

NUMERICAL METHODS BEKG2452 NUMERICAL INTEGRATION

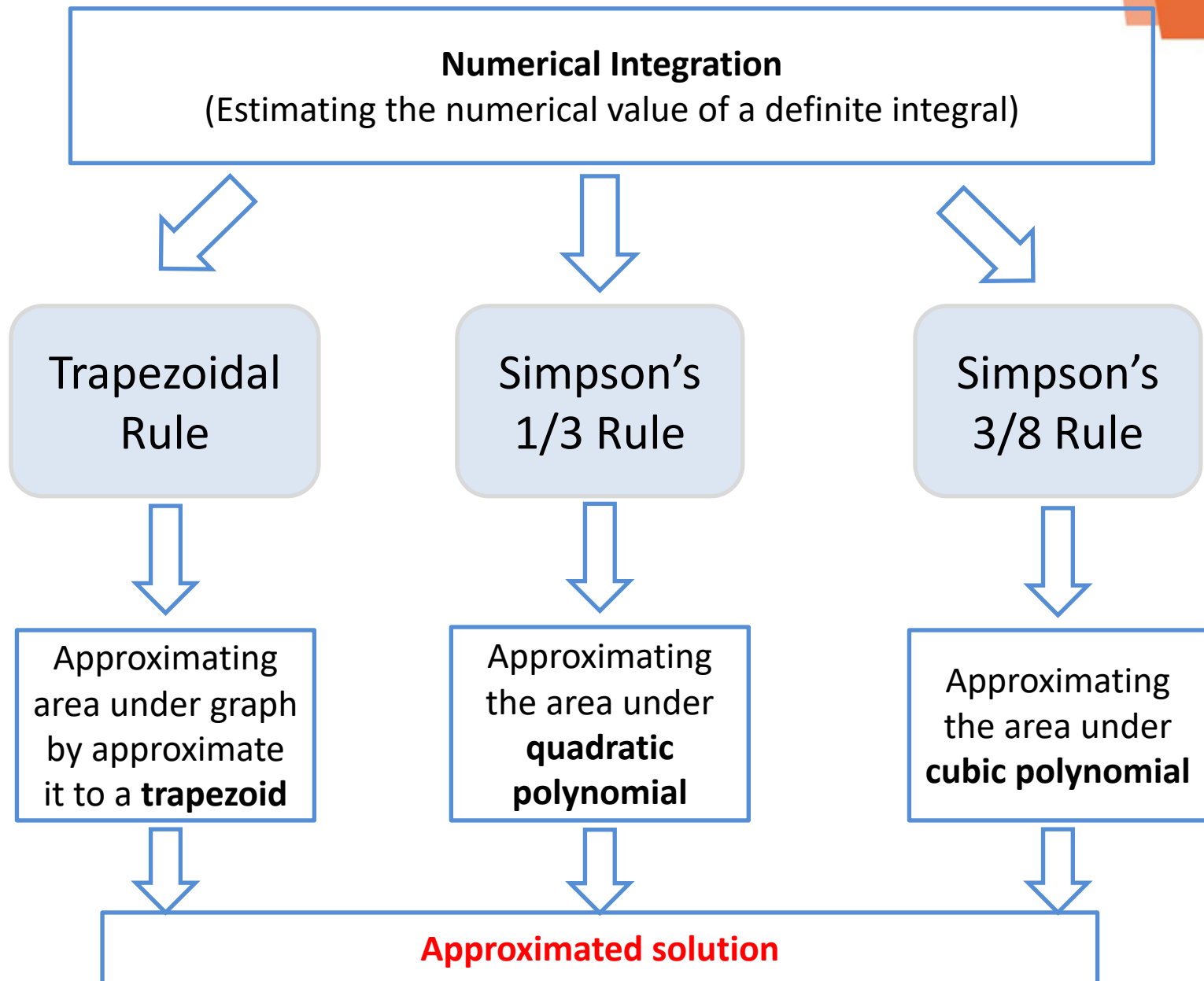
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Learning Outcomes

At the end of this chapter, student should be able to:

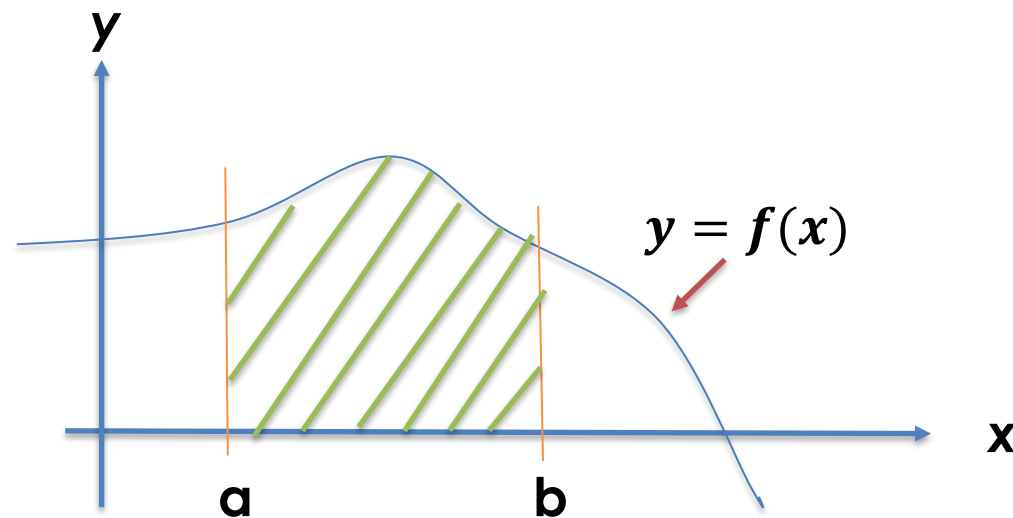
1. Solve the definite integral using Trapezoidal, Simpson's $1/3$ and Simpson's $3/8$ method.
2. Calculate area under a curve.



6.1 Introduction

- In calculus, integration is the reverse of differentiation.
- It represents area under a curve.

$$I = \int_a^b f(x) \, dx$$



6.1 Introduction

When we need numerical integration?

Situation 1

Analytical integration may be impossible, e.g.

$$\int_0^1 e^{x^2} dx$$

or complicated, e.g.

$$\int_0^3 e^{\frac{1}{x}} \sin \sqrt{\ln(3x + x^2)} dx$$

Situation 2

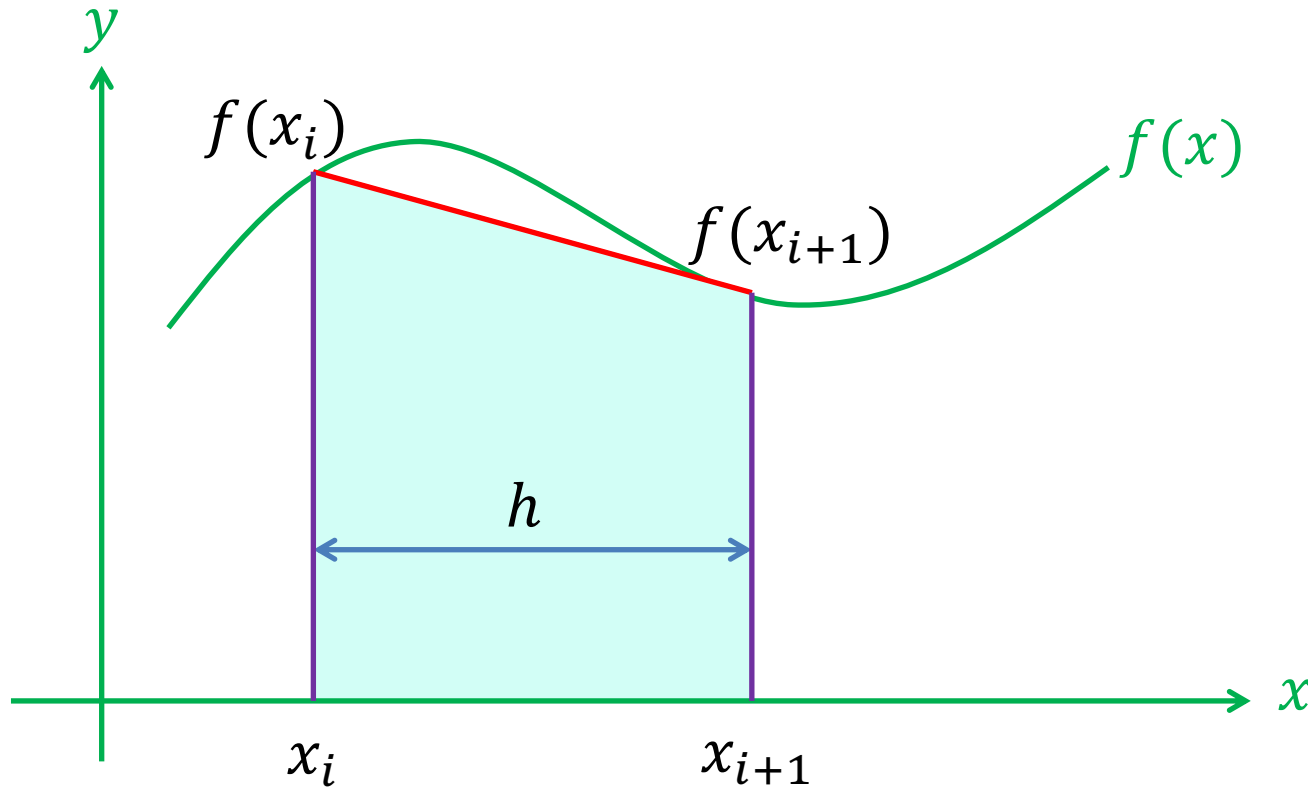
Given tabulated data, e.g.

x	1.0	1.5	2.0
$f(x)$	3.4	5.7	7.3

Estimate

$$\int_{1.0}^{2.0} [f(x)]^3 dx$$

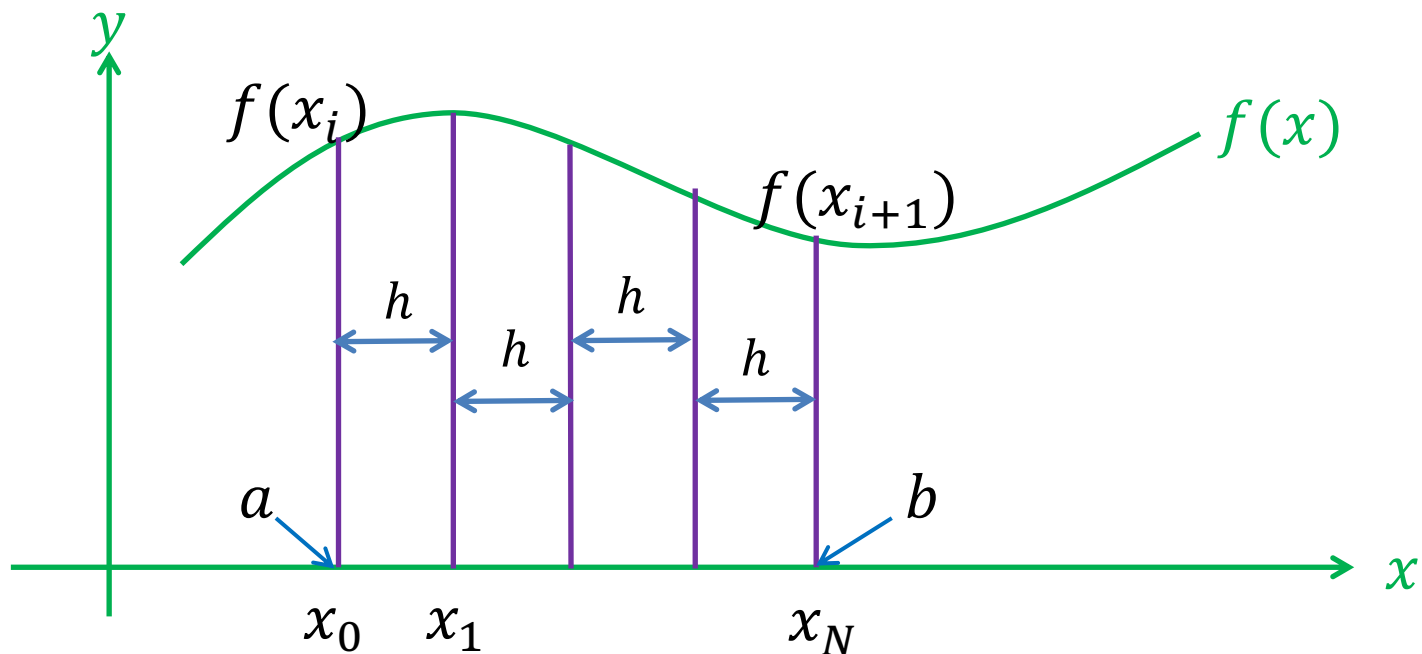
6.2 Trapezoidal Rule



$$\text{Area} = \int_{x_i}^{x_{i+1}} f(x) dx \approx \frac{h}{2} [f_i + f_{i+1}] \quad \leftarrow \text{Area of trapezoid}$$

6.2 Trapezoidal Rule

When the interval $[a, b]$ is divided into N equal strips of width h such that $x_i = a + ih, i = 0, 1, 2, 3, \dots, N$ and $b = a + Nh$,



$$\int_{x_0}^{x_N} f(x) dx = \int_{x_0}^{x_1} f(x) dx + \int_{x_1}^{x_2} f(x) dx + \dots + \int_{x_{N-1}}^{x_N} f(x) dx$$

6.2 Trapezoidal Rule

$$\begin{aligned}
 \int_{x_0}^{x_N} f(x) dx &= \int_{x_0}^{x_1} f(x) dx + \int_{x_1}^{x_2} f(x) dx + \cdots + \int_{x_{N-1}}^{x_N} f(x) dx \\
 &= \frac{h}{2} [f_0 + f_1] + \frac{h}{2} [f_1 + f_2] + \cdots + \frac{h}{2} [f_{N-1} + f_N]
 \end{aligned}$$

Simplify and yields **Trapezoidal Composite Rule**:

$$\int_{x_0}^{x_N} f(x) dx = \frac{h}{2} [f_0 + f_N + 2(f_1 + f_2 + \cdots f_{N-1})]$$

6.2 Trapezoidal Rule

Example 1:

Given the following data:

x	1.2	1.3	1.4	1.5	1.6	1.7
$f(x)$	2.2	2.1	4.7	6.2	5.5	5.0

- 1) Use Trapezoidal Rule to estimate the area, $A = \int_{1.2}^{1.7} f(x) dx$.
Let number of intervals be $N = 5$.

- 2) Use Trapezoidal Rule to estimate the volume,

$$V = \pi \int_{1.2}^{1.7} [f(x)]^2 dx$$

with $h = 0.1$.

6.2 Trapezoidal Rule

Solution:

1) $A = \int_{1.2}^{1.7} f(x) dx, N = 5.$

i	x_i	H&T $f(x_i)$	$f(x_i)$
0	1.2	2.2	
1	1.3		2.1
2	1.4		4.7
3	1.5		6.2
4	1.6		5.5
5	1.7	5.0	
Sum		7.2	18.5

Trapezoidal Rule

$$\begin{aligned}
 A &= \int_{1.2}^{1.7} f(x) dx \\
 &= \frac{h}{2} [f_0 + f_5 + 2(f_1 + f_2 + f_3 + f_4)] \\
 &= \frac{0.1}{2} [7.2 + 2(18.5)] \\
 &= 2.21
 \end{aligned}$$

6.2 Trapezoidal Rule

Solution:

$$2) \quad V = \pi \int_{1.2}^{1.7} [f(x)]^2 dx, h = 0.1.$$

i	x_i	$f(x_i)$	H&T $[f(x_i)]^2$	$[f(x_i)]^2$
0	1.2	2.2	4.84	
1	1.3	2.1		4.41
2	1.4	4.7		22.09
3	1.5	6.2		38.44
4	1.6	5.5		30.25
5	1.7	5.0	25.0	
	Sum		29.84	95.19

Trapezoidal Rule

$$V = \pi \int_{1.2}^{1.7} [f(x)]^2 dx$$

$$= \frac{h}{2} [f_0 + f_5 + 2(f_1 + f_2 + f_3 + f_4)] \pi$$

$$= \frac{0.1}{2} [29.84 + 2(95.19)] \pi$$

$$= 34.5921$$

6.2 Trapezoidal Rule

Example 2:

Use the Trapezoidal Rule to estimate the following definite integral with $h = 0.1$.

$$\int_0^{0.8} \sqrt{1 + x + x^2} \, dx$$

6.2 Trapezoidal Rule

Solution:

x_i	H&T $f(x_i)$	$f(x_i)$
0	1.0000	
0.1		1.0536
0.2		1.1136
0.3		1.1790
0.4		1.2490
0.5		1.3229
0.6		1.4000
0.7		1.4799
0.8	1.5620	
Sum	2.5620	8.7978

$$\int_0^{0.8} \sqrt{1+x+x^2} dx, h = 0.1.$$

Trapezoidal Rule

$$\int_0^{0.8} \sqrt{1+x+x^2} dx$$

$$= \frac{0.1}{2} [2.5620 + 2(8.7978)]$$

$$= 1.0079$$

6.2 Trapezoidal Rule

Exercise

Use the Trapezoidal Rule to estimate the following definite integral

$$1) \int_0^{\frac{\pi}{2}} \frac{\sin x}{x} dx, \quad h = \frac{\pi}{8}$$

$$2) \int_1^3 \frac{1}{x+2} dx, \quad N = 4$$

$$3) \int_1^2 x e^{\sqrt{x}} dx, \quad h = 0.2$$

6.2 Trapezoidal Rule

ANSWER

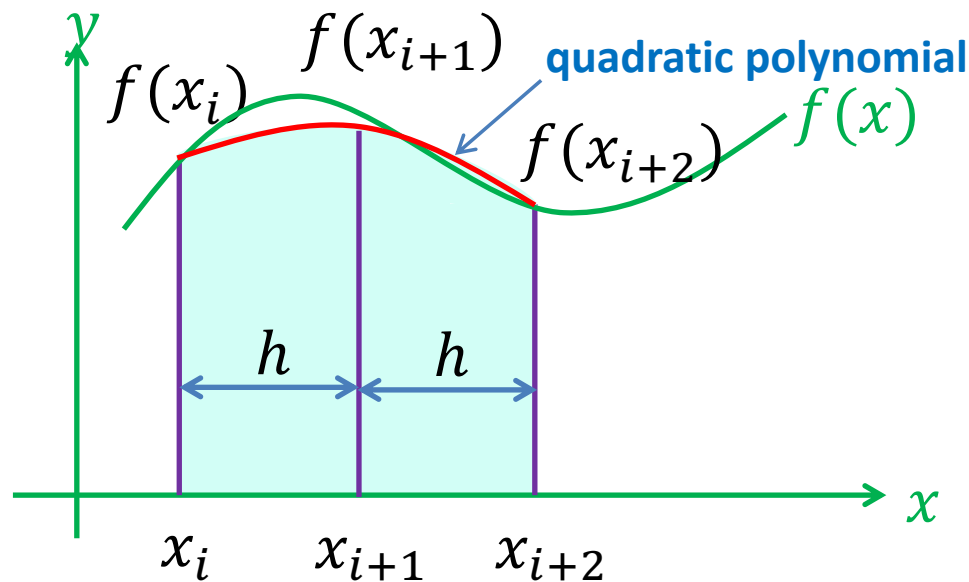
$$1) \int_0^{\frac{\pi}{2}} \frac{\sin x}{x} dx = 1.3655$$

$$2) \int_1^3 \frac{1}{x+2} dx = 0.5123$$

$$3) \int_1^2 x e^{\sqrt{x}} dx = 5.2372$$

6.3 Simpson's 1/3 Rule

Simpson's 1/3 rule approximating the integral of a function using **quadratic polynomials** (i.e., parabolic arcs instead of the straight line segments used in the Trapezoidal Rule).



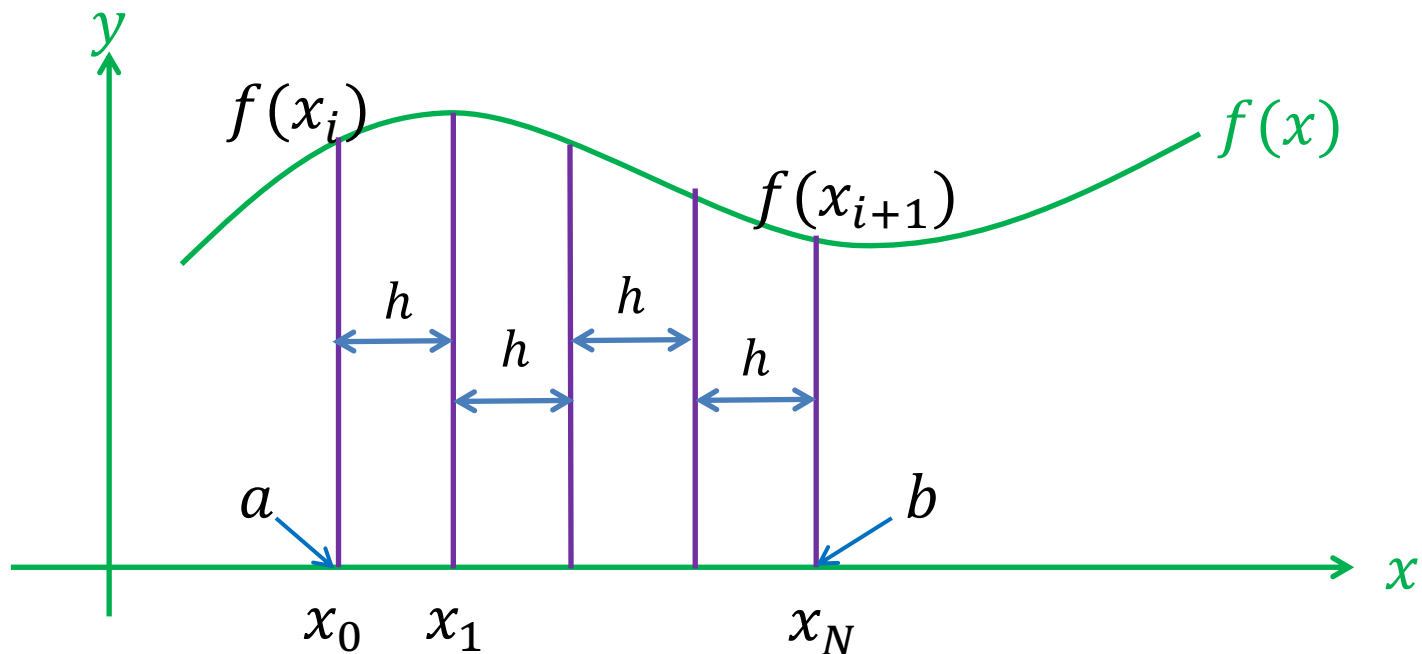
Also known as
Simpson's rule

From Lagrange
Interpolation

$$\text{Area} = \int_{x_i}^{x_{i+2}} f(x) dx \approx \frac{h}{3} [f_i + 4f_{i+1} + f_{i+2}]$$

6.3 Simpson's 1/3 Rule

When the interval $[a, b]$ is divided into N equal strips of width h such that $x_i = a + ih, i = 0, 1, 2, 3, \dots, N$ and $b = a + Nh$,



$$\int_{x_0}^{x_N} f(x) dx = \int_{x_0}^{x_1} f(x) dx + \int_{x_1}^{x_2} f(x) dx + \dots + \int_{x_{N-1}}^{x_N} f(x) dx$$

6.3 Simpson's 1/3 Rule

$$\begin{aligned}
 \int_{x_0}^{x_N} f(x) dx &= \int_{x_0}^{x_2} f(x) dx + \int_{x_2}^{x_4} f(x) dx + \cdots + \int_{x_{N-2}}^{x_N} f(x) dx \\
 &= \frac{h}{3} [f_0 + 4f_1 + f_2] + \frac{h}{3} [f_2 + 4f_3 + f_4] + \\
 &\quad \cdots + \frac{h}{3} [f_{N-2} + 4f_{N-1} + f_N]
 \end{aligned}$$

Simplify and yields **Simpson's Composite Rule**:

$$\begin{aligned}
 \int_{x_0}^{x_N} f(x) dx &= \frac{h}{3} [f_0 + f_N + 4(f_1 + f_3 + f_5 + \cdots f_{N-1}) \\
 &\quad + 2(f_2 + f_4 + f_6 + \cdots f_{N-2})]
 \end{aligned}$$

6.3 Simpson's 1/3 Rule

Example:

Given the following data:

x	2	3	4	5	6
$f(x)$	2.35	4.62	5.84	4.20	1.58

Let $h = 1$, use Simpson's Rule to estimate

$$\int_2^6 f(x) dx$$

6.3 Simpson's 1/3 Rule

Solution:

$$\int_2^6 f(x) dx, h = 1.$$

i	x_i	H&T $f(x_i)$	Odd $f(x_i)$	Even $f(x_i)$
0	2	2.35		
1	3		4.62	
2	4			5.84
3	5		4.2	
4	6	1.58		
Sum		3.93	8.82	5.84

Simpson's Rule

$$\int_2^6 f(x) dx$$

$$= \frac{h}{3} [f_0 + f_4 + 4(f_1 + f_3) + 2f_2]$$

$$= \frac{1}{3} [3.93 + 4(8.82) + 2(5.84)]$$

$$= 16.9633$$

6.3 Simpson's 1/3 Rule

Exercise:

Approximate

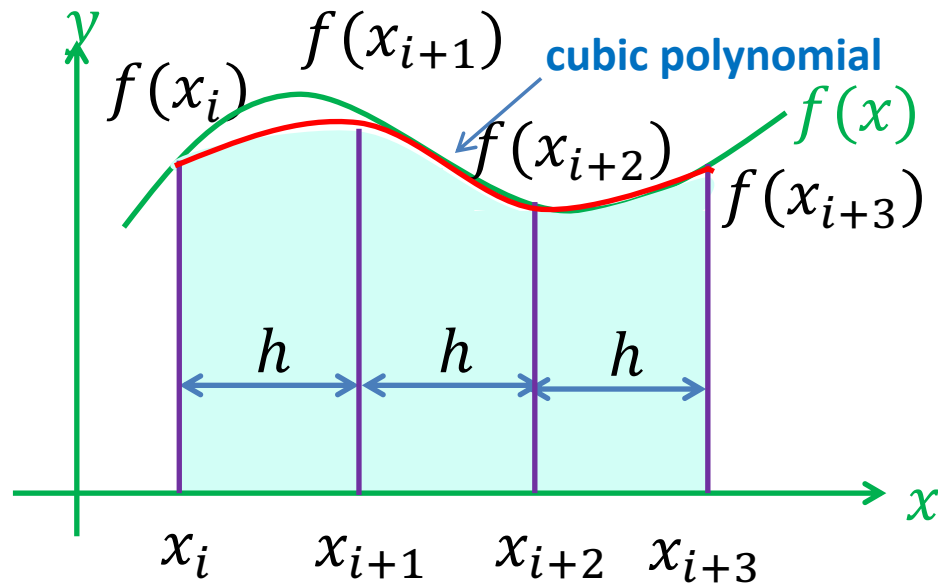
$$\int_0^2 x^2 e^{-x^2} dx$$

by using Simpson's Rule with $h = 0.25$

ANSWER: $\int_0^2 x^2 e^{-x^2} dx = 0.4227$

6.4 Simpson's 3/8 Rule

Simpson's 3/8 rule approximating the integral of a function using **cubic polynomials** (i.e., cubic arcs)



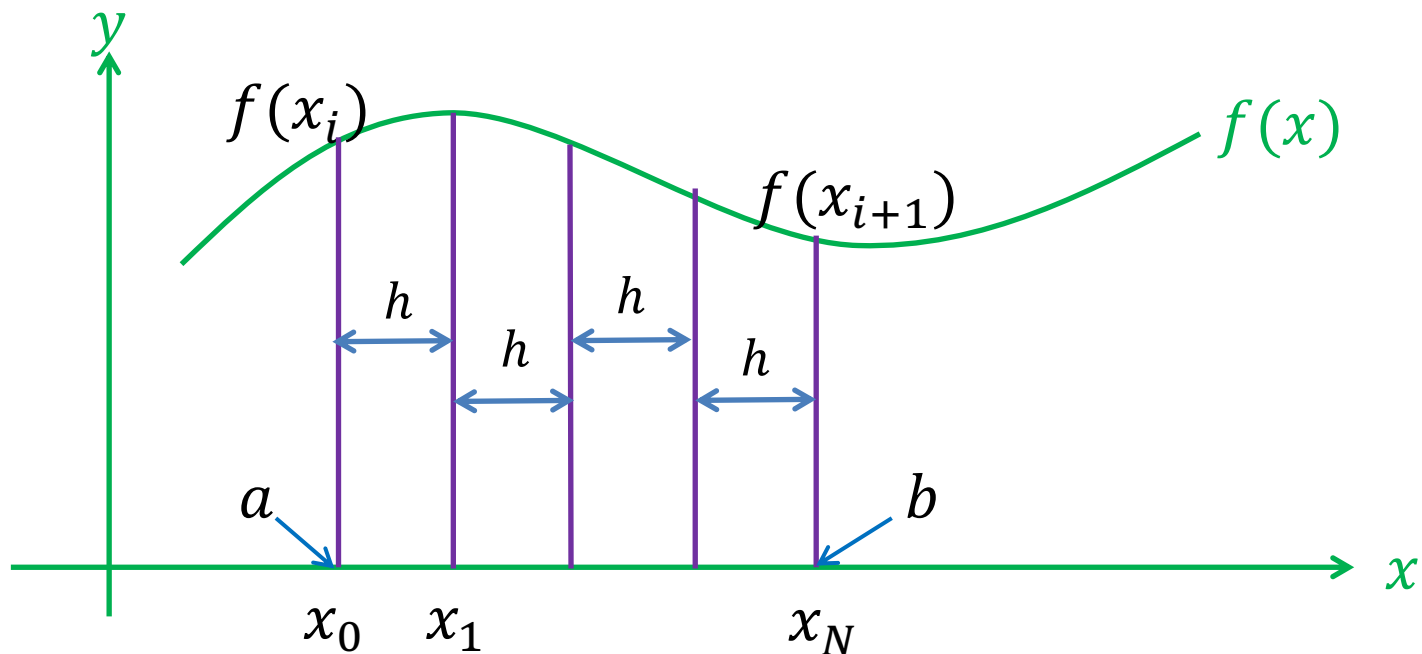
Also known as
Simpson's Second Rule

From Lagrange
Interpolation

$$\text{Area} = \int_{x_i}^{x_{i+3}} f(x) dx \approx \frac{3h}{8} [f_i + 3f_{i+1} + 3f_{i+2} + f_{i+3}]$$

6.4 Simpson's 3/8 Rule

When the interval $[a, b]$ is divided into N equal strips of width h such that $x_i = a + ih, i = 0, 1, 2, 3, \dots, N$ and $b = a + Nh$,



$$\int_{x_0}^{x_N} f(x) dx = \int_{x_0}^{x_1} f(x) dx + \int_{x_1}^{x_2} f(x) dx + \dots + \int_{x_{N-1}}^{x_N} f(x) dx$$

6.4 Simpson's 3/8 Rule

$$\begin{aligned}\int_{x_0}^{x_N} f(x) dx &= \int_{x_0}^{x_3} f(x) dx + \int_{x_3}^{x_6} f(x) dx + \cdots + \int_{x_{N-3}}^{x_N} f(x) dx \\ &= \frac{3h}{8} [f_0 + 3f_1 + 3f_2 + f_3] + \frac{3h}{8} [f_3 + 3f_4 + 3f_5 + f_6] + \\ &\quad \cdots + \frac{3h}{8} [f_{N-3} + 3f_{N-2} + 3f_{N-1} + f_N]\end{aligned}$$

Simplify and yields **Simpson's 3/8 Composite Rule**:

$$\begin{aligned}\int_{x_0}^{x_N} f(x) dx &= \frac{3h}{8} [f_0 + f_N + 2(f_3 + f_6 + f_9 + \cdots) \\ &\quad + 3(f_1 + f_2 + f_4 + \cdots)]\end{aligned}$$

6.4 Simpson's 3/8 Rule

Example:

Given the following data:

x	2.0	2.2	2.4	2.6	2.8	3.0	3.2
$f(x)$	1	0.5	0.333	0.25	0.2	0.167	0.143

Use Simpson's 3/8 Rule to estimate $\int_2^{3.2} f(x) dx$ with

- 1) $h = 0.2$
- 2) $h = 0.4$

If $f(x) = \frac{1}{5x-9}$, compute absolute error for each of the cases.

6.4 Simpson's 3/8 Rule

Solution:

1) $\int_2^{3.2} f(x) dx, h = 0.2.$

i	x_i	H&T $f(x)$	x3 $f(x)$	$f(x)$
0	2.0	1		
1	2.2			0.5
2	2.4			0.333
3	2.6		0.25	
4	2.8			0.2
5	3.0			0.167
6	3.2	0.143		
	Sum	1.143	0.25	1.2

Simpson's 3/8 Rule

$$\begin{aligned}
 & \int_2^{3.2} f(x) dx \\
 &= \frac{3h}{8} [f_0 + f_6 + 2(f_3) + 3(f_1 + f_2 \\
 &+ f_4 + f_5)] \\
 &= \frac{0.6}{8} [1.143 + 2(0.25) + 3(1.2)] \\
 &= 0.3932
 \end{aligned}$$

6.4 Simpson's 3/8 Rule

Solution:

2) $\int_2^{3.2} f(x) dx, h = 0.4.$

i	x	H&T $f(x)$	x^3 $f(x)$	$f(x)$
0	2.0	1		
1	2.4			0.333
2	2.8			0.2
3	3.2	0.143		
	Sum	1.143		0.533

Simpson's 3/8 Rule

$$\int_2^{3.2} f(x) dx$$

$$= \frac{3h}{8} [f_0 + f_3 + 3(f_1 + f_2)]$$

$$= \frac{1.2}{8} [1.143 + 3(0.533)]$$

$$= 0.4113$$

6.4 Simpson's 3/8 Rule

Solution:

Error

$$\begin{aligned}\int_2^{3.2} f(x) dx &= \int_2^{3.2} \frac{1}{5x-9} dx \\ &= \left[\frac{1}{5} \ln(5x-9) \right]_2^{3.2} = 0.3892\end{aligned}$$

	Estimated value	Absolute error
$h = 0.2$	0.3932	0.0040
$h = 0.4$	0.4113	0.0221

Conclusion: $h = 0.2$ gives better approximation

6.4 Simpson's 3/8 Rule

Example:

Use Simpson's 3/8 Rule to approximate

1) $\int_1^4 \frac{x}{x^2+4} dx$ with $h = 0.5$

2) $\int_{-1}^5 x^3 e^{-x} dx$ with $N = 9$

6.4 Simpson's 3/8 Rule

Solution:

1) $\int_1^4 \frac{x}{x^2+4} dx, h = 0.5.$

i	x_i	H&T $f(x_i)$	x3 $f(x_i)$	$f(x_i)$
0	1	0.2000		
1	1.5			0.24
2	2			0.25
3	2.5		0.2439	
4	3			0.2308
5	3.5			0.2154
6	4	0.2000		
Sum		0.4000	0.2439	0.9362

Simpson's 3/8 Rule

$$\int_1^4 \frac{x}{x^2+4} dx$$

$$= \frac{3h}{8} [f_0 + f_6 + 2(f_3) + 3(f_1 + f_2 + f_4 + f_5)]$$

$$= \frac{1.5}{8} [0.4 + 2(0.2439) + 3(0.9362)]$$

$$= 0.6931$$

6.4 Simpson's 3/8 Rule

Solution:

$$2) \int_{-1}^5 x^3 e^{-x} dx, N = 9.$$

Simpson's 3/8 Rule

$$\int_{-1}^5 x^3 e^{-x} dx$$

$$= \frac{3h}{8} [f_0 + f_9 + 2(f_3 + f_6) + 3(f_1 + f_2 + f_4 + f_5 + f_7 + f_8)]$$

$$= \frac{2}{8} [-1.8760 + 2(1.7121) + 3(4.4092)]$$

$$= 3.6940$$

i	x_i	H&T $f(x_i)$	x^3 $f(x_i)$	$f(x_i)$
0	-1.0000	-2.7183		
1	-0.3333			-0.0517
2	0.3333			0.0265
3	1.0000		0.3679	
4	1.6667			0.8744
5	2.3333			1.2319
6	3.0000		1.3443	
7	3.6667			1.2601
8	4.3333			1.0679
9	5.0000	0.8422		
	Sum	-1.8760	1.7121	4.4092

6.4 Simpson's 3/8 Rule

Exercise:

Use Simpson's 3/8 Rule to estimate the following definite integral

1) $\int_1^3 \frac{1}{x+2} dx, N = 6$

2) $\int_1^2 x e^{\sqrt{x}} dx, N = 9$