CALCULUS FOR TECHNOLOGY (BETU 1023)

WEEK 6

GRADIENT OF CURVE AT POINT AND MAXIMUM MINUMUM

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Gradient

Let $f(x)$, be a linear function, such that:

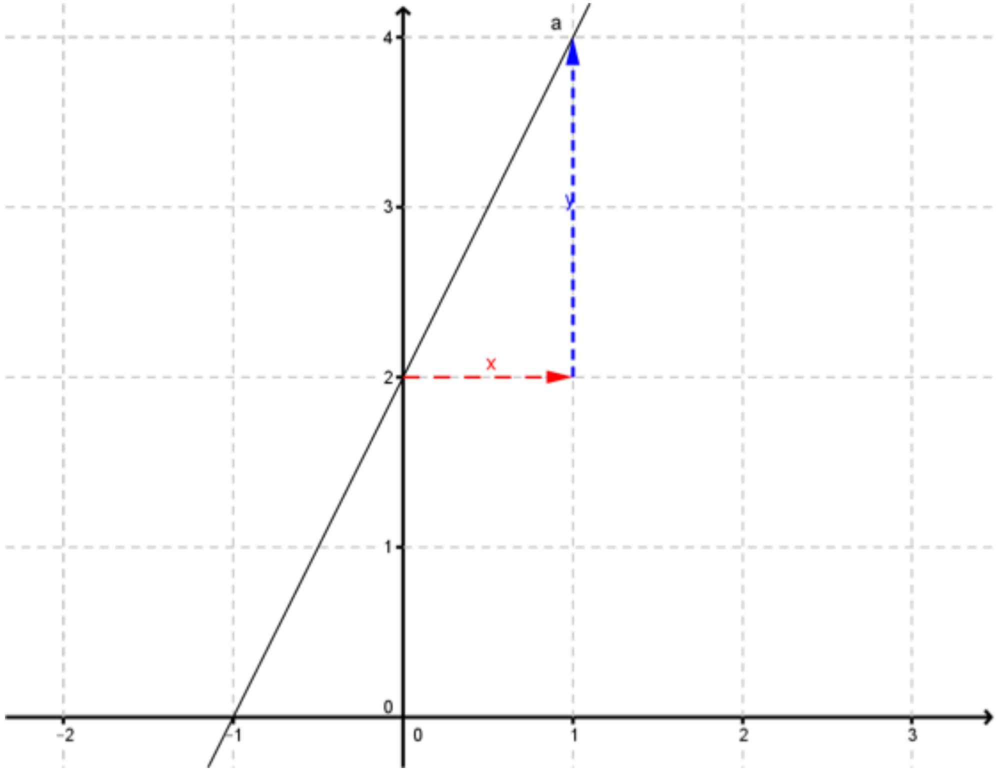
$$f(x) = ax + b$$

a is called slope, and b is called intercept.

Slope is also know as ***gradient*** is a number that represent how slanted is our linear function, and denoted by m.

Example

$$f(x) = 2x + 2$$



For the function above the gradient is 2, so:

$$m = 2 = \frac{2}{1} = \frac{\Delta y}{\Delta x}$$

Δy is called the difference of y, and Δx is the difference of x. If you see from the figure beside, every time x change 1 unit the number of y will change with 2 units.

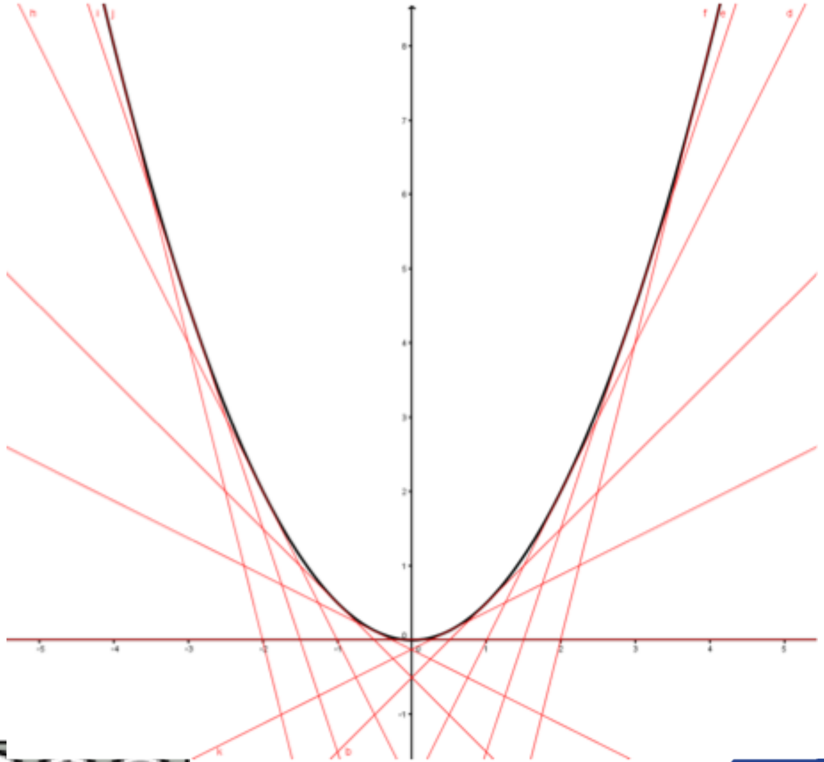
Tangent Line

Definition:

Let $f(x)$ is a differentiable non linear function for $\forall x \in R$, ***tangent*** line at given point is a straight line that touch curve $f(x)$ exactly at that point.

Example

$$f(x) = \frac{x^2}{2}$$



The red lines on figure beside are some example of tangent line at some certain points. Tangent lines are never cross the curve $f(x) = \frac{x^2}{2}$, they just "touches" the curve.

As we can see all the tangent lines are linear, means that they can represent in form of;

$$mx + b$$

Relation between Tangent Line and First Derivatives

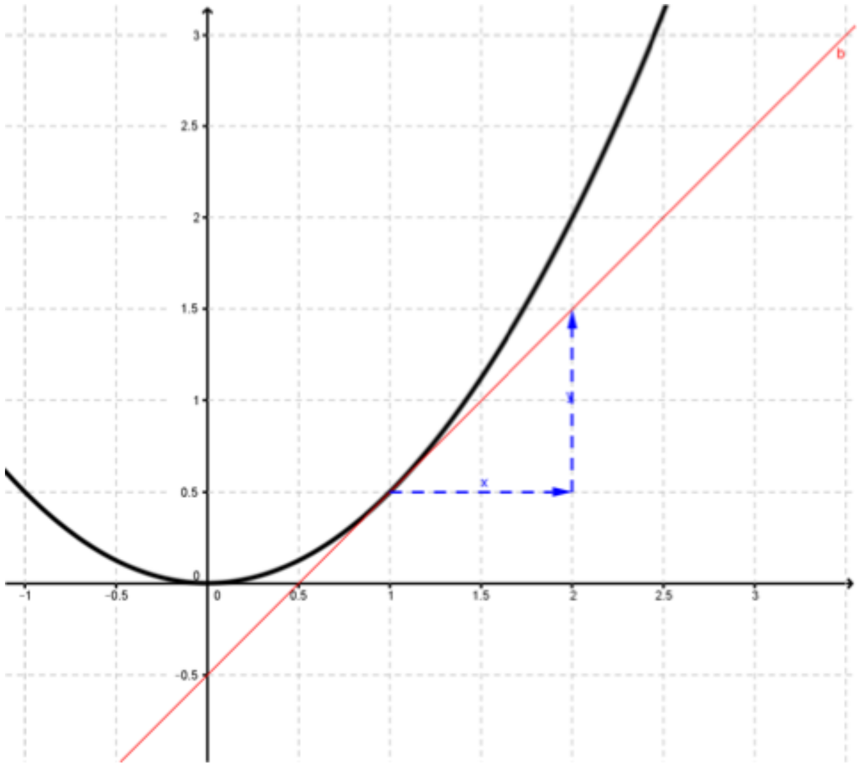
As we can see previously the tangent line has a form of $mx + b$, this means that every tangent line will have a gradient (m).

In the sense of calculus, slope or gradient (m) of tangent line at $x = c$, is the first derivatives of $f(x)$,

$$m = f'(c)$$

Example

Find the gradient of tangent line of $f(x)=\frac{x^2}{2}$, when $x=1$?



The gradient of tangent line (red line) by definition, using blue line, is

$$m = 1$$

If we find using first derivative we will get

$$f'(x) = x$$
$$m = f'(1) = 1$$

Hence the gradient is the first derivatives of $f(x)$

Absolute Optimum

Let a be a number, and $a \in S$, and let $f(x)$ is a function. Then $f(a)$ is:

- **absolute maximum** on S if $f(a) \geq f(x)$ for $\forall x \in S$.
- **absolute minimum** on S if $f(a) \leq f(x)$ for $\forall x \in S$.

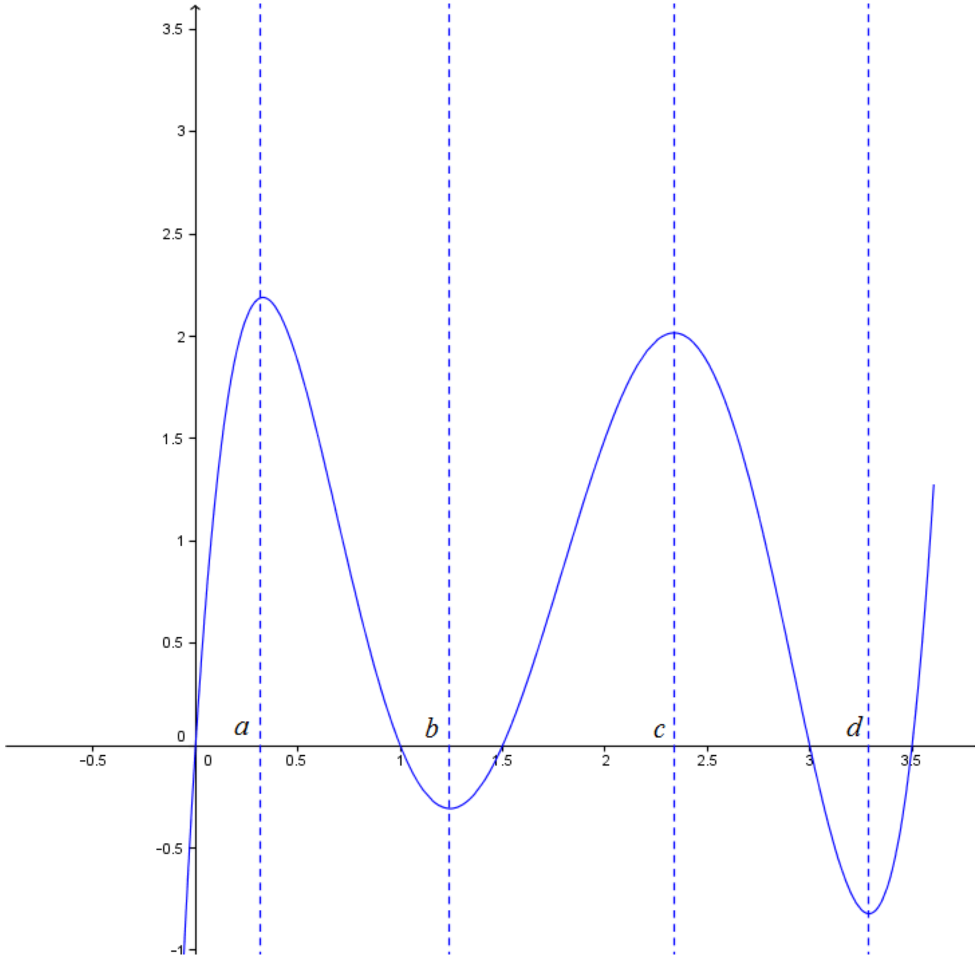
Local Optimum

Definition:

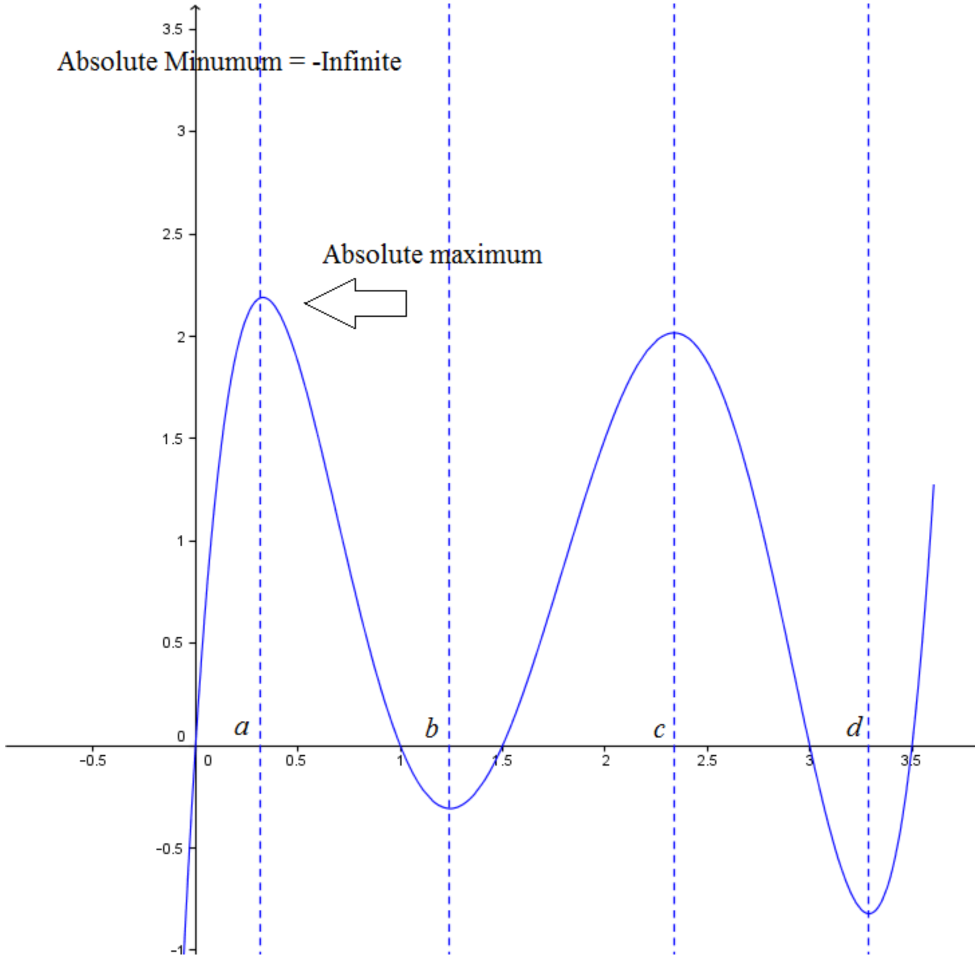
$f(a)$ is a

- **local maximum,** if $f(a) \geq f(x)$ when x near a.
- **local minimum,** if $f(a) \leq f(x)$ when x near a.

Example

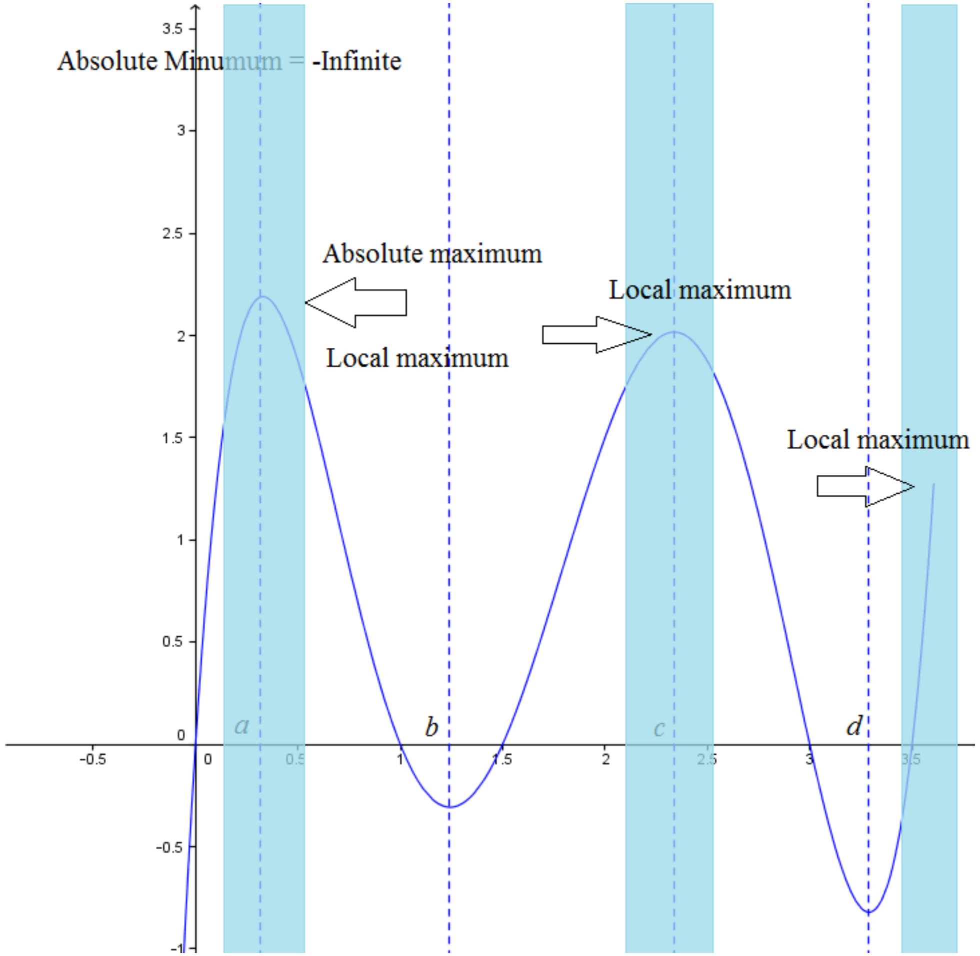


Example



Absolute Minumum = -Infinite
Absolute maximum
a
b
c
d
3.5
3
2.5
2
1.5
1
0.5
0
-0.5
-1
-0.5
0
0.5
1
1.5
2
2.5
3
3.5

Example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Local maximum
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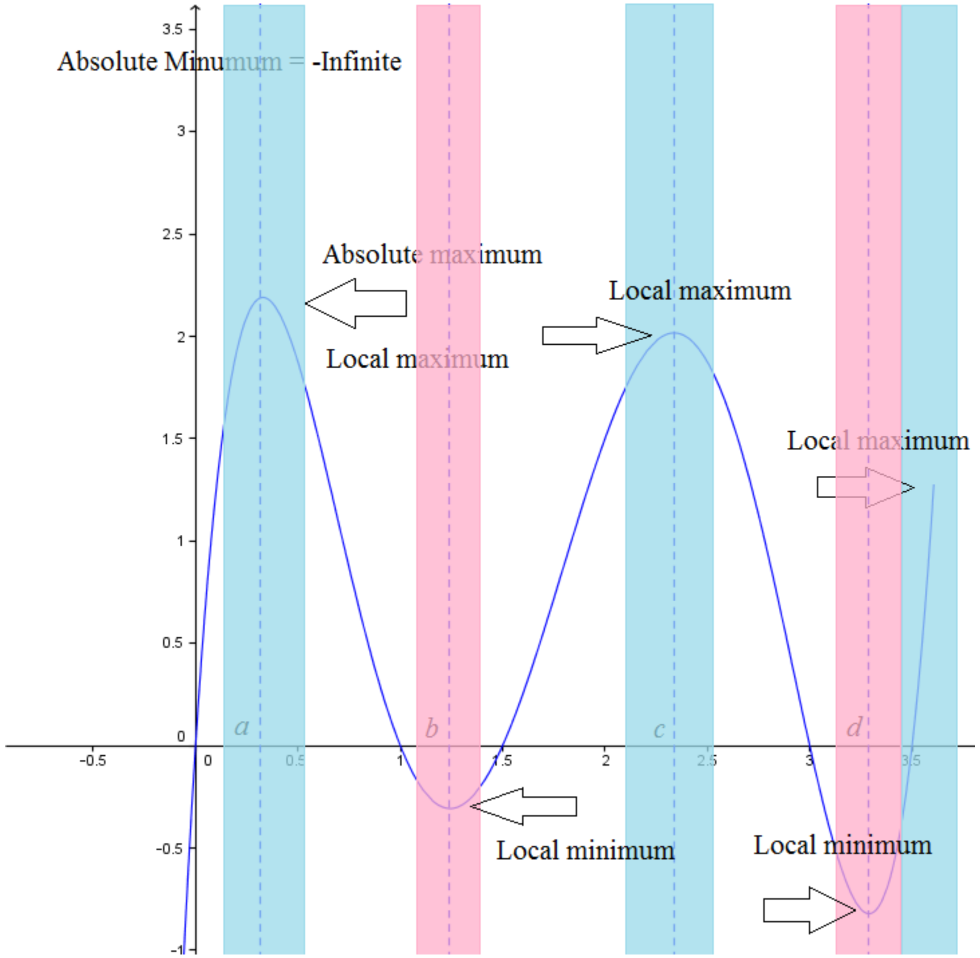
3
3.5

Example



Absolute Minumum = -Infinite
Absolute maximum
Local maximum
Local maximum
Local maximum
Local minimum
Local minimum
a
b
c
d

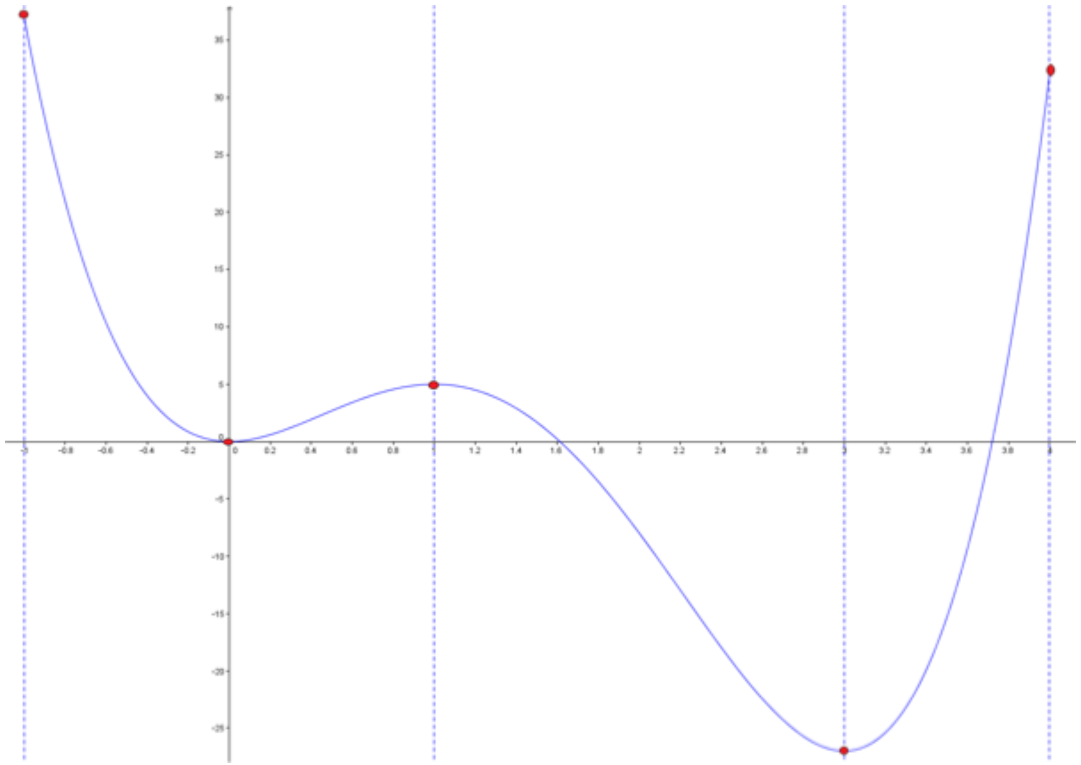
The Extreme Value Theorem

Let $f(x)$ is a continuous function on closed interval $[a,b]$, then $f(x)$ have an absolute maximum value $f(c)$ and absolute minimum value $f(d)$ at some number c and d in $[a,b]$.

Example

Find the absolute maximum and absolute minimum of

$$f(x) = 3x^4 - 16x^3 + 18x^2, \qquad -1 \le x \le 4$$



Example

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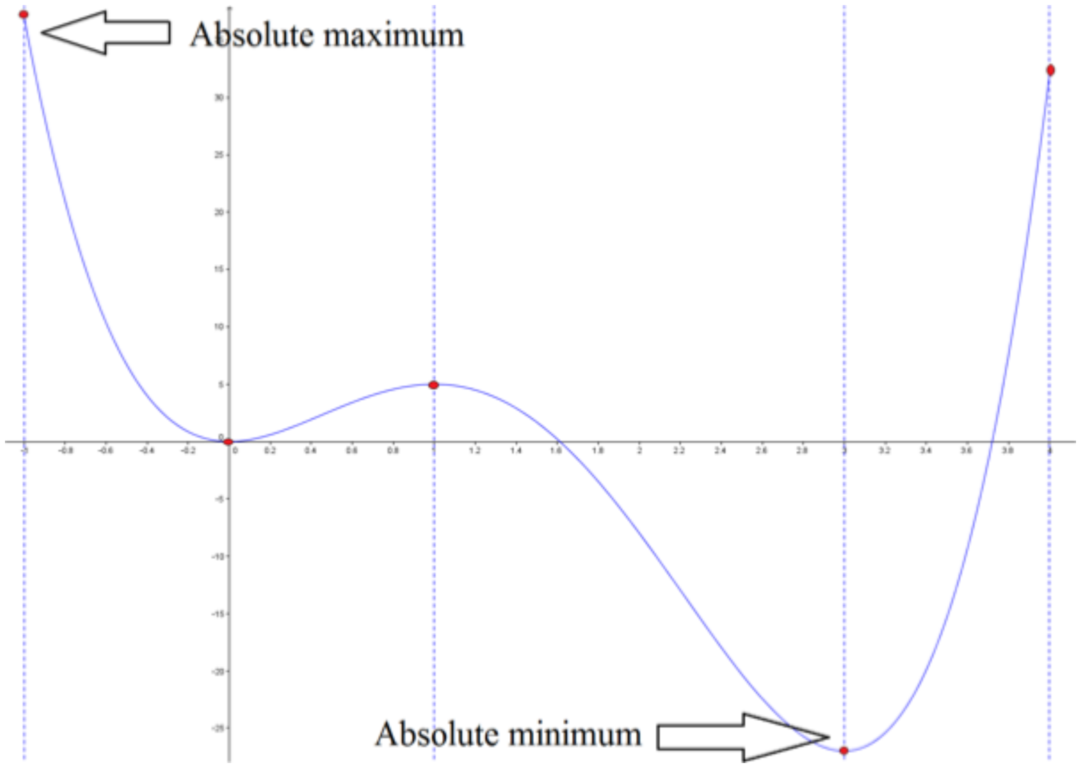
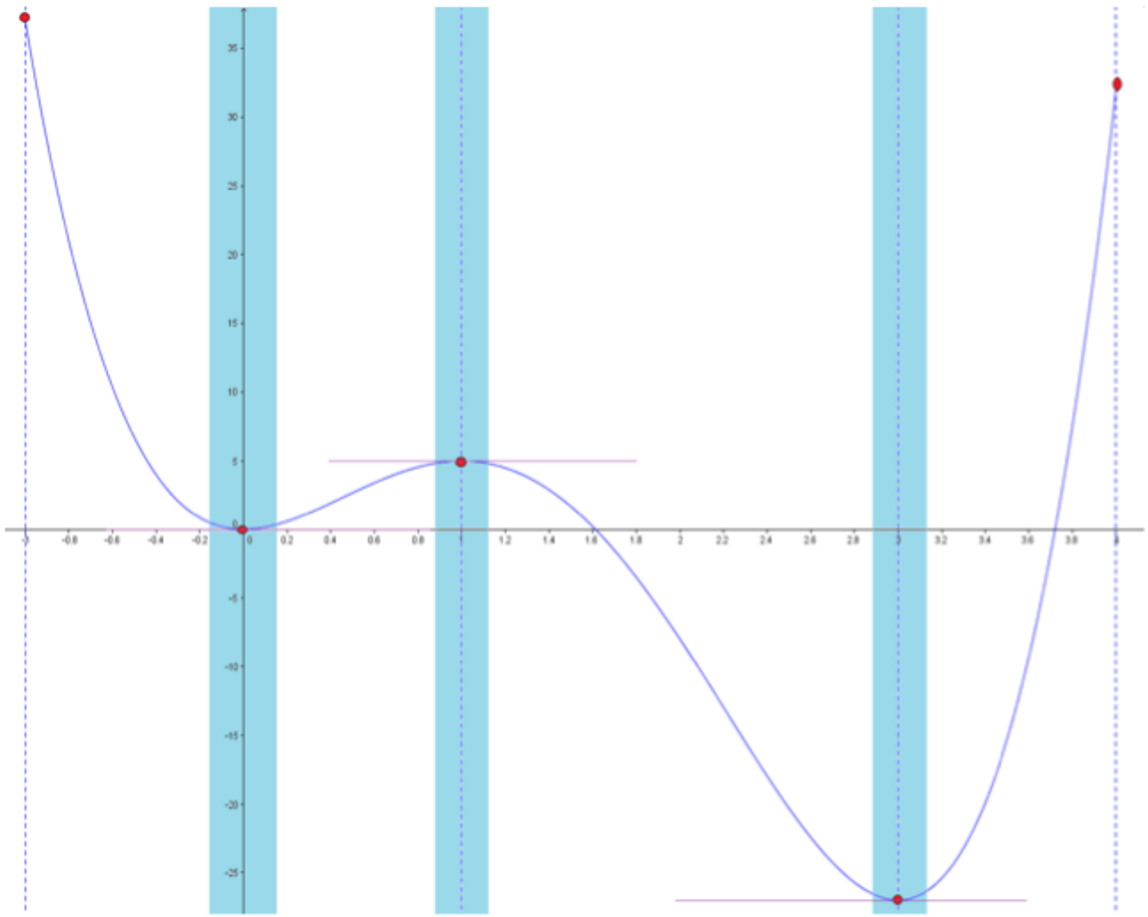


Relation Between Local Optimum and Derivatives



Let focus on three optimum points on figure beside. As we can see the tangent lines for all points are **horizontal line**, which mean slope at those points are 0, or if optimum values lies on c then,

$$f'(c) = 0$$

Fermat's Theorem

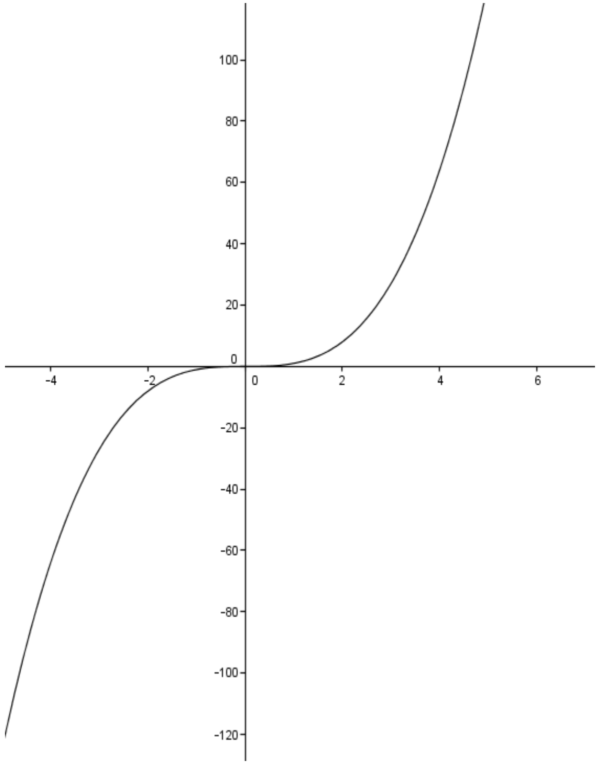
If f has local minimum or maximum at c, and if $f'(c)$ exist, then $f'(c) = 0$.

Note: The theorem does not mean that if $f'(c) = 0$, then f has local maximum or minimum.

Example

Examine the graph of

$$f(x) = x^3$$



The first derivatives of the function above is $f'(x) = 3x^2$ and if $f'(x) = 0$, then we get $x = 0$. from the graph beside the function is not maximum and not minimum at $x = 0$, saddle point, so it has shown that: Fermat's theorem does not mean that if $f'(c) = 0$, then f is local maximum or minimum at c.

Critical Number

Let f be a function and c is a number in the domain of f such that either $f'(c) = 0$ of $f'(c)$ does not exist, then we called c as **critical number**.

Example

Find all the critical numbers of $f(x) = x^{3/5}(4-x)$

Finding Absolute Maximum and Minimum

The Closed Interval Method

To find the *absolute* maximum and minimum values of a continuous function f on closed interval $[a, b]$:

1. Find the values of f at the critical numbers of f in (a, b).
2. Find the values of f at the endpoints of the interval.
3. The smallest value from step 1 and 2 is the absolute minimum value; The largest value from step 1 and 2 is the absolute maximum value.

Example

Find the absolute maximum and absolute minimum of

$$f(x) = x^3 - 3x + 1, \qquad -\frac{1}{2} \le x \le 4$$

Increasing/ Decreasing Test

Theorem:

- If $f'(x) > 0$ on an interval, then f is **increasing** on that interval
- If $f'(x) > 0$ on an interval, then f is **decreasing** on that interval

Example

Find the interval where the function

$$f(x) = 3x^4 - 4x^3 - 12x^2 + 5$$

is increasing and the interval where it is decreasing.

The First Derivatives Test

Theorem:

Suppose that c is a critical number of a continuous function f.

- If f' change from positive to negative at c, then f has a **local maximum** at c.
- If f' change from negative to positive at c, then f has a **local minimum** at c.
- If f' does not change sign at c (for example, if f' is positive on both sides of c or negative on both sides), then f has **no local maximum or minimum** at c.

Example

Find the local maximum and minimum of

$$f(x) = 3x^4 - 4x^3 - 12x^2 + 5$$

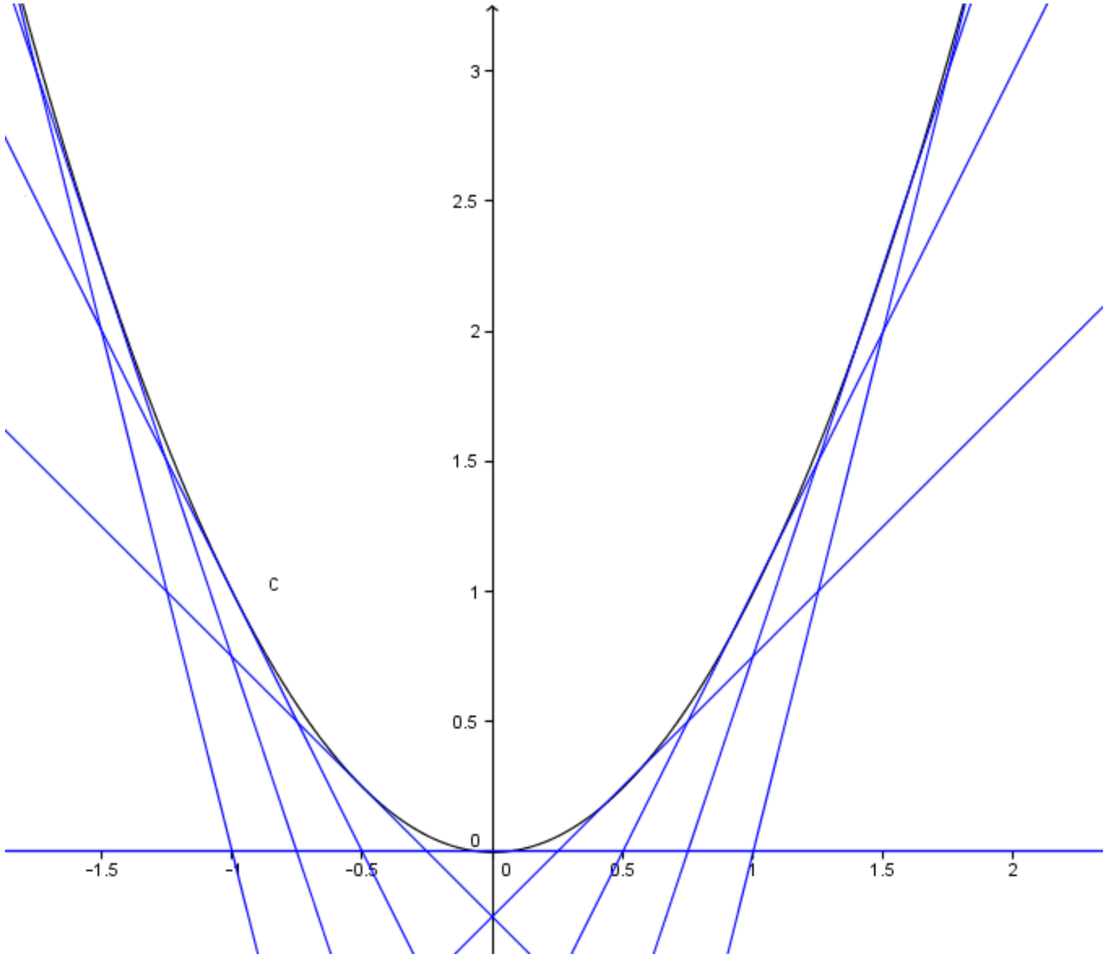
by using the first derivatives test.

Concave

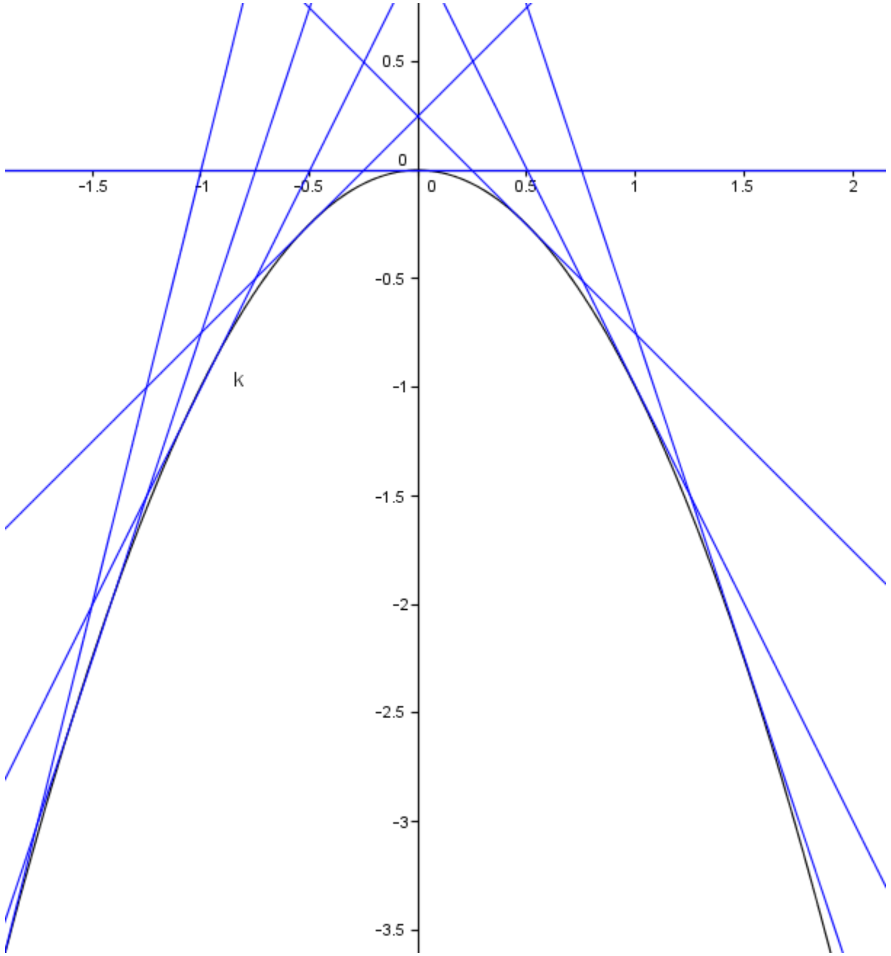
Definition:

- Let f be a graph in the interval I, f is called **concave up** on I if all the tangent line of f inside the interval I lies **above** f.
- Let f be a graph in the interval I, f is called **concave down** on I if all the tangent line of f inside the interval I lies **below** f.

Concave Up



Concave Down



Concavity Test

- A graph is **concave up** on I, If $f''(x) > 0, \forall x \in I$.
- A graph is **concave Down** on I, If $f''(x) > 0, \forall x \in I$.

Inflection Point

$P(x_p, y_p)$ on $y = f(x)$ is called an **inflection point** if f is continuous at P and the y change from concave up to concave down or vice versa. If using second derivatives terminology means,

$$f''(x_p) = 0$$

The Second Derivative Test

Let f is continuous function near c.

- If $f'(x)=0$ and $f''(c)>0$, then at c $f(x)$ is **local minimum**.
- If $f'(x)=0$ and $f''(c)<0$, then at c $f(x)$ is **local maximum**.

Example

Discuss the curve

$$y = x^4 - 4x^3$$

Summary

- Understand what is Gradient and tangent line.
- Understand the relation between gradient, tangent line, and first derivatives.
- Can find the gradient of tangent line.
- Understand the maximum and minimum values in a function.
- Able to distinguish the local optimum and absolute optimum.
- Able to find critical values.
- Able to find the optimum value of a function using several tests given.

REFERENCES

- James, S. (2012). *Calculus* (7th ed.). Cengage Learning.
- Bivens, I.C., Stephen, D., & Howard, A. (2012). Calculus Early Transcedentals (10th ed.). John Willey & Sons Inc.