



OPENCOURSEWARE

NUMERICAL METHODS BEKG2452 SOLUTION OF ORDINARY DIFFERENTIAL EQUATION (Euler's Method & Heun's Method)

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Learning Outcomes

At the end of this topic, student should be able to:

1. Solve ordinary differential equation by using Euler's Method.
2. Solve ordinary differential equation by using Heun's Method.

Solution of Ordinary Differential Equation

Analytical Methods

- Separable
- Exact
- Homogeneous
- Linear
- Bernoulli

Exact solution

Numerical Methods

Euler's
Method

Heun's
Method

Runge-Kutta
Method

- Order 2
- Order 4

Approximated solution

7.1 Euler's Method

- ✓ Most basic explicit method for solving ordinary differential equation (ODE)
- ✓ The simplest Runge-Kutta Method

Formula:

$$y_{i+1} = y_i + hf(x_i, y_i)$$

7.1 Euler's Method

Draw a line structure and pull in the initial conditions



Rearrange the given ODE in the form of

$$y' = f(x, y)$$

Discretize the equation into nth terms:

$$y'_i = f(x_i, y_i); \quad i = 0, 1, 2, \dots, n$$

Compute the next term by using formula

$$y_{i+1} = y_i + hf(x_i, y_i); \quad i = 0, 1, 2, \dots, n$$

7.1 Euler's Method

Example 1:

Use Euler's method to numerically integrate

$$\frac{dy}{dx} + y = 3x^3 - 7x^2 + 5x, \quad y(0) = 1$$

from $x = 0$ to $x = 2$ with a step size of 0.5.

7.1 Euler's Method

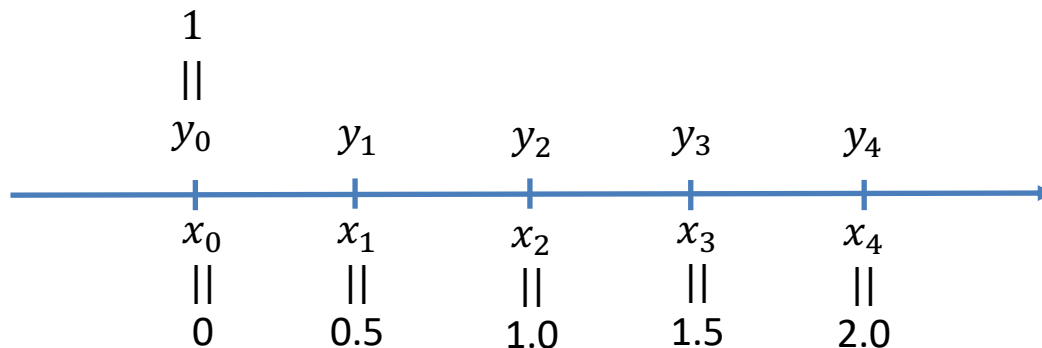
Solution:

Reminder:
 $y' = f(x, y)$

$$\frac{dy}{dx} = f(x, y) = 3x^3 - 7x^2 + 5x - y$$

$$0 \leq x \leq 2, \quad h = 0.5, \quad y(0) = 1$$

Construct the time line:



Let $i = 0$,

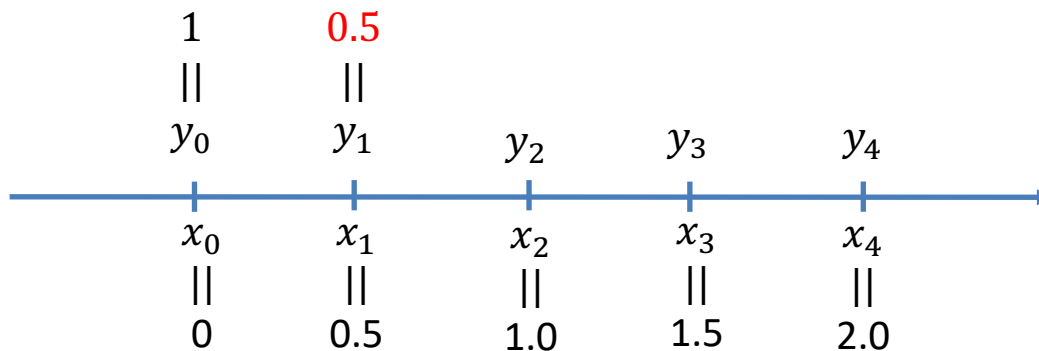
$$\begin{aligned}
 y_1 &= y_0 + hf(x_0, y_0) \\
 &= 1 + 0.5f(0, 1) \\
 &= 1 + 0.5[3(0)^3 - 7(0)^2 + 5(0) - 1] \\
 &= 0.5
 \end{aligned}$$

7.1 Euler's Method

Solution: (cont.)

$$\frac{dy}{dx} = f(x, y) = 3x^3 - 7x^2 + 5x - y$$

Update the time line:



Let $i = 1$,

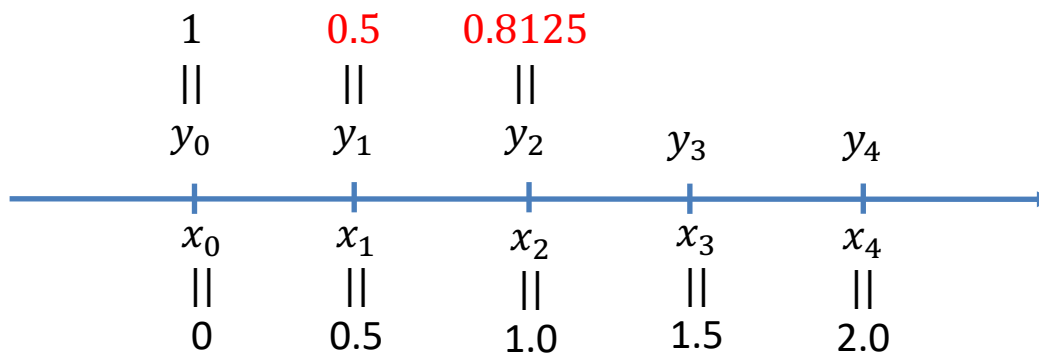
$$\begin{aligned}
 y_2 &= y_1 + hf(x_1, y_1) \\
 &= 0.5 + 0.5f(0.5, 0.5) \\
 &= 0.5 + 0.5[3(0.5)^3 - 7(0.5)^2 + 5(0.5) - 0.5] \\
 &= 0.8125
 \end{aligned}$$

7.1 Euler's Method

Solution: (cont.)

$$\frac{dy}{dx} = f(x, y) = 3x^3 - 7x^2 + 5x - y$$

Update the time line:



Let $i = 2$,

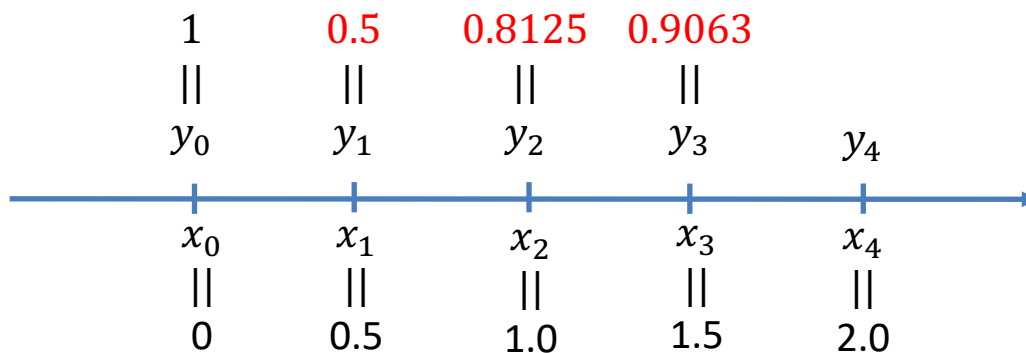
$$\begin{aligned}
 y_3 &= y_2 + hf(x_2, y_2) \\
 &= 0.8125 + 0.5f(1, 0.8125) \\
 &= 0.8125 + 0.5[3(1)^3 - 7(1)^2 + 5(1) - 0.8125] \\
 &= 0.9063
 \end{aligned}$$

7.1 Euler's Method

Solution: (cont.)

$$\frac{dy}{dx} = f(x, y) = 3x^3 - 7x^2 + 5x - y$$

Update the time line:



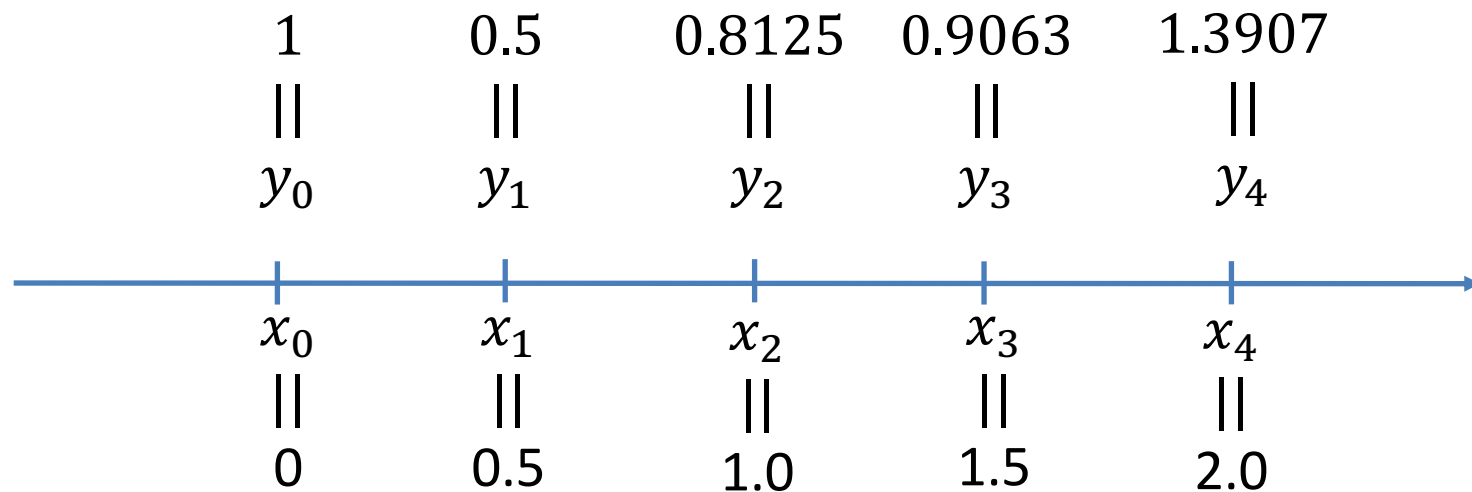
Let $i = 3$,

$$\begin{aligned}
 y_4 &= y_3 + hf(x_3, y_3) \\
 &= 0.9063 + 0.5f(1.5, 0.9063) \\
 &= 0.9063 + 0.5[3(1.5)^3 - 7(1.5)^2 + 5(1.5) - 0.9063] \\
 &= 1.3907
 \end{aligned}$$

7.1 Euler's Method

Solution: (cont.)

Update the time line:



7.1 Euler's Method

Example 2:

Use Euler's method to solve the initial value problem

$$\frac{dy}{dt} = \frac{t - y}{2}$$

on $0 \leq t \leq 1$ with $h=0.25$. The initial condition at $t=0$ is $y=1$.

7.1 Euler's Method

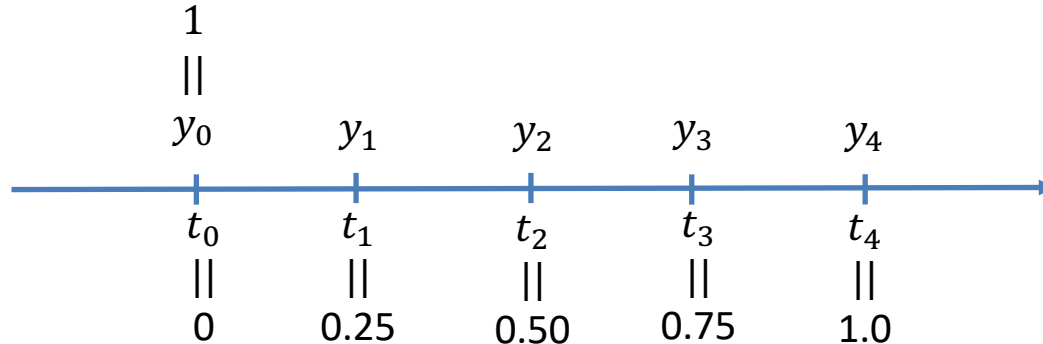
Solution:

$$\frac{dy}{dt} = f(t, y) = \frac{t-y}{2} \quad 0 \leq t \leq 1,$$

$$h = 0.25, \quad y(0) = 1$$

Reminder:
 $y' = f(x, y)$

Construct the time line:



Let $i = 0$,

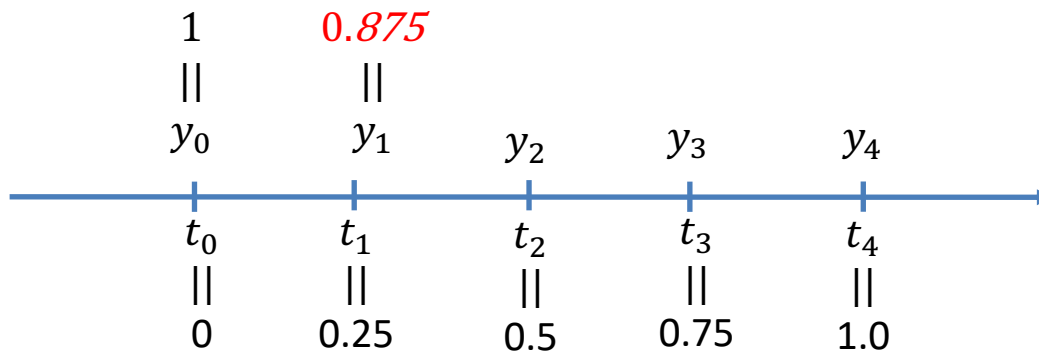
$$\begin{aligned} y_1 &= y_0 + hf(t_0, y_0) \\ &= 1 + 0.25f(0, 1) \\ &= 1 + 0.25 \left[\frac{0-1}{2} \right] \\ &= 0.875 \end{aligned}$$

7.1 Euler's Method

Solution: (cont.)

$$\frac{dy}{dt} = f(t, y) = \frac{t - y}{2}$$

Update the time line:



Let $i = 1$,

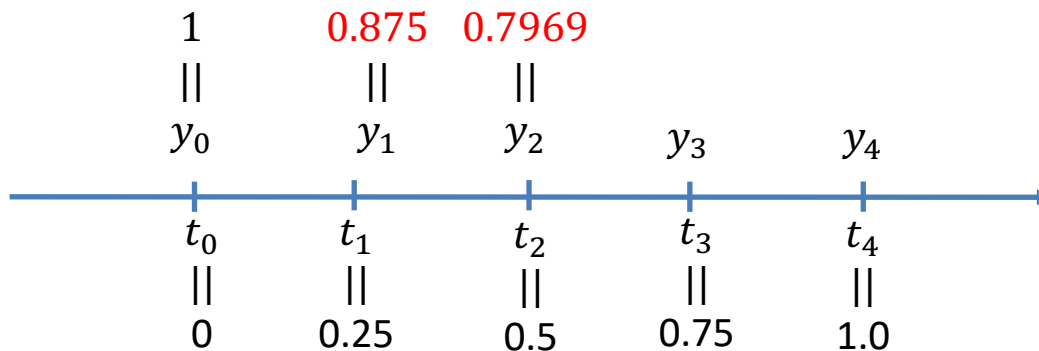
$$\begin{aligned} y_2 &= y_1 + hf(t_1, y_1) \\ &= 0.875 + 0.25f(0.25, 0.875) \\ &= 0.875 + 0.25 \left[\frac{0.25 - 0.875}{2} \right] \\ &= 0.7969 \end{aligned}$$

7.1 Euler's Method

Solution: (cont.)

$$\frac{dy}{dt} = f(t, y) = \frac{t - y}{2}$$

Update the time line:



Let $i = 2$,

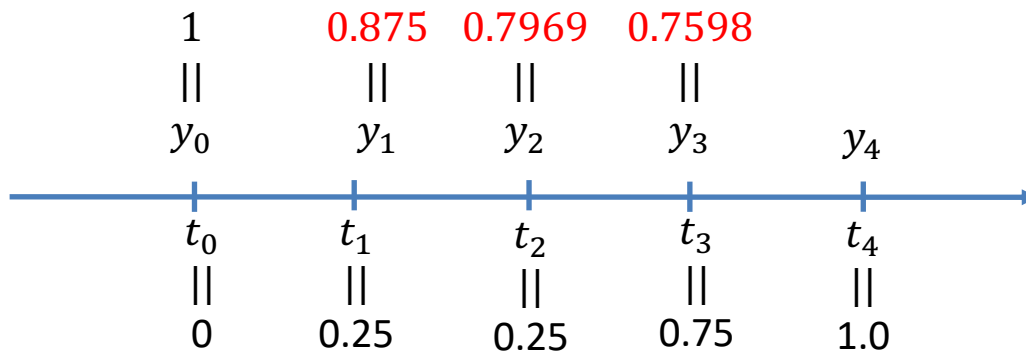
$$\begin{aligned} y_3 &= y_2 + hf(t_2, y_2) \\ &= 0.7969 + 0.25f(0.5, 0.7969) \\ &= 0.7969 + 0.25 \left[\frac{0.5 - 0.7969}{2} \right] \\ &= 0.7598 \end{aligned}$$

7.1 Euler's Method

Solution: (cont.)

$$\frac{dy}{dt} = f(t, y) = \frac{t - y}{2}$$

Update the time line:



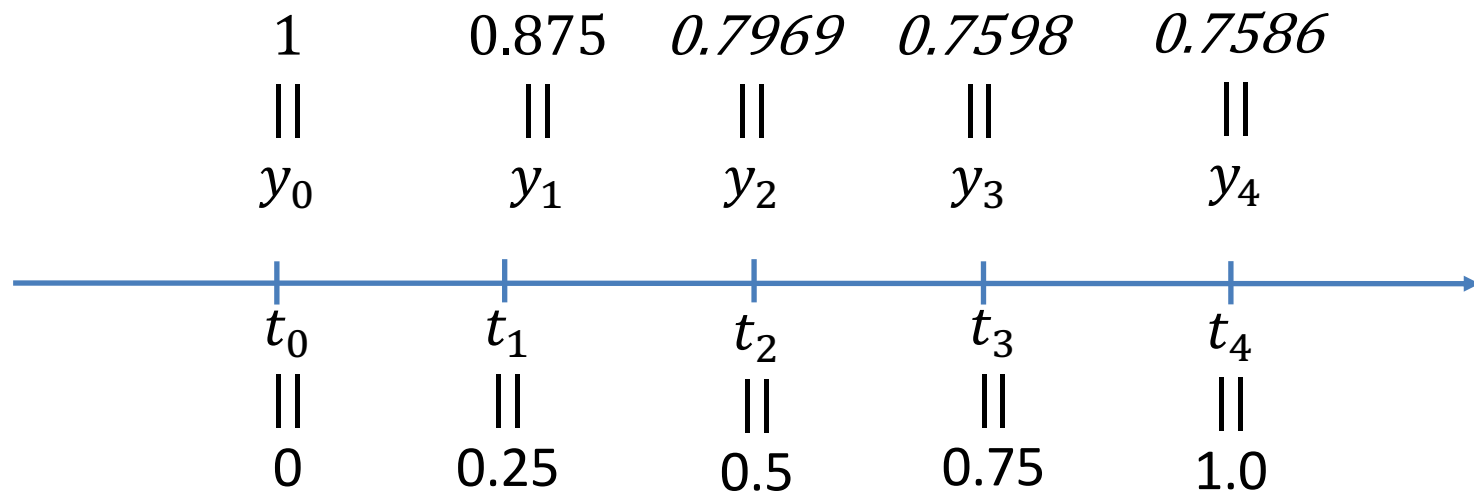
Let $i = 3$,

$$\begin{aligned}
 y_4 &= y_3 + hf(t_3, y_3) \\
 &= 0.7598 + 0.25f(0.75, 0.7598) \\
 &= 0.7598 + 0.25 \left[\frac{0.75 - 0.7598}{2} \right] \\
 &= 0.7586
 \end{aligned}$$

7.1 Euler's Method

Solution: (cont.)

Update the time line:



7.1 Euler's Method

Exercise:

Use Euler's method to numerically integrate

$$\frac{dy}{dx} = 3y^2 - 5x - 1, \quad y(0) = 0$$

from $x = 0$ to $x = 1$ with a step size of 0.25.

7.1 Euler's Method

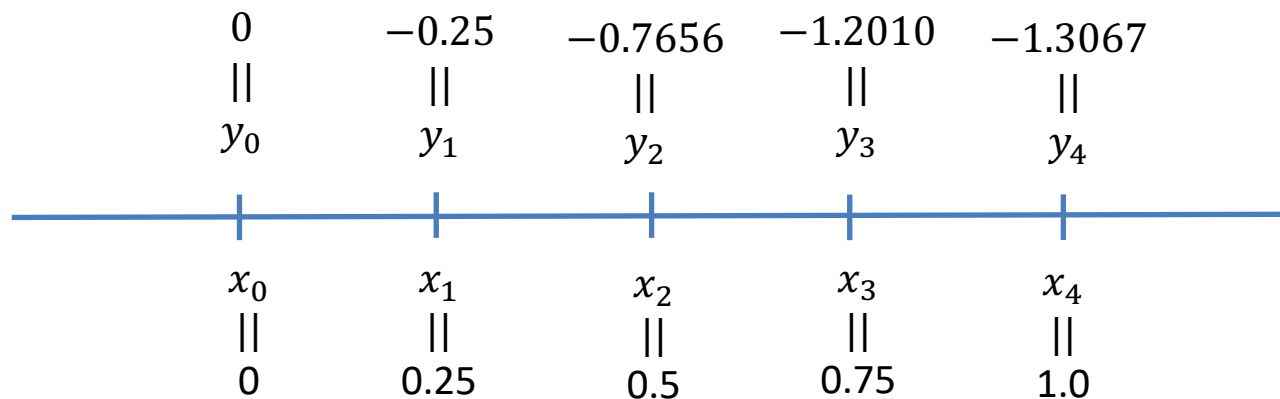
Answer:

Reminder:
 $y' = f(x, y)$

$$\frac{dy}{dx} = f(x, y) = 3y^2 - 5x - 1$$

$$0 \leq x \leq 1, \quad h = 0.25, \quad y(0) = 0$$

Time line:



7.2 Heun's Method

✓ Improved/modified Euler's Method

Formula

$$y_{i+1} = y_i + \frac{h}{2} [f(x_i, y_i) + f(x_{i+1}, y_{i+1}^{\circ})] \quad i = 0, 1, 2, 3, \dots$$

where

$$y_{i+1}^{\circ} = y_i + hf(x_i, y_i) \quad (\text{predictor estimate})$$

7.2 Heun's Method

Line Structure



Rearrange: $y' = f(x, y)$



Discretizing: $y'_i = f(x_i, y_i)$
& compute predictor estimate y_{i+1}^0



Compute y_{i+1}

7.2 Heun's Method

Example 1:

Use Heun's method to numerically integrate

$$y' + 3y = e^x, \quad y(0) = 1$$

from $x = 0$ to $x = 2$ with a step size of 0.5.

7.2 Heun's Method

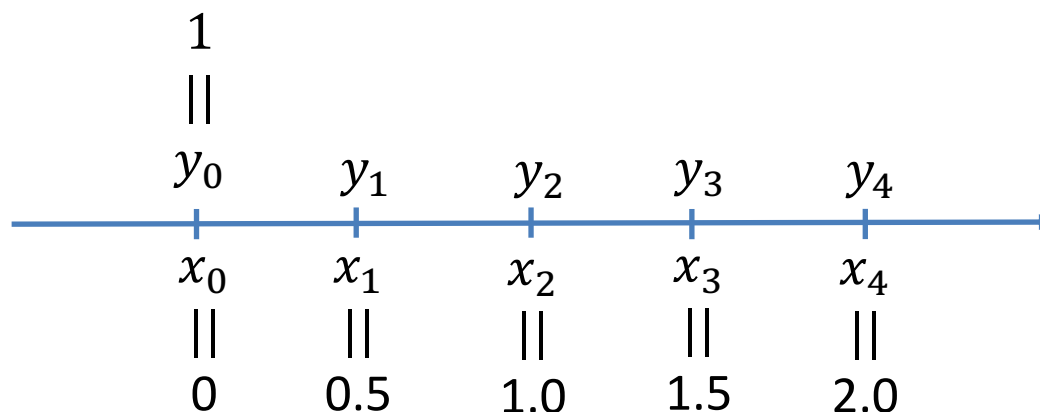
Reminder:
 $y' = f(x, y)$

Solution:

$$y' = f(x, y) = -3y + e^x$$

$$0 \leq x \leq 2, \quad h = 0.5, \quad y(0) = 1$$

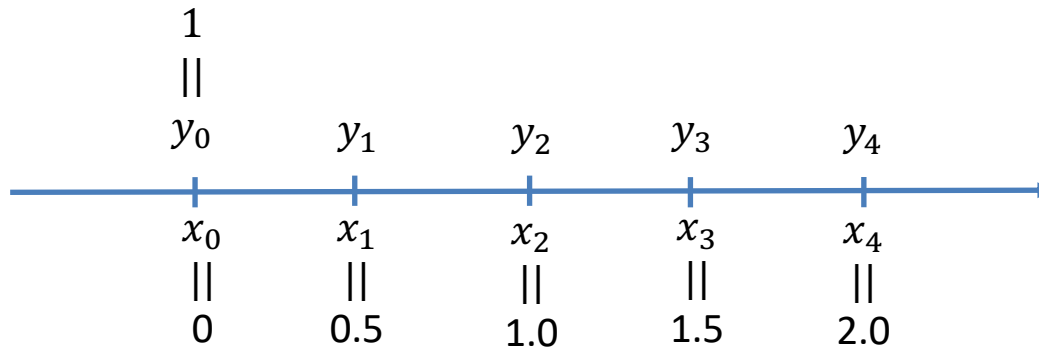
Construct the time line:



7.2 Heun's Method

Solution: (cont.)

$$y' = f(x, y) = -3y + e^x$$



Let $i = 0$,

$$\begin{aligned} f(x_0, y_0) &= f(0, 1) \\ &= -3(1) + e^0 = -2 \end{aligned}$$

$$\begin{aligned} y_1^\circ &= y_0 + hf(x_0, y_0) \\ &= 1 + 0.5f(0, 1) \\ &= 0 \end{aligned}$$

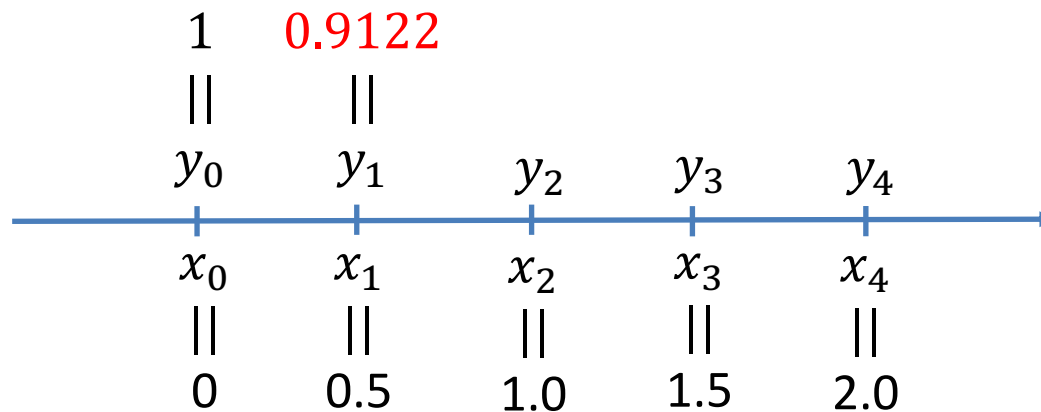
$$\begin{aligned} f(x_1, y_1^\circ) &= f(0.5, 0) = -3(0) + e^{0.5} \\ &= 1.6487 \end{aligned}$$

$$\begin{aligned} y_1 &= y_0 + \frac{h}{2} [f(x_0, y_0) + f(x_1, y_1^\circ)] \\ &= 1 + \frac{0.5}{2} (-2 + 1.6487) \\ &= 0.9122 \end{aligned}$$

7.2 Heun's Method

Solution: (cont.)

$$y' = f(x, y) = -3y + e^x$$



Let $i = 1$,

$$\begin{aligned} f(x_1, y_1) &= f(0.5, 0.9122) \\ &= -3(0.9122) + e^{0.5} \\ &= -1.0879 \end{aligned}$$

$$\begin{aligned} y_2^\circ &= y_1 + hf(x_1, y_1) \\ &= 0.9122 + 0.5f(0.5, 0.9122) \\ &= 0.3683 \end{aligned}$$

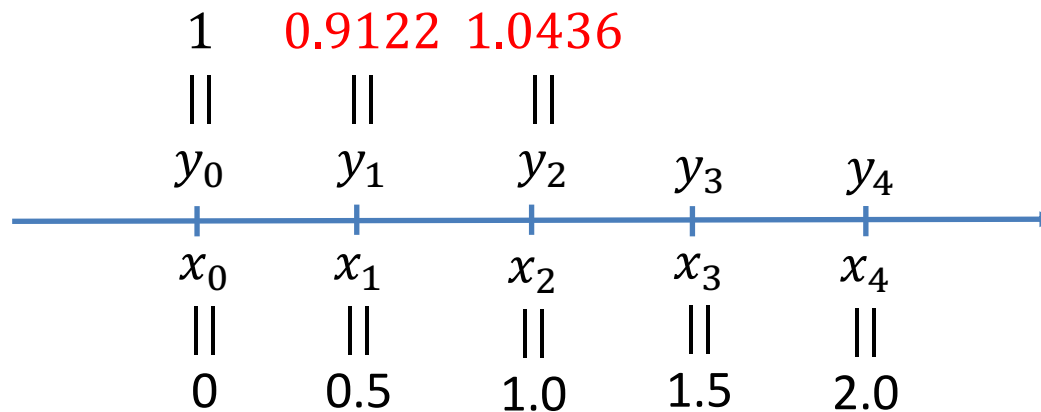
$$\begin{aligned} f(x_2, y_2^\circ) &= f(1, 0.3683) \\ &= -3(0.3683) + e^1 \\ &= 1.6134 \end{aligned}$$

$$\begin{aligned} y_2 &= y_1 + \frac{h}{2} [f(x_1, y_1) + f(x_2, y_2^\circ)] \\ &= 0.9122 + \frac{0.5}{2} (-1.0879 + 1.6134) \\ &= 1.0436 \end{aligned}$$

7.2 Heun's Method

Solution: (cont.)

$$y' = f(x, y) = -3y + e^x$$



Let $i = 2$,

$$\begin{aligned} f(x_2, y_2) &= f(1, 1.0436) \\ &= -3(1.0436) + e^1 \\ &= -0.4125 \end{aligned}$$

$$\begin{aligned} y_3^\circ &= y_2 + hf(x_2, y_2) \\ &= 1.0436 + 0.5 f(1, 1.0436) \\ &= 0.8374 \end{aligned}$$

$$\begin{aligned} f(x_3, y_3^\circ) &= f(1.5, 0.8374) \\ &= -3(0.8374) + e^{1.5} \\ &= 1.9695 \end{aligned}$$

$$\begin{aligned} y_3 &= y_2 + \frac{h}{2} [f(x_2, y_2) + f(x_3, y_3^\circ)] \\ &= 1.0436 + \frac{0.5}{2} (-0.4125 + 1.9695) \\ &= 1.4329 \end{aligned}$$

7.2 Heun's Method

Solution: (cont.)

$$y' = f(x, y) = -3y + e^x$$

1	0.9122	1.0436	1.4329	
y_0	y_1	y_2	y_3	y_4
x_0	x_1	x_2	x_3	x_4
0	0.5	1.0	1.5	2.0

Let $i = 3$,

$$\begin{aligned} f(x_3, y_3) &= f(1.5, 1.4329) \\ &= -3(1.4329) + e^{1.5} \\ &= 0.183 \end{aligned}$$

$$\begin{aligned} y_4^\circ &= y_3 + hf(x_3, y_3) \\ &= 1.4329 + 0.5 f(1.5, 1.4329) \\ &= 1.5244 \end{aligned}$$

$$\begin{aligned} f(x_4, y_4^\circ) &= f(2, 1.5244) \\ &= -3(1.5244) + e^2 \\ &= 2.8159 \end{aligned}$$

$$\begin{aligned} y_4 &= y_3 + \frac{h}{2} [f(x_3, y_3) + f(x_4, y_4^\circ)] \\ &= 1.4329 + \frac{0.5}{2} (0.183 + 2.8159) \\ &= 2.1826 \end{aligned}$$

7.2 Heun's Method

Solution: (cont.)

$$y' = f(x, y) = -3y + e^x$$

1	0.9122	1.0436	1.4329	2.1826
y_0	y_1	y_2	y_3	y_4
x_0	x_1	x_2	x_3	x_4
0	0.5	1.0	1.5	2.0

7.2 Heun's Method

Example 2:

Use Heun's method to numerically integrate

$$\frac{dy}{dt} = \frac{t - y}{2}$$

on $0 \leq t \leq 1$ with $h=0.25$. The initial condition at $t=0$ is $y=1$.

7.2 Heun's Method

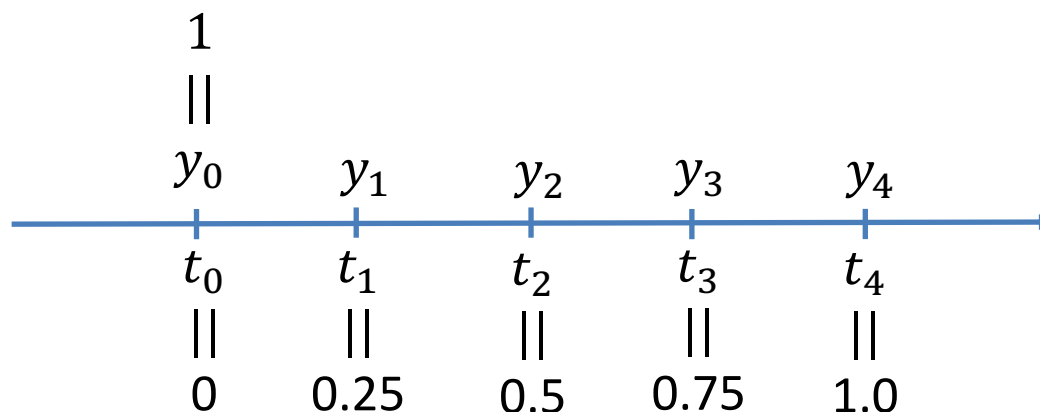
Reminder:
 $y' = f(x, y)$

Solution:

$$y' = f(t, y) = \frac{dy}{dt} = \frac{t - y}{2}$$

$$0 \leq t \leq 1, \quad h = 0.25, \quad y(0) = 1$$

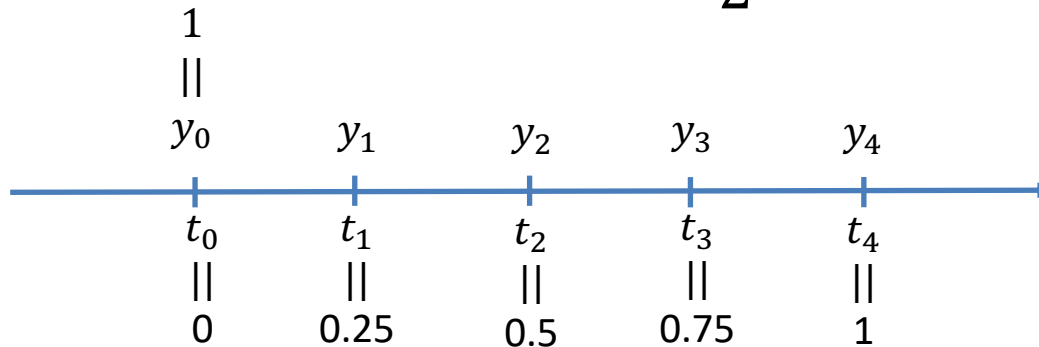
Construct the time line:



7.2 Heun's Method

Solution: (cont.)

$$y' = f(t, y) = \frac{t - y}{2}$$



Let $i = 0$,

$$\begin{aligned} f(t_0, y_0) &= f(0, 1) \\ &= \frac{0-1}{2} = -0.5 \end{aligned}$$

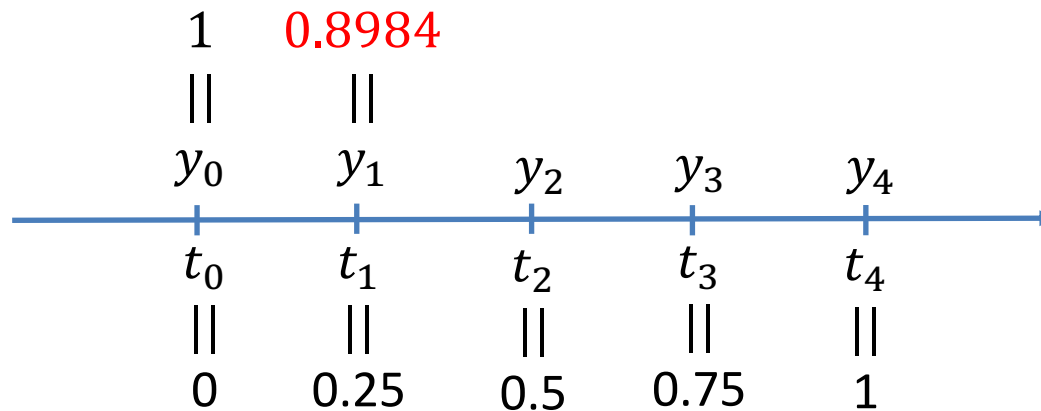
$$\begin{aligned} y_1^\circ &= y_0 + hf(t_0, y_0) \\ &= 1 + 0.25f(0, 1) \\ &= 0.875 \end{aligned}$$

$$\begin{aligned} f(t_1, y_1^\circ) &= f(0.25, 0.875) \\ &= -0.3125 \end{aligned}$$

$$\begin{aligned} y_1 &= y_0 + \frac{h}{2} [f(t_0, y_0) + f(t_1, y_1^\circ)] \\ &= 1 + \frac{0.25}{2} [f(0, 1) + f(0.25, 0.875)] \\ &= \mathbf{0.8984} \end{aligned}$$

7.2 Heun's Method

Solution: (cont.) $y' = f(t, y) = \frac{t-y}{2}$



Let $i = 1$,

$$f(t_1, y_1) = f(0.25, 0.8984) \\ = -0.3242$$

$$y_2^{\circ} = y_1 + hf(t_1, y_1) \\ = 0.8984 + 0.25f(0.25, 0.8984) \\ = 0.8174$$

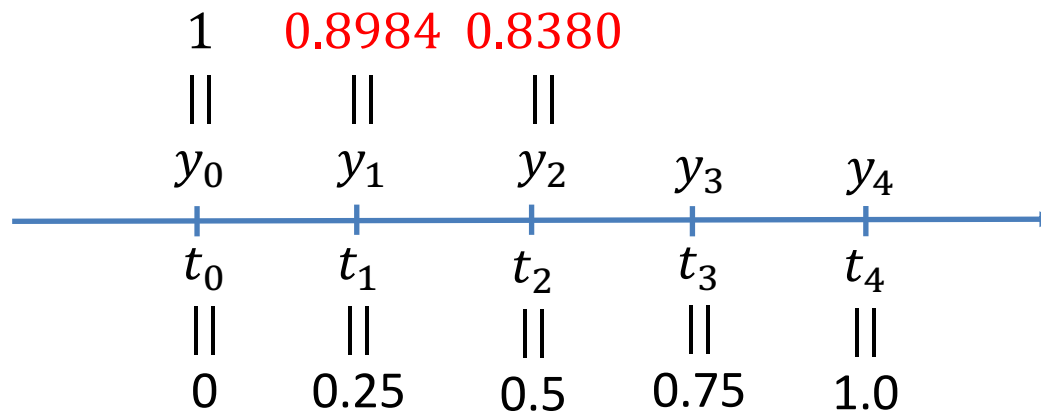
$$f(t_2, y_2^{\circ}) = f(0.5, 0.8174) \\ = -0.1587$$

$$y_2 = y_1 + \frac{h}{2} [f(t_1, y_1) + f(t_2, y_2^{\circ})] \\ = 0.8984 + \frac{0.25}{2} [f(0.25, 0.8984) + f(0.5, 0.8174)] \\ = 0.8380$$

7.2 Heun's Method

Solution: (cont.)

$$y' = f(t, y) = \frac{t-y}{2}$$



Let $i = 2$,

$$\begin{aligned} f(t_2, y_2) &= f(0.5, 0.8380) \\ &= -0.169 \end{aligned}$$

$$\begin{aligned} y_3^\circ &= y_2 + hf(t_2, y_2) \\ &= 0.8380 + 0.25 f(0.5, 0.8380) \\ &= 0.7958 \end{aligned}$$

$$\begin{aligned} f(t_3, y_3^\circ) &= f(0.75, 0.7958) \\ &= -0.0229 \end{aligned}$$

$$\begin{aligned} y_3 &= y_2 + \frac{h}{2} [f(t_2, y_2) + f(t_3, y_3^\circ)] \\ &= 0.8380 + \frac{h}{2} [f(0.5, 0.8380) + f(0.75, 0.7958)] \\ &= 0.8140 \end{aligned}$$

7.2 Heun's Method

Solution: (cont.) $y' = f(t, y) = \frac{t-y}{2}$

1	0.8984	0.8380	0.8140	
y_0	y_1	y_2	y_3	y_4
t_0	t_1	t_2	t_3	t_4
0	0.25	0.5	0.75	1

Let $i = 3$,

$$f(t_3, y_3) = f(0.75, 0.8140)$$

$$= -0.032$$

$$y_4^\circ = y_3 + hf(t_3, y_3)$$

$$= 0.8140 + 0.25 f(0.75, 0.8140)$$

$$= 0.8060$$

$$f(t_4, y_4^\circ) = f(1, 0.8060)$$

$$= 0.097$$

$$y_4 = y_3 + \frac{h}{2} [f(t_3, y_3) + f(t_4, y_4^\circ)]$$

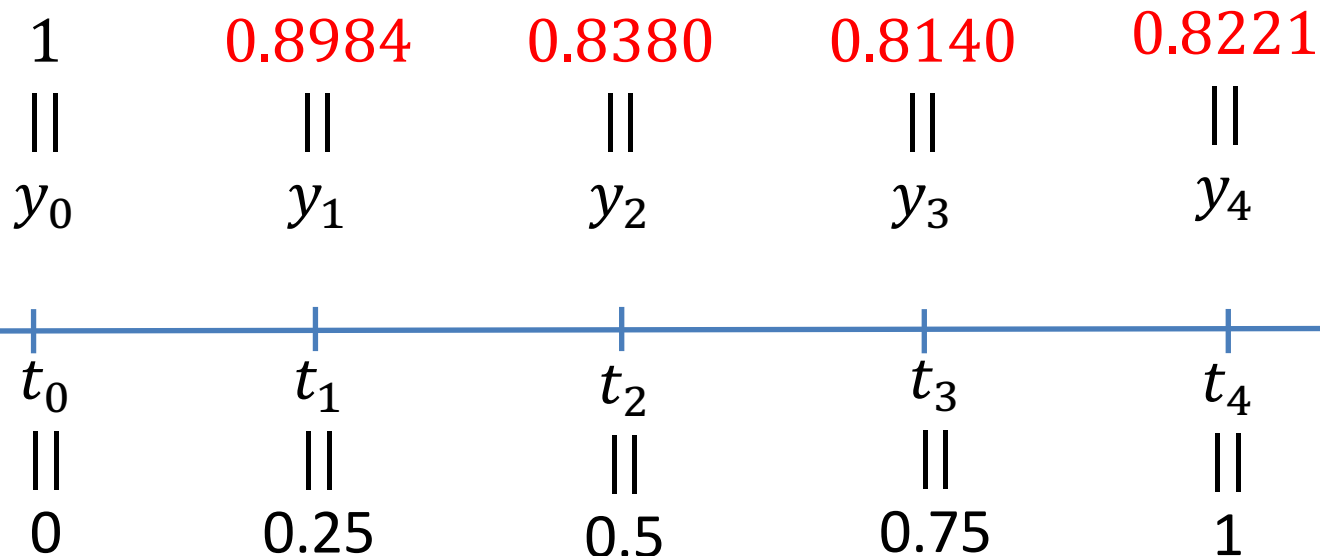
$$= 0.8140 + \frac{0.25}{2} [f(0.75, 0.8140) + f(1, 0.8060)]$$

$$= 0.8221$$

7.2 Heun's Method

Solution: (cont.)

$$y' = f(t, y) = \frac{t - y}{2}$$



7.2 Heun's Method

Exercise:

Use Heun's method to numerically integrate

$$\frac{dy}{dx} - 2y = 3x, \quad y(0) = 2$$

from $x = 0$ to $x = 1$ with a step size of 0.25.

7.2 Heun's Method

Answer:

$$y' = f(x, y) = 3x + 2y$$

