NUMERICAL METHODS
BEKG 2452
SOLUTION OF DIFFERENTIAL EQUATIONS
(Runge-Kutta Method)

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Lesson Outcome

Upon completion of this lesson, the student should be able to:

1. Solve numerically first order linear differential equations using Runge-Kutta Order 2 (RK2).
2. Solve numerically first order linear differential equations using Runge-Kutta Order 4 (RK4).

Solution of Differential Equation

Analytical Methods

- Separable
- Exact
- Homogeneous
- Linear
- Bernoulli

Exact solution

Numerical Methods

Euler's Method

Heun's Method

Runge-Kutta Method
- Order 2
- Order 4

Approximated solution

7.3 Runge-Kutta Method

A method of numerically integrating ODE by using a trial step at the midpoint of an interval to cancel out lower-order error terms.

Second Order Formula: (known as RK2 or Ralson's Method)

$$y_{i+1} = y_i + \left[\frac{1}{3}k_1 + \frac{2}{3}k_2\right]h$$

where

$$k_1 = f(x_i, y_i)$$

$$k_2 = f\left(x_i + \frac{3}{4}h, y_i + \frac{3}{4}k_1 h\right)$$

7.3 Runge-Kutta Method

Forth Order Formula: (Known as RK4)

$$y_{i+1} = y_i + \frac{1}{6}h(k_1 + 2k_2 + 2k_3 + k_4)\,; \quad i = 0,1,2,3,\ldots$$

where

$$k_1 = f(x_i, y_i\,)$$

$$k_2 = f\left(x_i + \frac{h}{2}, y_i + \frac{k_1 h}{2}\right)$$

$$k_3 = f\left(x_i + \frac{h}{2}, y_i + \frac{k_2 h}{2}\right)$$

$$k_4 = f(x_i + h, y_i + k_3 h\,)$$

7.3.1 Second Order Runge-Kutta Method (RK2)

The steps are similar with Heun's method:

Line Structure	→	Rearrange: $y' = f(x, y)$	→	Discretizing: $y_i' = f(x_i, y_i)$ & compute values of k	→	Compute y_{i+1}

Illustrative Example 1:

Use the second order Runge-Kutta method to numerically integrate

$$y' - y = -x^2, \qquad y(0) = 0$$

from $x = 0$ to $x = 2$ with a step size of 0.5.

7.3.1 Second Order Runge-Kutta Method (RK2)

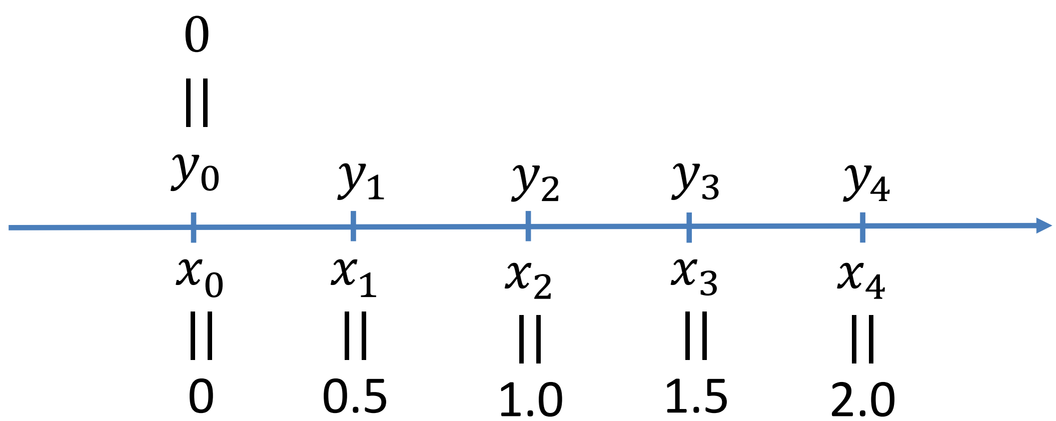
Reminder:
$y' = f(x, y)$

Solution:

$$y' = f(x, y) = y - x^2$$

$$0 \le x \le 2, \quad h = 0.5, \quad y(0) = 0$$

Construct the time line:



7.3.1 Second Order Runge-Kutta Method (RK2)

Solution: (cont.)

$$y' = f(x,y) = y - x^2$$

$0 = y_0$	y_1	y_2	y_3	y_4
$x_0 = 0$	$x_1 = 0.5$	$x_2 = 1.0$	$x_3 = 1.5$	$x_4 = 2.0$

Let $i = 0$,

$$k_1 = f(x_0, y_0) = f(0,0)$$
$$= 0$$

$$k_2 = f\left(x_0 + \frac{3}{4}h, y_0 + \frac{3}{4}k_1 h\right)$$
$$= f(0.375, 0\,)$$
$$= -0.1406$$

$$y_1 = y_0 + \left[\frac{1}{3}k_1 + \frac{2}{3}k_2\right]h$$
$$= 0 + \left[\frac{1}{3}(0) + \frac{2}{3}(-0.1406)\right](0.5)$$
$$= -0.0469$$

7.3.1 Second Order Runge-Kutta Method (RK2)

Solution: (cont.)

$$y' = f(x, y) = y - x^2$$

0	-0.0469			
$\|\|$	$\|\|$			
y_0	y_1	y_2	y_3	y_4
x_0	x_1	x_2	x_3	x_4
$\|\|$	$\|\|$	$\|\|$	$\|\|$	$\|\|$
0	0.5	1.0	1.5	2.0

Let $i = 1$,

$$k_1 = f(x_1, y_1)$$
$$= f(0.5, -0.0469)$$
$$= -0.2969$$

$$k_2 = f\left(x_1 + \frac{3}{4}h, y_1 + \frac{3}{4}k_1 h\right)$$
$$= f(0.875, -0.1582)$$
$$= -0.9238$$

$$y_2 = y_1 + \left[\frac{1}{3}k_1 + \frac{2}{3}k_2\right]h$$
$$= -0.0469 + \left[\frac{1}{3}(-0.2969) + \frac{2}{3}(-0.9238)\right](0.5)$$
$$= -0.4043$$

7.3.1 Second Order Runge-Kutta Method (RK2)

Solution: (cont.)

$$y' = f(x, y) = y - x^2$$

$$y_0 = 0, \quad y_1 = -0.0469, \quad y_2 = -0.4043, \quad y_3, \quad y_4$$

$$x_0 = 0, \quad x_1 = 0.5, \quad x_2 = 1.0, \quad x_3 = 1.5, \quad x_4 = 2.0$$

Let $i = 2$,

$$\begin{aligned} k_1 &= f(x_2, y_2) \\ &= f(1, -0.4043) \\ &= -1.4043 \end{aligned}$$

$$\begin{aligned} k_2 &= f\left(x_2 + \frac{3}{4}h, y_2 + \frac{3}{4}k_1 h\right) \\ &= f(1.375, -0.9309\,) \\ &= -2.8215 \end{aligned}$$

$$\begin{aligned} y_3 &= y_2 + \left[\frac{1}{3}k_1 + \frac{2}{3}k_2\right]h \\ &= -0.4043 \\ &\quad + \left[\tfrac{1}{3}(-1.4043) + \tfrac{2}{3}(-2.8215)\right](0.5) \\ &= -1.5789 \end{aligned}$$

7.3.1 Second Order Runge-Kutta Method (RK2)

Solution: (cont.)

$$y' = f(x, y) = y - x^2$$

$$\begin{array}{ccccc}
0 & -0.0469 & -0.4043 & -1.5789 & \\
\| & \| & \| & \| & \\
y_0 & y_1 & y_2 & y_3 & y_4 \\
x_0 & x_1 & x_2 & x_3 & x_4 \\
\| & \| & \| & \| & \| \\
0 & 0.5 & 1.0 & 1.5 & 2.0
\end{array}$$

Let $i = 3$,

$$\begin{aligned}
k_1 &= f(x_3, y_3) \\
&= f(1.5, -1.5789) \\
&= -3.8289
\end{aligned}$$

$$\begin{aligned}
k_2 &= f\left(x_3 + \frac{3}{4}h, y_3 + \frac{3}{4}k_1 h\right) \\
&= f(1.875, -3.0147) \\
&= -6.5303
\end{aligned}$$

$$\begin{aligned}
y_4 &= y_3 + \left[\frac{1}{3}k_1 + \frac{2}{3}k_2\right]h \\
&= -1.5789 \\
&\quad + \left[\frac{1}{3}(-3.8289) + \frac{2}{3}(-6.5303)\right](0.5) \\
&= -4.3938
\end{aligned}$$

7.3.1 Second Order Runge-Kutta Method (RK2)

Solution: (cont.)

$$y' = f(x,y) = y - x^2$$

0	-0.0469	-0.4043	-1.5789	-4.3938
$\vert\vert$	$\vert\vert$	$\vert\vert$	$\vert\vert$	$\vert\vert$
y_0	y_1	y_2	y_3	y_4
x_0	x_1	x_2	x_3	x_4
$\vert\vert$	$\vert\vert$	$\vert\vert$	$\vert\vert$	$\vert\vert$
0	0.5	1.0	1.5	2.0

7.3.1 Second Order Runge-Kutta Method (RK2)

Illustrative Example 2:

Use the second order Runge-Kutta method to numerically integrate

$$y' - \cos 2x - \sin 3y = 0, \qquad y(0) = 1$$

from $x = 0$ to $x = 1$ with a step size of 0.2.

7.3.1 Second Order Runge-Kutta Method (RK2)

Solution:

Reminder:
$y' = f(x, y)$

$$y' = f(x, y) = \cos 2x + \sin 3y$$

$$0 \le x \le 1, \quad h = 0.2, \quad y(0) = 1$$

Construct the time line:

1					
$\parallel$					
y_0	y_1	y_2	y_3	y_4	y_5
x_0	x_1	x_2	x_3	x_4	x_5
$\parallel$	$\parallel$	$\parallel$	$\parallel$	$\parallel$	$\parallel$
0	0.2	0.4	0.6	0.8	1.0

7.3.1 Second Order Runge-Kutta Method (RK2)

Solution: (cont.)

$$y' = f(x,y) = \cos 2x + \sin 3y$$

1					
$\parallel$					
y_0	y_1	y_2	y_3	y_4	y_5
x_0	x_1	x_2	x_3	x_4	x_5
$\parallel$	$\parallel$	$\parallel$	$\parallel$	$\parallel$	$\parallel$
0	0.2	0.4	0.6	0.8	1.0

Let $i = 0$,

$$\begin{aligned} k_1 &= f(x_0, y_0) = f(0,1) \\ &= 1.1411 \end{aligned}$$

$$\begin{aligned} k_2 &= f\left(x_0 + \frac{3}{4}h, y_0 + \frac{3}{4}k_1 h\right) \\ &= f(0.15, 1.1712\,) \\ &= 0.5919 \end{aligned}$$

$$\begin{aligned} y_1 &= y_0 + \left[\frac{1}{3}k_1 + \frac{2}{3}k_2\right]h \\ &= 1 + \left[\frac{1}{3}(1.1411) + \frac{2}{3}(0.5919)\right](0.2) \\ &= 1.1550 \end{aligned}$$

7.3.1 Second Order Runge-Kutta Method (RK2)

Solution: (cont.)

$$y' = f(x,y) = \cos 2x + \sin 3y$$

1	1.1550				
$\vert\vert$	$\vert\vert$				
y_0	y_1	y_2	y_3	y_4	y_5
x_0	x_1	x_2	x_3	x_4	x_5
$\vert\vert$	$\vert\vert$	$\vert\vert$	$\vert\vert$	$\vert\vert$	$\vert\vert$
0	0.2	0.4	0.6	0.8	1.0

Let $i = 1$,

$$k_1 = f(x_1, y_1) = f(0.2, 1.1550) = 0.6033$$

$$k_2 = f\left(x_1 + \frac{3}{4}h, y_1 + \frac{3}{4}k_1 h\right) = f(0.35, 1.2455) = 0.2044$$

$$y_2 = y_1 + \left[\frac{1}{3}k_1 + \frac{2}{3}k_2\right]h = 1.1550 + \left[\frac{1}{3}(0.6033) + \frac{2}{3}(0.2044)\right](0.2) = 1.2225$$

7.3.1 Second Order Runge-Kutta Method (RK2)

Solution: (cont.)

$$y' = f(x, y) = \cos 2x + \sin 3y$$

1	1.1550	1.2225			
$\parallel$	$\parallel$	$\parallel$			
y_0	y_1	y_2	y_3	y_4	y_5
x_0	x_1	x_2	x_3	x_4	x_5
$\parallel$	$\parallel$	$\parallel$	$\parallel$	$\parallel$	$\parallel$
0	0.2	0.4	0.6	0.8	1.0

Let $i = 2$,

$$\begin{aligned} k_1 &= f(x_2, y_2) \\ &= f(0.4, 1.2225) \\ &= 0.1947 \end{aligned}$$

$$\begin{aligned} k_2 &= f\left(x_2 + \frac{3}{4}h, y_2 + \frac{3}{4}k_1 h\right) \\ &= f(0.55, 1.2517) \\ &= -0.1221 \end{aligned}$$

$$\begin{aligned} y_3 &= y_2 + \left[\frac{1}{3}k_1 + \frac{2}{3}k_2\right]h \\ &= 1.2225 \\ &\quad + \left[\frac{1}{3}(0.1947) + \frac{2}{3}(-0.1221)\right](0.2) \\ &= 1.2192 \end{aligned}$$

7.3.1 Second Order Runge-Kutta Method (RK2)

Solution: (cont.)

$$y' = f(x, y) = \cos 2x + \sin 3y$$

1	1.1550	1.2225	1.2192		
$\parallel$	$\parallel$	$\parallel$	$\parallel$		
y_0	y_1	y_2	y_3	y_4	y_5
x_0	x_1	x_2	x_3	x_4	x_5
$\parallel$	$\parallel$	$\parallel$	$\parallel$	$\parallel$	$\parallel$
0	0.2	0.4	0.6	0.8	1.0

Let $i = 3$,

$$\begin{aligned} k_1 &= f(x_3, y_3) \\ &= f(0.6, 1.2192) \\ &= -0.1311 \end{aligned}$$

$$\begin{aligned} k_2 &= f\left(x_3 + \frac{3}{4}h, y_3 + \frac{3}{4}k_1 h\right) \\ &= f(0.75, 1.1995\,) \\ &= -0.3704 \end{aligned}$$

$$\begin{aligned} y_4 &= y_3 + \left[\frac{1}{3}k_1 + \frac{2}{3}k_2\right]h \\ &= 1.2192 \\ &\quad + \left[\frac{1}{3}(-0.1311) + \frac{2}{3}(-0.3704)\right](0.2) \\ &= 1.1611 \end{aligned}$$

7.3.1 Second Order Runge-Kutta Method (RK2)

Solution: (cont.)

$$y' = f(x, y) = \cos 2x + \sin 3y$$

1	1.1550	1.2225	1.2192	1.1611	
$\parallel$	$\parallel$	$\parallel$	$\parallel$	$\parallel$	
y_0	y_1	y_2	y_3	y_4	y_5
x_0	x_1	x_2	x_3	x_4	x_5
$\parallel$	$\parallel$	$\parallel$	$\parallel$	$\parallel$	$\parallel$
0	0.2	0.4	0.6	0.8	1.0

Let $i = 4$,

$$\begin{aligned} k_1 &= f(x_4, y_4) \\ &= f(0.8, 1.1611) \\ &= -0.3643 \end{aligned}$$

$$\begin{aligned} k_2 &= f\left(x_4 + \frac{3}{4}h, y_4 + \frac{3}{4}k_1 h\right) \\ &= f(0.95, 1.1065) \\ &= -0.5003 \end{aligned}$$

$$\begin{aligned} y_5 &= y_4 + \left[\frac{1}{3}k_1 + \frac{2}{3}k_2\right]h \\ &= 1.1661 \\ &\quad + \left[\tfrac{1}{3}(-0.3643) + \tfrac{2}{3}(-0.5003)\right](0.2) \\ &= 1.0751 \end{aligned}$$

7.3.1 Second Order Runge-Kutta Method (RK2)

Solution: (cont.)

$$y' = f(x, y) = \cos 2x + \sin 3y$$

1	1.1550	1.2225	1.2192	1.1611	1.0751
||	||	||	||	||	||
y_0	y_1	y_2	y_3	y_4	y_5
x_0	x_1	x_2	x_3	x_4	x_5
||	||	||	||	||	||
0	0.2	0.4	0.6	0.8	1.0

7.3.1 Second Order Runge-Kutta Method (RK2)

Exercise 7.3:

Use the second order Runge-Kutta method to numerically integrate

$$\frac{dy}{dx} + y = \sin x, \qquad y(0) = 1$$

from $x = 0$ to $x = 3$ with a step size of $h = 1$.

7.3.1 Second Order Runge-Kutta Method (RK2)

Solution:

$$y' = f(x, y) = (\sin x) - y$$

1	0.9544	0.9930	0.5994
$\|\|$	$\|\|$	$\|\|$	$\|\|$
y_0	y_1	y_2	y_3
x_0	x_1	x_2	x_3
$\|\|$	$\|\|$	$\|\|$	$\|\|$
0	1	2	3

$$k_1 = -1$$
$$k_2 = 0.4316$$

$$k_1 = -0.1130$$
$$k_2 = 0.1143$$

$$k_1 = -0.0837$$
$$k_2 = -0.5486$$

7.3.2 Forth Order Runge-Kutta Method (RK4)

The steps are similar with Heun's method:

Line Structure → Rearrange: $y' = f(x, y)$ → Discretizing: $y_i' = f(x_i, y_i)$ & compute values of k → Compute y_{i+1}

Illustrative Example 1:

Use the forth order Runge-Kutta method to numerically integrate

$$y' - y = x, \qquad y(0) = 1$$

from $x = 0$ to $x = 1$ with a step size of 0.5.

7.3.2 Forth Order Runge-Kutta Method (RK4)

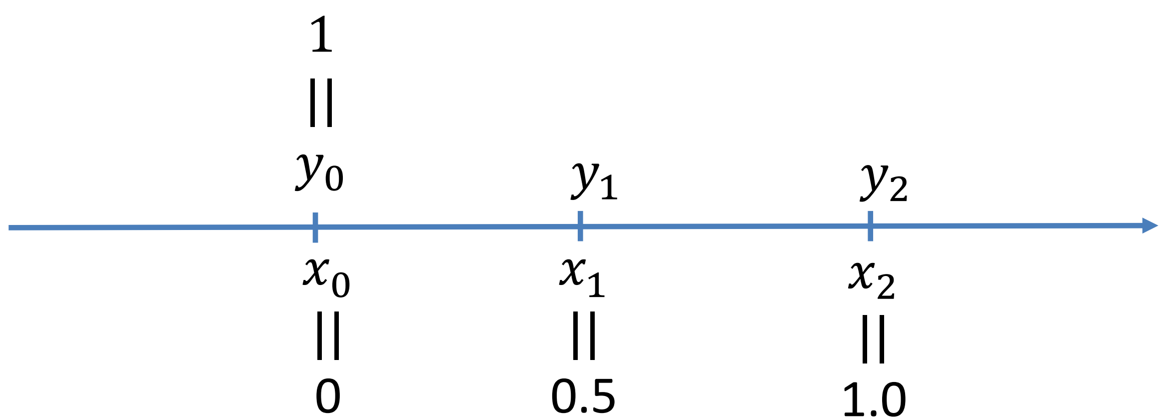
Reminder:
$y' = f(x, y)$

Solution:

$$y' = f(x, y) = x + y$$

$$0 \le x \le 1, \quad h = 0.5, \quad y(0) = 1$$

Construct the time line:



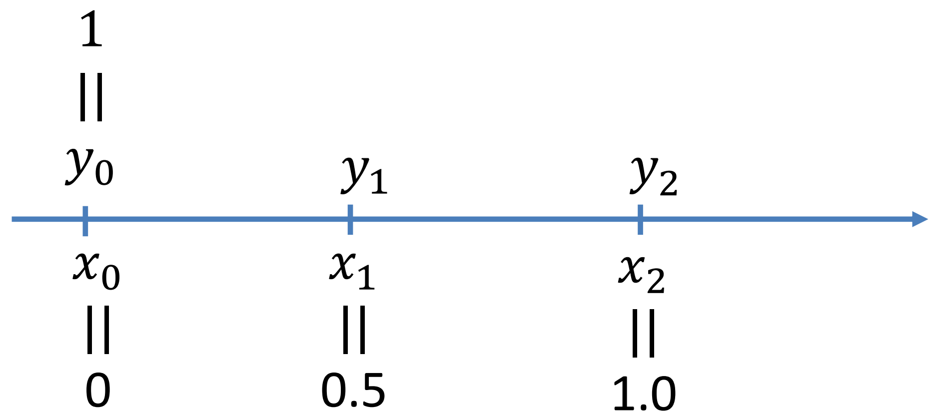
7.3.2 Forth Order Runge-Kutta Method (RK4)

Solution: (cont.)

Let $i = 0$,

$$y' = f(x, y) = x + y$$

$$\begin{aligned}
k_1 &= f(x_0, y_0) \\
&= f(0, 1) \\
&= 1 \\
k_2 &= f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1 h}{2}\right) \\
&= f(0.25, 1.25) \\
&= 1.5
\end{aligned}$$



$$\begin{aligned}
k_3 &= f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2 h}{2}\right) \\
&= f(0.25, 1.375) \\
&= 1.625
\end{aligned}$$

$$\begin{aligned}
k_4 &= f(x_0 + h, y_0 + k_3 h) \\
&= f(0.5, 1.8125) \\
&= 2.3125
\end{aligned}$$

$$\begin{aligned}
y_1 &= y_0 + \frac{1}{6} h(k_1 + 2k_2 + 2k_3 + k_4) \\
&= 1 + \frac{1}{6}(0.5)(1 + 2(1.5) + 2(1.625) + 2.3125) \\
&= 1.7969
\end{aligned}$$

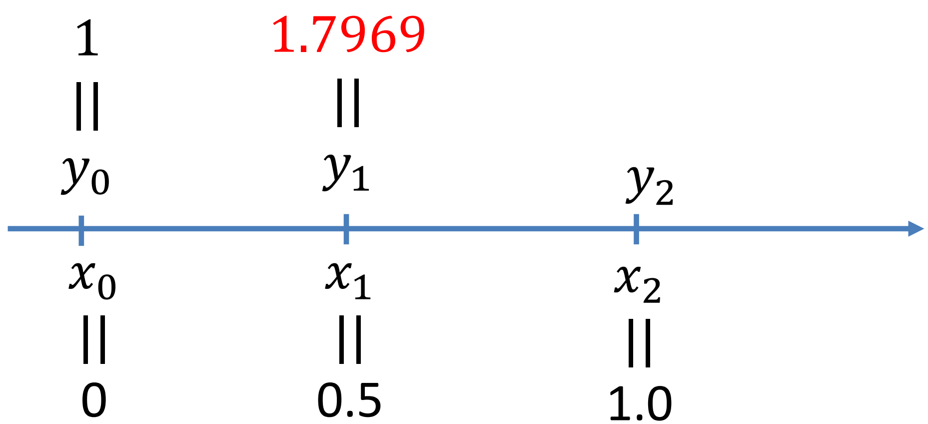
7.3.2 Forth Order Runge-Kutta Method (RK4)

Solution: (cont.)

Let $i = 1$,

$$y' = f(x, y) = x + y$$

$$\begin{aligned} k_1 &= f(x_1, y_1) \\ &= f(0.5, 1.7969) \\ &= 2.2969 \end{aligned}$$



$$\begin{aligned} k_2 &= f\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1 h}{2}\right) \\ &= f(0.75,\ 2.3711\ ) \\ &= 3.1211 \end{aligned}$$

$$\begin{aligned} k_3 &= f\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2 h}{2}\right) \\ &= f(0.75, 2.5772\ ) \\ &= 3.3272 \end{aligned}$$

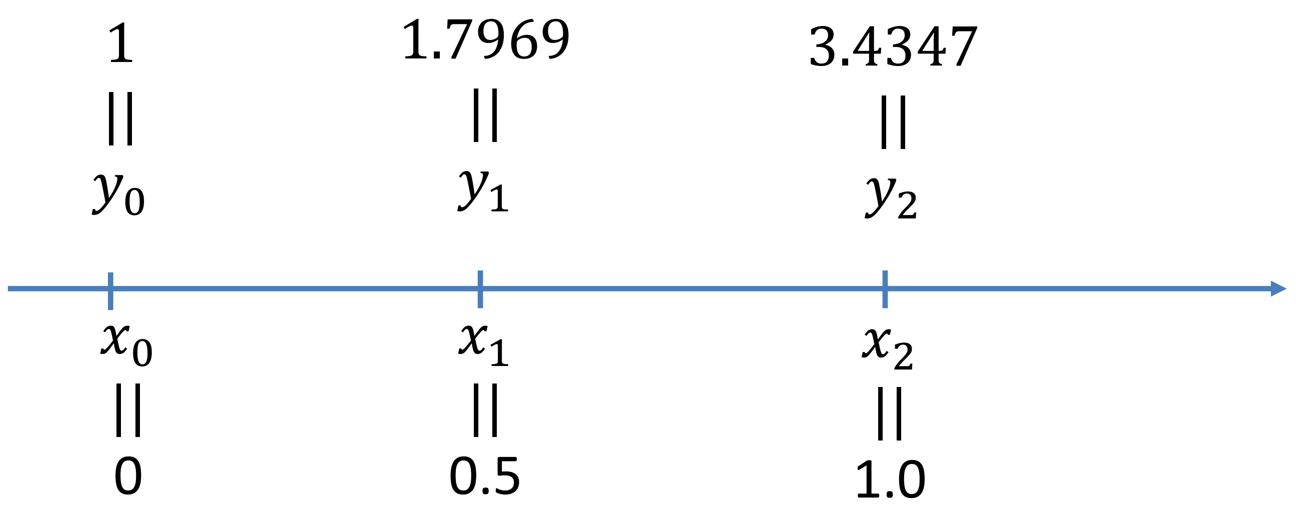
$$\begin{aligned} k_4 &= f(x_1 + h, y_1 + k_3 h\ ) \\ &= f(1, 3.4605\ ) \\ &= 4.4605 \end{aligned}$$

$$\begin{aligned} y_2 &= y_1 + \frac{1}{6} h(k_1 + 2k_2 + 2k_3 + k_4) \\ &= 1.7969 + \frac{1}{6}(0.5)(2.2969 + 2(3.1211) + 2(3.3272) + 4.4605) \\ &= 3.4347 \end{aligned}$$

7.3.2 Forth Order Runge-Kutta Method (RK4)

Solution: (cont.)

$$y' = f(x, y) = x + y$$



7.3.2 Forth Order Runge-Kutta Method (RK4)

Illustrative Example 2:

Use the forth order Runge-Kutta method to numerically integrate

$$y' + 3x = \cos 4y\,, \qquad\qquad y(0) = 0.3$$

from $x = 0$ to $x = 1$ with a step size of 0.25.

7.3.2 Forth Order Runge-Kutta Method (RK4)

Solution:

Reminder:
$y' = f(x, y)$

$$y' = f(x, y) = -3x + \cos 4y$$

$$0 \le x \le 1, \quad h = 0.25, \quad y(0) = 0.3$$

Construct the time line:

0.3				
$\|\|$				
y_0	y_1	y_2	y_3	y_4
x_0	x_1	x_2	x_3	x_4
$\|\|$	$\|\|$	$\|\|$	$\|\|$	$\|\|$
0	0.25	0.5	0.75	1.0

7.3.2 Forth Order Runge-Kutta Method (RK4)

Solution: (cont.)

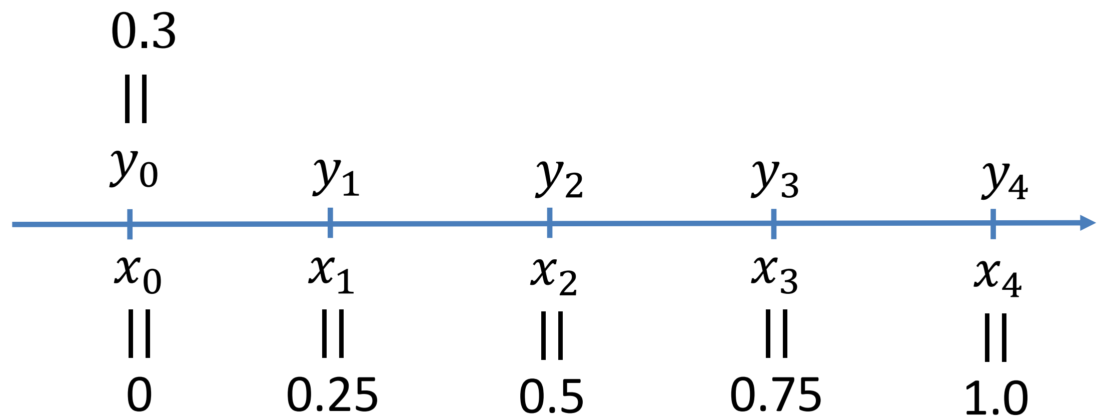
Let $i = 0$, $\qquad y' = f(x, y) = -3x + \cos 4y$

$$k_1 = f(x_0, y_0) = f(0, 0.3) = 0.3624$$

$$k_2 = f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1 h}{2}\right) = f(0.125, 0.3453) = -0.1865$$

$$k_3 = f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2 h}{2}\right) = f(0.125, 0.2767) = 0.0725$$

$$k_4 = f(x_0 + h, y_0 + k_3 h) = f(0.25, 0.3181) = -0.4560$$



$$y_1 = y_0 + \frac{1}{6}h(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= 0.3 + \frac{1}{6}(0.25)(0.3624 + 2(-0.1865) + 2(0.0725) + (-0.4560))$$

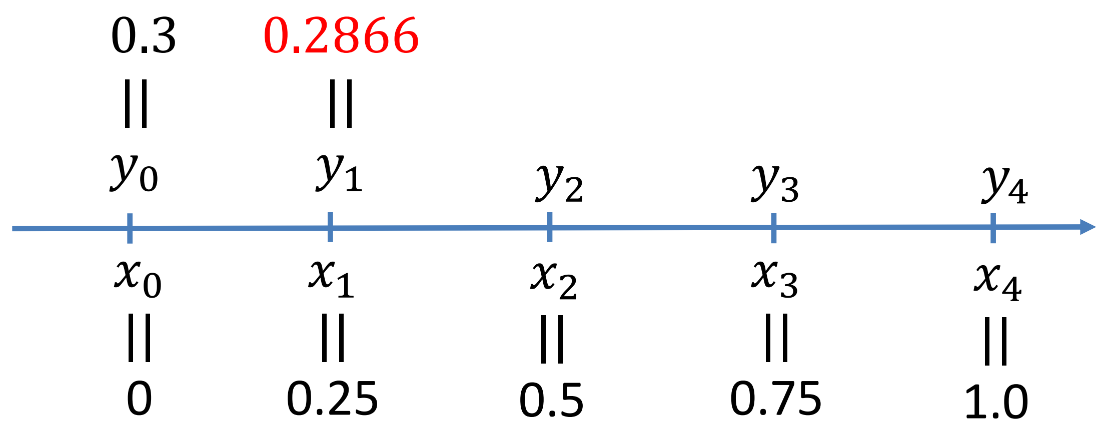
$$= 0.2866$$

7.3.2 Forth Order Runge-Kutta Method (RK4)

Solution: (cont.)

Let $i = 1$, $\quad y' = f(x,y) = -3x + \cos 4y$

$$k_1 = f(x_1, y_1) = f(0.25, 0.2866) = -0.3382$$



$$k_2 = f\left(x_1 + \frac{h}{2}, y_1 + \frac{k_1 h}{2}\right) = f(0.375, 0.2443) = -0.5657$$

$$k_3 = f\left(x_1 + \frac{h}{2}, y_1 + \frac{k_2 h}{2}\right) = f(0.375, 0.2159) = -0.4753$$

$$k_4 = f(x_1 + h, y_1 + k_3 h) = f(0.5, 0.1678) = -0.7169$$

$$y_2 = y_1 + \frac{1}{6}h(k_1 + 2k_2 + 2k_3 + k_4)$$
$$= 0.2866 + \frac{1}{6}(0.25)(-0.3382 + 2(-0.5657) + 2(-0.4753) + (-0.7169))$$
$$= 0.1559$$

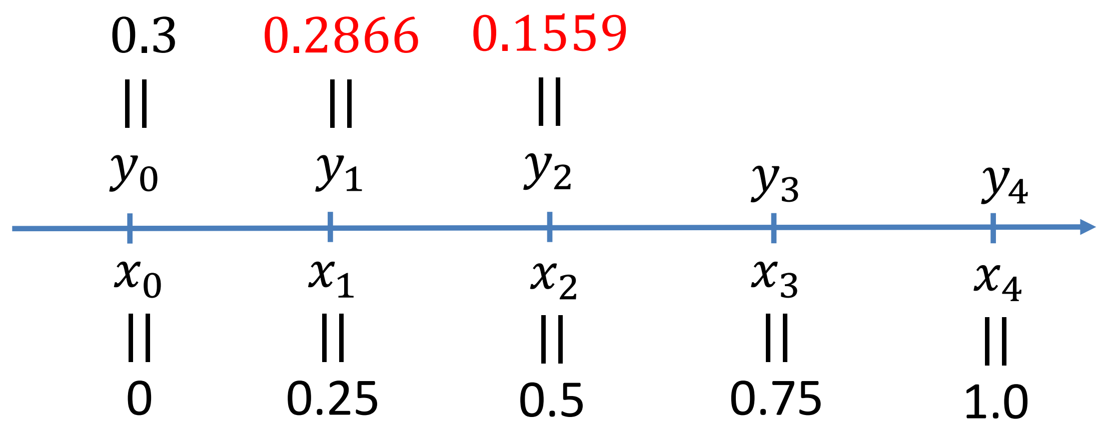
7.3.2 Forth Order Runge-Kutta Method (RK4)

Solution: (cont.)

Let $i = 1$,

$$y' = f(x, y) = -3x + \cos 4y$$

$$\begin{aligned} k_1 &= f(x_2, y_2) \\ &= f(0.5, 0.1559) \\ &= -0.6882 \end{aligned}$$



$$\begin{aligned} k_2 &= f\left(x_2 + \frac{h}{2}, y_2 + \frac{k_1 h}{2}\right) \\ &= f(0.625, 0.0699) \\ &= -0.9138 \end{aligned}$$

$$\begin{aligned} k_3 &= f\left(x_2 + \frac{h}{2}, y_2 + \frac{k_2 h}{2}\right) \\ &= f(0.625, 0.0417) \\ &= -0.8889 \end{aligned}$$

$$\begin{aligned} k_4 &= f(x_2 + h, y_2 + k_3 h) \\ &= f(0.75, -0.0663) \\ &= -1.2850 \end{aligned}$$

$$\begin{aligned} y_3 &= y_2 + \frac{1}{6} h(k_1 + 2k_2 + 2k_3 + k_4) \\ &= 0.1559 + \frac{1}{6}(0.25)(-0.6882 + 2(-0.9138) + 2(-0.8889) + (-1.2850)) \\ &= -0.0765 \end{aligned}$$

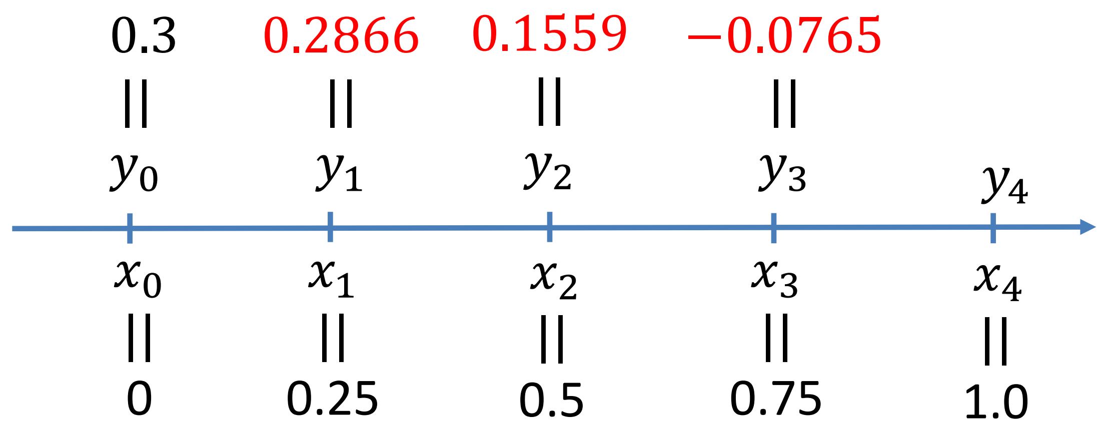
7.3.2 Forth Order Runge-Kutta Method (RK4)

Solution: (cont.)

Let $i = 1$,

$$y' = f(x, y) = -3x + \cos 4y$$

$$\begin{aligned} k_1 &= f(x_3, y_3) \\ &= f(0.75, -0.0765) \\ &= -1.2965 \end{aligned}$$



$$\begin{aligned} k_2 &= f\left(x_3 + \frac{h}{2}, y_3 + \frac{k_1 h}{2}\right) \\ &= f(0.875, -0.2386) \\ &= -2.0469 \end{aligned}$$

$$\begin{aligned} k_3 &= f\left(x_3 + \frac{h}{2}, y_3 + \frac{k_2 h}{2}\right) \\ &= f(0.875, -0.3324) \\ &= -2.3861 \end{aligned}$$

$$\begin{aligned} k_4 &= f(x_3 + h, y_3 + k_3 h) \\ &= f(1.0, -0.6731) \\ &= -3.9008 \end{aligned}$$

$$\begin{aligned} y_4 &= y_3 + \frac{1}{6} h(k_1 + 2k_2 + 2k_3 + k_4) \\ &= -0.0765 + \frac{1}{6}(0.25)(-1.2965 + 2(-2.0469) + 2(-2.3861) + (-3.9008)) \\ &= -0.6625 \end{aligned}$$

7.3.2 Forth Order Runge-Kutta Method (RK4)

Solution: (cont.)

$$y' = f(x, y) = -3x + \cos 4y$$

0.3	0.2866	0.1559	-0.0765	-0.6625
$\parallel$	$\parallel$	$\parallel$	$\parallel$	$\parallel$
y_0	y_1	y_2	y_3	y_4
x_0	x_1	x_2	x_3	x_4
$\parallel$	$\parallel$	$\parallel$	$\parallel$	$\parallel$
0	0.25	0.5	0.75	1.0

7.3.2 Forth Order Runge-Kutta Method (RK4)

Exercise 7.4:

Use the forth order Runge-Kutta method to numerically integrate

$$\frac{dy}{dx} + y = 1 - x^2, \qquad y(0) = 0.5$$

from $x = 0$ to $x = 0.2$ with a step size of $h = 0.1$.

7.3.2 Forth Order Runge-Kutta Method (RK4)

Solution:

$$y' = f(x, y) = 1 - x^2 - y$$

0.5	0.5473	0.5881
$\Vert$	$\Vert$	$\Vert$
y_0	y_1	y_2
x_0	x_1	x_2
$\Vert$	$\Vert$	$\Vert$
0	0.1	0.2

$$k_1 = 0.5 \qquad k_1 = 0.4427$$
$$k_2 = 0.4725 \qquad k_2 = 0.4081$$
$$k_3 = 0.4739 \qquad k_3 = 0.4098$$
$$k_4 = 0.4426 \qquad k_4 = 0.3718$$