ from difference formula.

By central-difference formulas,

$$\frac{\partial u}{\partial t}(x_i, 0) = \left(\frac{\partial u}{\partial t}\right)_{i,j=0} \approx \frac{u_{i,j+1} - u_{i,j-1}}{2k} = \frac{u_{i,1} - u_{i,-1}}{2(1/3)} = 4$$

is approximated to

$$u_{i,1} - u_{i,-1} = 8/3$$
$$u_{i,-1} = u_{i,1} - 8/3$$

8.3 Hyperbolic Equation – Wave Equation

Solution:

Step 5: Compute equations for internal points $u_{i,j}$ (Refer to grid in Step 3).

$$u_{i,j+1} = 2u_{i,j} - u_{i,j-1} + (16/9)(u_{i+1,j} - 2u_{i,j} + u_{i-1,j})$$

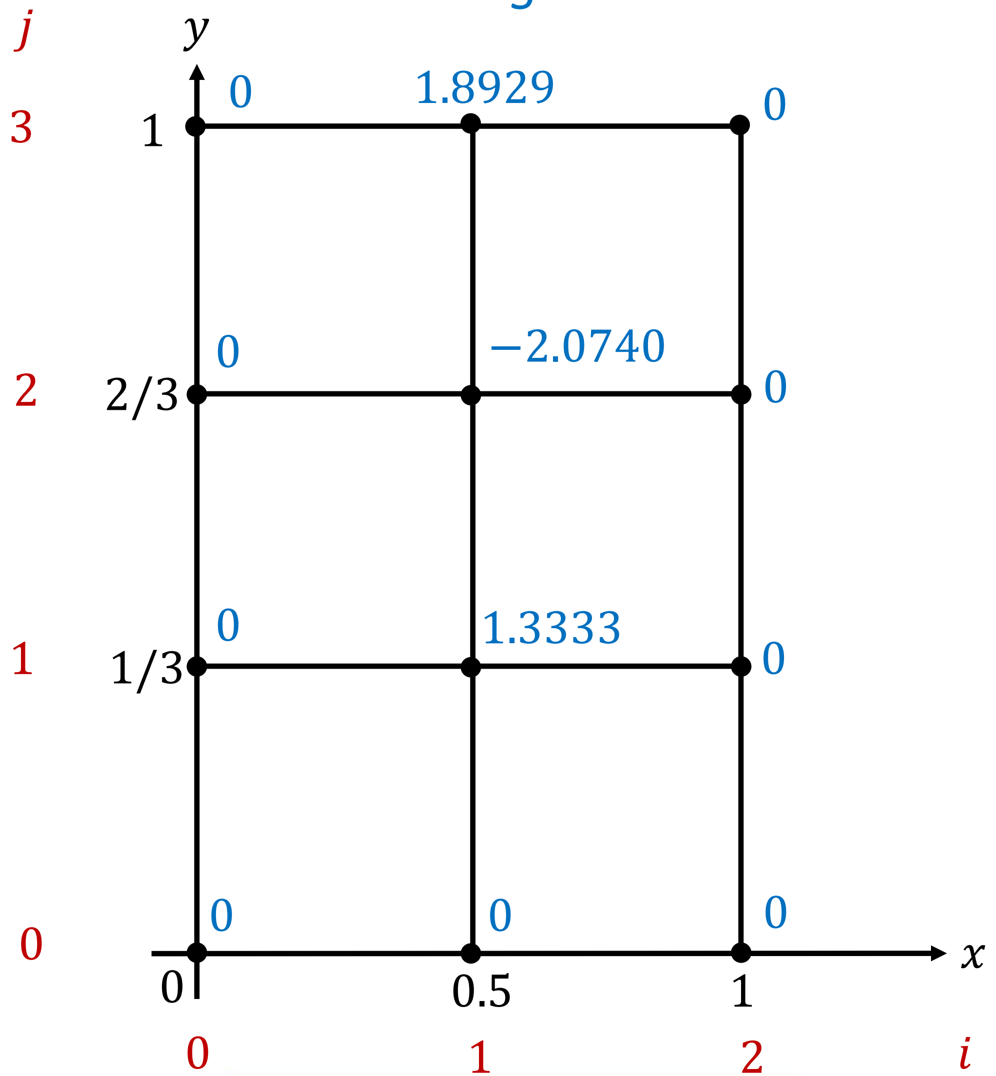
$$u_{1,1} = 2u_{1,0} - u_{1,-1} + (16/9)(u_{2,0} - 2u_{1,0} + u_{0,0})$$
$$u_{1,1} = 2(0) - (u_{1,1} - 8/3) + (16/9)[0 - 2(0) + 0]$$
$$2u_{1,1} = 2.6667$$
$$u_{1,1} = 1.3333$$

$$u_{1,2} = 2u_{1,1} - u_{1,0} + (16/9)(u_{2,1} - 2u_{1,1} + u_{0,1})$$
$$u_{1,2} = 2(1.3333) - (0) + (16/9)[0 - 2(1.3333) + 0]$$
$$u_{1,2} = -2.0740$$

$$u_{1,3} = 2u_{1,2} - u_{1,1} + (16/9)(u_{2,2} - 2u_{1,2} + u_{0,2})$$
$$u_{1,2} = 2(-2.0740) - (1.3333) + (16/9)[0 - 2(-2.0740) + 0]$$
$$u_{1,2} = 1.8929$$

Solution:

Step 6: Fill in the values into the grid.



8.3 Hyperbolic Equation – Wave Equation

Illustrative Example 2:

Find approximation solutions for Wave Equation

$$\frac{\partial^2 u}{\partial t^2} = 9\frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < 1, \qquad t > 0$$

with boundary condition

$$u(0,t) = u(1,t) = 0, \qquad 0 \le t \le 1$$

and initial condition

$$u(x,0) = x(x-1), \qquad 0 \le x \le 1$$

$$\frac{\partial u}{\partial t}(x,0) = 1, \qquad 0 \le x \le 1$$

by using central-difference formula. Given $h = \frac{1}{3}$ and $k = 0.5$.

8.3 Hyperbolic Equation – Wave Equation

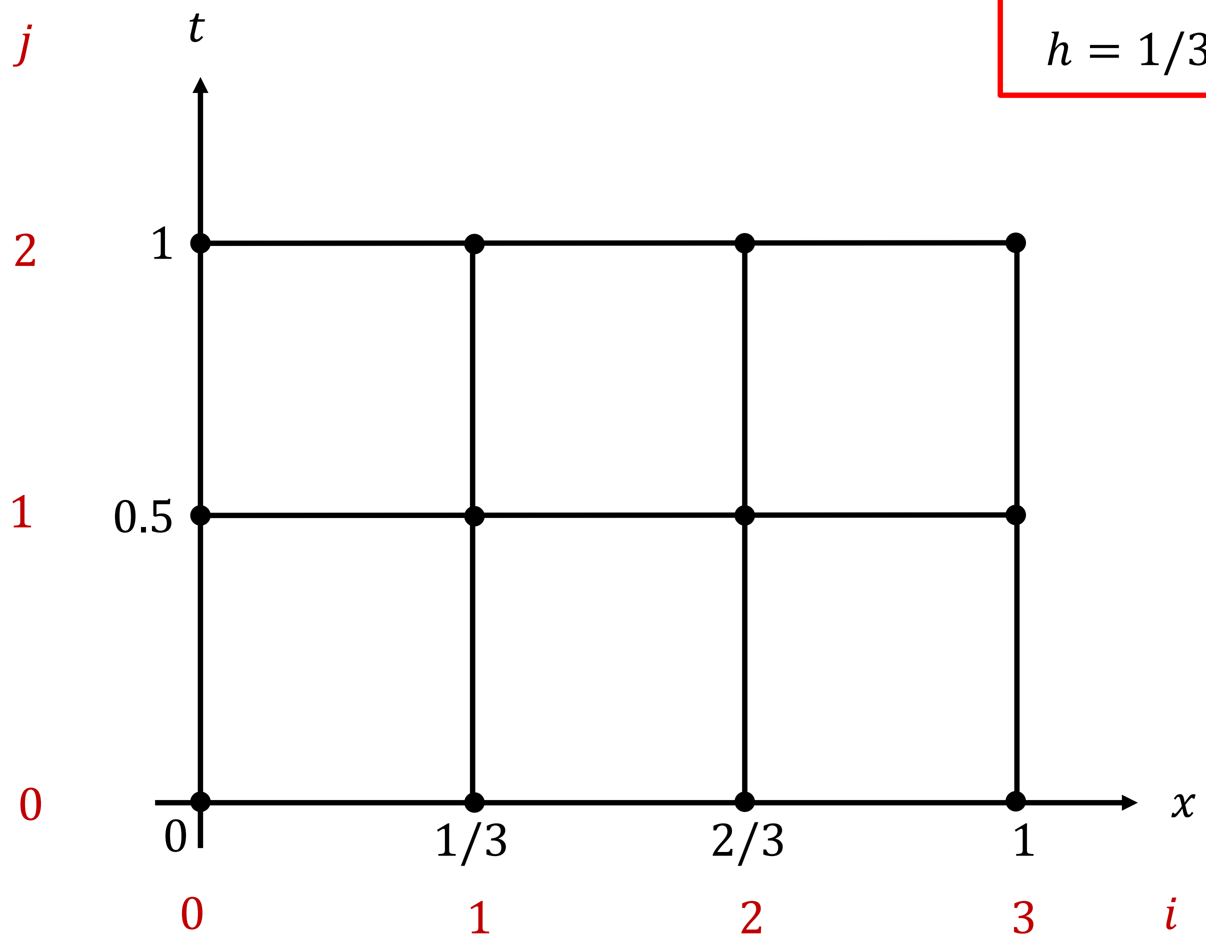
Solution:

Step 1: Sketch the grid points.

Recall:

$0 \leq x \leq 1, \qquad 0 \leq t \leq 1$

$h = 1/3$ and $k = 0.5$



8.3 Hyperbolic Equation – Wave Equation

Solution:

Step 2: Compute boundary and initial values.

Recall:
$h = 1/3$ and $k = 0.5$

$$\boldsymbol{u(x,0) = x(x-1), \qquad 0 \le x \le 1}$$

$$u(0,0) = 0$$
$$u(0.3333,0) = -0.2222$$
$$u(0.6667,0) = -0.2222$$
$$u(1,0) = 0$$

$$\boldsymbol{u(0,t) = 0, \qquad u(1,t) = 0, \qquad 0 \le t \le 1}$$

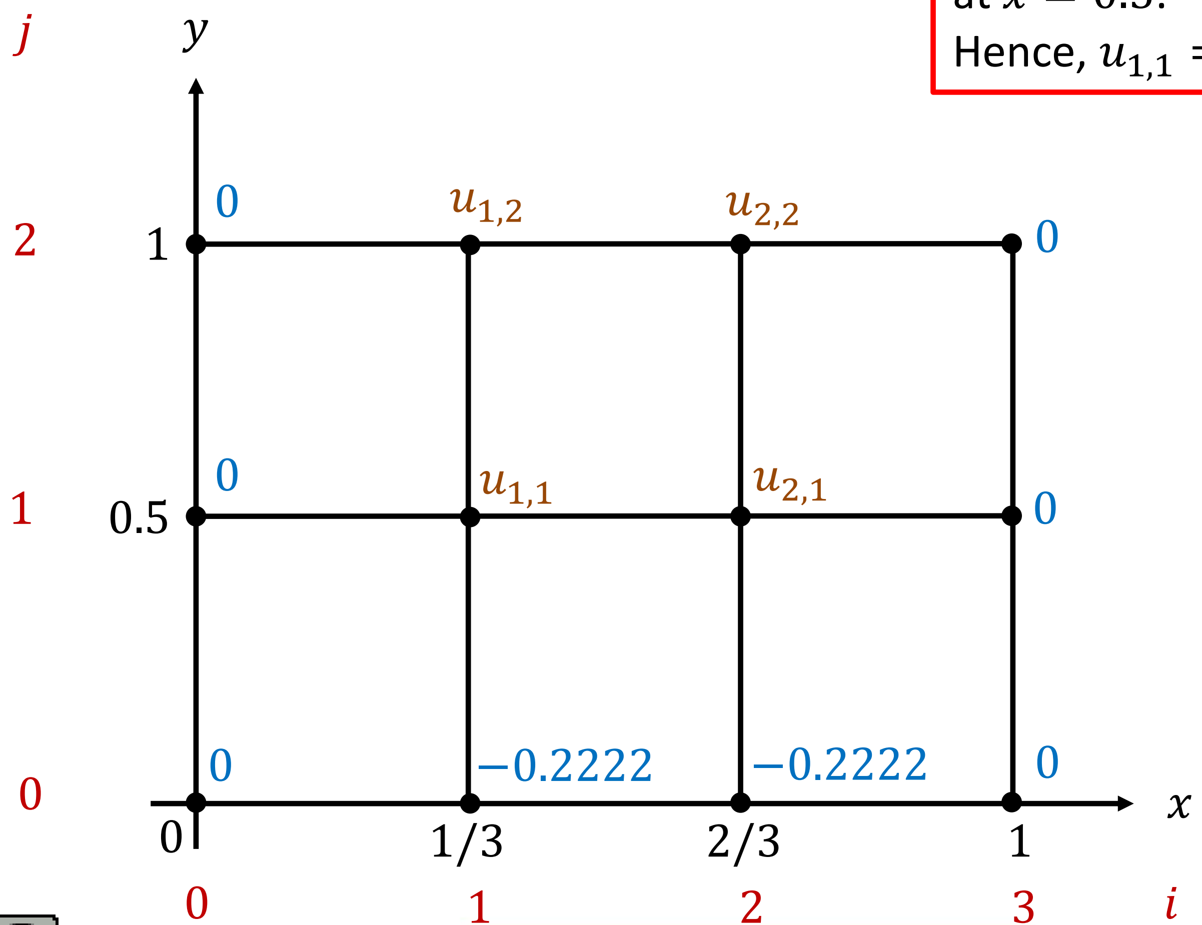
$$u(0,0) = 0 \qquad u(1,0) = 0$$
$$u(0,0.5) = 0 \qquad u(1,0.5) = 0$$
$$u(0,1) = 0 \qquad u(1,1) = 0$$

8.3 Hyperbolic Equation – Wave Equation

Solution:

Step 3: Fill in the values into the grid.

Boundary conditions are symmetry at $x = 0.5$.
Hence, $u_{1,1} = u_{2,1}$ and $u_{1,2} = u_{2,2}$



8.3 Hyperbolic Equation – Wave Equation

Solution:

Step 4: Obtain formula $u_{i,j}$ from difference formula.

By central-difference formulas,

$$\left(\frac{\partial^2 u}{\partial t^2}\right)_{i,j} - 9\left(\frac{\partial^2 u}{\partial x^2}\right)_{i,j} = 0$$

Recall:
$h = 1/3$ and $k = 0.5$

is approximated to

$$\frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{k^2} - 9\left(\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2}\right) = 0$$

$$\frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{(0.5)^2} - 9\left(\frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{(1/3)^2}\right) = 0$$

$$u_{i,j+1} - 2u_{i,j} + u_{i,j-1} - \frac{9(0.5)^2}{(1/3)^2}(u_{i+1,j} - 2u_{i,j} + u_{i-1,j}) = 0$$

$$u_{i,j+1} = 2u_{i,j} - u_{i,j-1} + 20.25(u_{i+1,j} - 2u_{i,j} + u_{i-1,j})$$

8.3 Hyperbolic Equation – Wave Equation

Solution:

Step 4: Obtain formula $u_{i,j}$ from difference formula.

By central-difference formulas,

$$\frac{\partial u}{\partial t}(x_i, 0) = \left(\frac{\partial u}{\partial t}\right)_{i,j=0} \approx \frac{u_{i,j+1} - u_{i,j-1}}{2k} = \frac{u_{i,1} - u_{i,-1}}{2(0.5)} = 1$$

is approximated to

$$u_{i,1} - u_{i,-1} = 1$$
$$u_{i,-1} = u_{i,1} - 1$$

8.3 Hyperbolic Equation – Wave Equation

Solution:

Step 5: Compute equations for internal points $u_{i,j}$ (Refer to grid in Step 3).

$$u_{i,j+1} = 2u_{i,j} - u_{i,j-1} + 20.25(u_{i+1,j} - 2u_{i,j} + u_{i-1,j})$$

$u_{1,1} = 2u_{1,0} - u_{1,-1} + 20.25(u_{2,0} - 2u_{1,0} + u_{0,0})$
$u_{1,1} = 2(-0.2222) - (u_{1,1} - 1) + 20.25[-0.2222 - 2(-0.2222) + 0]$
$2u_{1,1} = 5.0552$
$u_{1,1} = 2.5276$

$u_{2,1} = 2u_{2,0} - u_{2,-1} + 20.25(u_{3,0} - 2u_{2,0} + u_{1,0})$
$u_{2,1} = 2(0.2222) - (u_{2,1} - 1) + 20.25[0 - 2(-0.2222) - 0.2222]$
$2u_{2,1} = 5.0552$
$u_{2,1} = 2.5276$

Boundary conditions are symmetry at $x = 0.5$.
Hence, $u_{1,1} = u_{2,1}$ and $u_{1,2} = u_{2,2}$

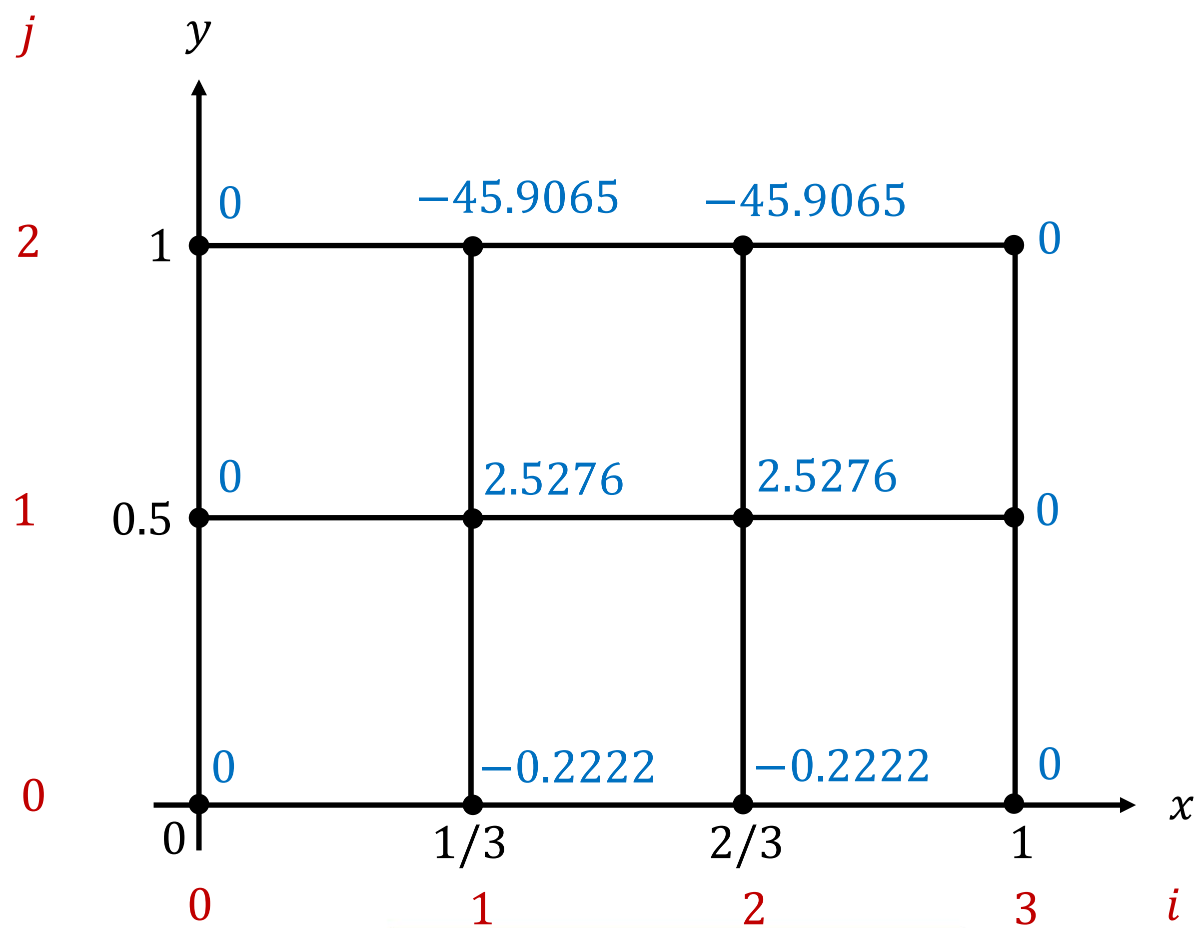
$u_{1,2} = 2u_{1,1} - u_{1,0} + 20.25(u_{2,1} - 2u_{1,1} + u_{0,1})$
$u_{1,2} = 2(2.5276) - (-0.2222) + 20.25[2.5276 - 2(2.5276) + 0]$
$u_{1,2} = -45.9065$

$u_{2,2} = 2u_{2,1} - u_{2,0} + 20.25(u_{3,1} - 2u_{2,1} + u_{1,1})$
$u_{2,2} = 2(2.5276) - (-0.2222) + 20.25[0 - 2(2.5276) + 2.5276]$
$u_{2,2} = -45.9065$

8.3 Hyperbolic Equation – Wave Equation

Solution:

Step 6: Fill in the values into the grid.



8.3 Hyperbolic Equation – Wave Equation

Exercise 8.2:

Find approximation solutions for Wave Equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}, \qquad 0 < x < 1, \qquad t > 0$$

with boundary condition

$$u(0,t) = u(1,t) = 0, \qquad 0 \leq t \leq 0.1$$

and initial condition

$$u(x,0) = \sin \pi x, \qquad 0 \leq x \leq 1$$

$$\frac{\partial u}{\partial t}(x,0) = 0, \qquad 0 \leq x \leq 1$$

by using central-difference formula. Given $h = 0.25$ and $k = 0.05$.