



OPENCOURSEWARE

BEKG 2452

NUMERICAL METHODS

Interpolation

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Lesson Outcome

Upon completion of this lesson, the student should be able to:

1. Determine polynomials using Newton's interpolation and Lagrange interpolation.
2. Estimate the value of a function for any intermediate value of the independent variable.

Interpolation

A method of estimating the value of a function for any intermediate value of the independent variable



Newton's
Interpolating
Polynomials



Lagrange
Interpolating
Polynomials



Curve Fitting

- Find the **polynomial function** that fits a given set of points



Use the polynomial function to constructing a new data point

4.1 Newton's Interpolating Polynomials

- Applicable for **equally and unequally spaced** data.
- A **straight line** from Newton's Interpolating Polynomial that passing through two points (x_0, y_0) and (x_1, y_1)

$$f(x) = b_0 + b_1(x - x_0)$$

4.1 Newton's Interpolating Polynomials

- A **parabola** from Newton's Interpolating Polynomial that passing through three points (x_0, y_0) , (x_1, y_1) and (x_2, y_2)

$$f(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1)$$

- **General formula** of Newton's Interpolating Polynomial that passing through n points (x_0, y_0) , (x_1, y_1) , ..., (x_{n-1}, y_{n-1})

$$f(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1) + \dots + b_{n-1}(x - x_0) \dots (x - x_{n-2})$$

4.1 Newton's Interpolating Polynomials

Example:

Construct the parabola that passing through (x_0, y_0) , (x_1, y_1) and (x_2, y_2) .

From Newton's Interpolation, the parabolic equation is

$$f(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1)$$

Derivation of coefficient b_0, b_1, b_2 :

Substitute (x_0, y_0) into x and y :

$$y_0 = b_0 + b_1(\cancel{x_0 - x_0}^0) + b_2(\cancel{x_0 - x_0}^0)(x_0 - x_1)$$
$$\therefore b_0 = y_0$$

4.1 Newton's Interpolating Polynomials

Substitute (x_1, y_1) into x and y :

$$y_1 = y_0 + b_1(x_1 - x_0) + b_2(x_1 - x_0)(x_1 - x_1)$$

$$\therefore b_1 = \frac{y_1 - y_0}{x_1 - x_0}$$

Substitute (x_2, y_2) into x and y :

$$y_2 = y_0 + \frac{y_1 - y_0}{x_1 - x_0}(x_2 - x_0) + b_2(x_2 - x_0)(x_2 - x_1)$$

$$\therefore b_2 = \frac{y_2 - y_0 - \frac{y_1 - y_0}{x_1 - x_0}(x_2 - x_0)}{(x_2 - x_0)(x_2 - x_1)} = \frac{\frac{y_2 - y_1}{x_2 - x_1} - \frac{y_1 - y_0}{x_1 - x_0}}{(x_2 - x_0)}$$

i	x_i	$f(x_i)$	Δf_i	$\Delta^2 f_i$	$\Delta^3 f_i$	$\Delta^4 f_i$
0	x_0	y_0				
			$\Delta f_0 = \frac{y_1 - y_0}{x_1 - x_0}$			
1	x_1	y_1		$\Delta^2 f_0 = \frac{\Delta f_1 - \Delta f_0}{x_2 - x_0}$		
			$\Delta f_1 = \frac{y_2 - y_1}{x_2 - x_1}$		$\Delta^3 f_0 = \frac{\Delta^2 f_1 - \Delta^2 f_0}{x_3 - x_0}$	
2	x_2	y_2		$\Delta^2 f_1 = \frac{\Delta f_2 - \Delta f_1}{x_3 - x_1}$		$\Delta^4 f_0 = \frac{\Delta^3 f_1 - \Delta^3 f_0}{x_4 - x_0}$
			$\Delta f_2 = \frac{y_3 - y_2}{x_3 - x_2}$		$\Delta^3 f_1 = \frac{\Delta^2 f_2 - \Delta^2 f_1}{x_4 - x_1}$	
3	x_3	y_3		$\Delta^2 f_2 = \frac{\Delta f_3 - \Delta f_2}{x_4 - x_2}$		
			$\Delta f_3 = \frac{y_4 - y_3}{x_4 - x_3}$			
4	x_4	y_4				

Note: Determination of b_i depends of the order

4.1 Newton's Interpolating Polynomials

4.1.1 Linear Interpolation

To connect **two data points** with the first order polynomial (**straight line**)

Formula:

$$f(x) = b_0 + b_1(x - x_0)$$

Table:

i	x_i	$f(x_i)$	Δf_i
0	x_0	y_0	
			$\Delta f_0 = \frac{y_1 - y_0}{x_1 - x_0}$
1	x_1	y_1	

4.1 Newton's Interpolating Polynomials

4.1.1 Linear Interpolation

Example 4.1:

Estimate natural logarithm of 2 ($\ln 2$) using linear interpolation given $\ln 1 = 0$ and $\ln 3 = 1.0986$.

Solution: $(1, 0), (3, 1.0986), (2, ?)$

i	x_i	$f(x_i)$	Δf_i
0	1	0.0000	
			$\Delta f_0 = \frac{1.0986 - 0}{3 - 1} = 0.5493$
1	3	1.0986	

$$f(x) = b_0 + b_1(x - x_0) = 0.5493(x - 1) \approx \ln x$$

$$\therefore f(2) = 0.5493$$

4.1 Newton's Interpolating Polynomials

4.1.2 Quadratic Interpolation

To connect **three data points** with the second order polynomial (**parabola**)

Formula:

$$f(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1)$$

Table:

i	x_i	$f(x_i)$	Δf_i	$\Delta^2 f_i$
0	x_0	y_0		
			$\Delta f_0 = \frac{y_1 - y_0}{x_1 - x_0}$	
1	x_1	y_1		$\Delta^2 f_0 = \frac{\Delta f_1 - \Delta f_0}{x_2 - x_0}$
			$\Delta f_1 = \frac{y_2 - y_1}{x_2 - x_1}$	
2	x_2	y_2		

4.1.2 Quadratic Interpolation

Example 4.2:

Estimate $f(1.7)$ using quadratic interpolation given $(0, 1), (2, -3), (4, -8)$.

Solution:

i	x_i	$f(x_i)$	Δf_i	$\Delta^2 f_i$
0	0	1		
			$\frac{-3 - 1}{2 - 0} = -2$	
1	2	-3		$\frac{-2.5 - (-2)}{4 - 0} = -0.125$
			$\frac{-8 - (-3)}{4 - 2} = -2.5$	
2	4	-8		

$$f(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1)$$

$$= 1 - 2x - 0.125x(x - 2)$$

$$\therefore f(1.7) = -2.3363$$

4.1 Newton's Interpolating Polynomials

4.1.3 Cubic Interpolation

To connect **four data points** with the third order polynomial (**cubic polynomial**)

Formula:

$$f(x) = b_0 + b_1(x - x_0) + b_2(x - x_0)(x - x_1) + b_3(x - x_0)(x - x_1)(x - x_2)$$

Table:

i	x_i	$f(x_i)$	Δf_i	$\Delta^2 f_i$	$\Delta^3 f_i$
0	x_0	y_0			
			$\Delta f_0 = \frac{y_1 - y_0}{x_1 - x_0}$		
1	x_1	y_1		$\Delta^2 f_0 = \frac{\Delta f_1 - \Delta f_0}{x_2 - x_0}$	
			$\Delta f_1 = \frac{y_2 - y_1}{x_2 - x_1}$		$\Delta^3 f_0 = \frac{\Delta^2 f_1 - \Delta^2 f_0}{x_3 - x_0}$
2	x_2	y_2		$\Delta^2 f_1 = \frac{\Delta f_2 - \Delta f_1}{x_3 - x_1}$	
			$\Delta f_2 = \frac{y_3 - y_2}{x_3 - x_2}$		
3	x_3	y_3			

Example 4.3:

Estimate $f(5.3)$ using cubic interpolation given $(1, 1), (4, 9), (7, 13), (10, 20)$.

Solution:

i	x_i	$f(x_i)$	Δf_i	$\Delta^2 f_i$	$\Delta^3 f_i$
0	1	1			
			2.6667		
1	4	9		-0.2222	
			1.3333		0.0432
2	7	13		0.1667	
			2.3333		
3	10	20			

$$f(x) = 1 + 2.6667(x - 1) - 0.2222(x - 1)(x - 4) + 0.0432(x - 1)(x - 4)(x - 7)$$

$$\therefore f(5.3) = 10.8142$$

Exercise:

Given the following data

i	0	1	2	3
x_i	0	1	2	3
$f(x_i)$	10	8	2	-9

- 1) By using Newton's Interpolating polynomial of order 2, estimate $f(0.6)$.
- 2) By using Newton's Interpolation, find the $P_3(x)$ and then use it to estimate $f(2.7)$.

[Ans: 9.28; -5.1156]

4.2 Lagrange Interpolating Polynomials

- Applicable for **equally spaced or not equally spaced** data.
- A **straight line** from Lagrange Interpolating Polynomial that passing through two points (x_0, y_0) and (x_1, y_1)

$$f(x) = \frac{x - x_1}{x_0 - x_1} f(x_0) + \frac{x - x_0}{x_1 - x_0} f(x_1)$$

4.2 Lagrange Interpolating Polynomials

- A **parabola** from Lagrange Interpolating Polynomial that passing through three points (x_0, y_0) , (x_1, y_1) and (x_2, y_2)

$$\begin{aligned} f(x) = & \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} f(x_0) \\ & + \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} f(x_1) \\ & + \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} f(x_2) \end{aligned}$$

4.2 Lagrange Interpolating Polynomials

General formula of Lagrange Interpolating Polynomial that passing through n points $(x_0, y_0), (x_1, y_1), \dots, (x_{n-1}, y_{n-1})$:

n is order of interpolation

$$f_n(x) = \sum_{i=0}^n L_i(x) f(x_i) = L_0(x) f(x_0) + \dots + L_n(x) f(x_n)$$

where

$$L_i(x) = \prod_{j=0, j \neq i}^n \frac{(x - x_j)}{(x_i - x_j)}$$

$$= \frac{(x - x_0)(x - x_1)}{(x_i - x_0)(x_i - x_1)} \cdots \frac{(x - x_{i-1})(x - x_{i+1})}{(x_i - x_{i-1})(x_i - x_{i+1})} \cdots \frac{(x - x_n)}{(x_i - x_n)}$$

4.2 Lagrange Interpolating Polynomials

Lagrange Interpolation Coefficient:

$$L_i(x) = \prod_{j=0, j \neq i}^n \frac{(x - x_j)}{(x_i - x_j)}$$

is a polynomial with degree n which satisfies

$$L_i(x_j) = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$$

and

$$\sum_{i=0}^n L_i(x_j) = 1$$

4.2 Lagrange Interpolating Polynomials

4.2.1 Linear Interpolation

To connect **two data points** with the first order polynomial (**straight line**)

Formula:

$$f(x) = L_0(x)f(x_0) + L_1(x)f(x_1)$$

where

$$L_0(x) = \frac{x - x_1}{x_0 - x_1}$$

$$L_1(x) = \frac{x - x_0}{x_1 - x_0}$$

Example 4.4:

Use Lagrange Interpolating Polynomial of degree 1 to estimate $f(4.7)$ given $f(4.3) = 0.1678$ and $f(4.9) = 0.2451$.

Solution:

$$f(x) = L_0(x)f(x_0) + L_1(x)f(x_1)$$

where

$$L_0(x) = \frac{x - x_1}{x_0 - x_1} = \frac{x - 4.9}{4.3 - 4.9}$$

$$L_1(x) = \frac{x - x_0}{x_1 - x_0} = \frac{x - 4.3}{4.9 - 4.3}$$

Solution: (cont.)

Recall:

$$f(4.3) = 0.1678$$

$$f(4.9) = 0.2451$$

$$f(x) = L_0(x)f(x_0) + L_1(x)f(x_1)$$

$$f(x) = \frac{x - 4.9}{4.3 - 4.9} (0.1678) + \frac{x - 4.3}{4.9 - 4.3} (0.2451)$$

$$\begin{aligned} f(4.7) &= \frac{4.7 - 4.9}{4.3 - 4.9} (0.1678) + \frac{4.7 - 4.3}{4.9 - 4.3} (0.2451) \\ &= 0.2193 \end{aligned}$$

4.2 Lagrange Interpolating Polynomials

4.2.2 Quadratic Interpolation

To connect **three data points** with the second order polynomial (**parabola**)

Formula:

$$f(x) = L_0(x)f(x_0) + L_1(x)f(x_1) + L_2(x)f(x_2)$$

where

$$L_0(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)}$$

$$L_1(x) = \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)}$$

$$L_2(x) = \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)}$$

Example 4.5:

Use Lagrange Interpolating Polynomial of degree 2 to estimate $f(-3.8)$ given $f(-10.3) = 1.0823$, $f(-5.1) = 4.7798$ and $f(-1.5) = 9.1514$.

Solution:

$$f(x) = L_0(x)f(x_0) + L_1(x)f(x_1) + L_2(x)f(x_2)$$

where

$$L_0(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} = \frac{(x + 5.1)(x + 1.5)}{(-10.3 + 5.1)(-10.3 + 1.5)}$$

$$L_1(x) = \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} = \frac{(x + 10.3)(x + 1.5)}{(-5.1 + 10.3)(-5.1 + 1.5)}$$

$$L_2(x) = \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} = \frac{(x + 10.3)(x + 5.1)}{(-1.5 + 10.3)(-1.5 + 5.1)}$$

Solution: (cont.)

Recall:

$$f(-10.3) = 1.0823$$

$$f(-5.1) = 4.7798$$

$$f(-1.5) = 9.1514$$

$$f(x) = L_0(x)f(x_0) + L_1(x)f(x_1) + L_2(x)f(x_2)$$

$$L_0(x) = \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} = \frac{(x + 5.1)(x + 1.5)}{(-10.3 + 5.1)(-10.3 + 1.5)} = \frac{(x + 5.1)(x + 1.5)}{45.76}$$

$$L_1(x) = \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} = \frac{(x + 10.3)(x + 1.5)}{(-5.1 + 10.3)(-5.1 + 1.5)} = -\frac{(x + 10.3)(x + 1.5)}{18.72}$$

$$L_2(x) = \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} = \frac{(x + 10.3)(x + 5.1)}{(-1.5 + 10.3)(-1.5 + 5.1)} = \frac{(x + 10.3)(x + 5.1)}{31.68}$$

Solution: (cont.)

$$\begin{aligned} f(x) &= L_0(x)f(x_0) + L_1(x)f(x_1) + L_2(x)f(x_2) \\ &= \frac{(x+5.1)(x+1.5)}{45.76} (1.0823) - \frac{(x+10.3)(x+1.5)}{18.72} (4.7798) \\ &\quad + \frac{(x+10.3)(x+5.1)}{31.68} (9.1514) \end{aligned}$$

$$\begin{aligned} \therefore f(-3.8) &= \frac{(-3.8+5.1)(-3.8+1.5)}{45.76} (1.0823) \\ &\quad - \frac{(-3.8+10.3)(-3.8+1.5)}{18.72} (4.7798) \\ &\quad + \frac{(-3.8+10.3)(-3.8+5.1)}{31.68} (9.1514) \\ &= 6.1874 \end{aligned}$$

Exercise:

Given the following data

i	0	1	2
x_i	-1	1	3
$f(x_i)$	11	8	3

By using Lagrange Interpolating polynomial of order 2, estimate $f(1.8)$.

[Ans: 6.24]

4.2 Lagrange Interpolating Polynomials

4.2.3 Cubic Interpolation

To connect **four data points** with the third order polynomial (**cubic polynomial**)

Formula:

$$f(x) = L_0(x)f(x_0) + L_1(x)f(x_1) + L_2(x)f(x_2) + L_3(x)f(x_3)$$

where

$$L_0(x) = \frac{(x - x_1)(x - x_2)(x - x_3)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)}$$

$$L_1(x) = \frac{(x - x_0)(x - x_2)(x - x_3)}{(x_1 - x_0)(x_1 - x_2)(x_1 - x_3)}$$

$$L_2(x) = \frac{(x - x_0)(x - x_1)(x - x_3)}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3)}$$

$$L_3(x) = \frac{(x - x_0)(x - x_1)(x - x_2)}{(x_3 - x_0)(x_3 - x_1)(x_3 - x_2)}$$

Example 4.6:

Construct the Lagrange Interpolation polynomial of degree 3 with the data set given below:

i	0	1	2	3
x_i	-1	0	1	2
$f(x_i)$	1	1	1	-5

Solution:

$$\begin{aligned}
 f(x) &= L_0(x)f(x_0) + L_1(x)f(x_1) + L_2(x)f(x_2) + L_3(x)f(x_3) \\
 &= \frac{(x-0)(x-1)(x-2)}{(-1-0)(-1-1)(-1-2)} (1) + \frac{(x+1)(x-1)(x-2)}{(0+1)(0-1)(0-2)} (1) \\
 &\quad + \frac{(x+1)(x-0)(x-2)}{(1+1)(1-0)(1-2)} (1) + \frac{(x+1)(x-0)(x-1)}{(2+1)(2-0)(2-1)} (-5) \\
 &= -\frac{x}{6}(x-1)(x-2) + \frac{1}{2}(x+1)(x-1)(x-2) \\
 &\quad -\frac{x}{2}(x+1)(x-2) - \frac{5x}{6}(x+1)(x-1)
 \end{aligned}$$

$$\therefore f(x) = 1 + x - x^3$$

Exercise:

Find the Lagrange approximation polynomial of degree 3 for the function $f(x) = \frac{x^2 - e^x + \ln x}{\sqrt{x + \sin x}}$.

Use the $x_0 = 1, x_1 = 3, x_2 = 4$ and $x_3 = 7$.

Then estimate $f(5)$.

[Ans: -73.2355]