OPENCOURSEWARE

CALCULUS FOR TECHNOLOGY (BETU 1023)

WEEK 1

LIMITS AND CONTINUITY

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LEARNING OUTCOMES

By the end of this topic, students are able to:

- Define limit intuitively
- Explain the concept of limits at any given point,
- Find limit using analytical method
- Explain the continuity of a functions

THE CONCEPT OF LIMITS??

If f is some function then $\lim_{x \to a} f(x) = L$ is read "the limit $f(x)$ as x approaches a is L". It means that if you choose values of x close but not equal to a, then will be close to the value L; moreover, $f(x)$ gets closer and closer to L as x gets closer and closer to a.

EXAMPLE 1

If $f(x)=x+3$ then $\lim_{x\to 4} f(x)=7$

Is true because if you substitute numbers x close to 4 in $f(x)=x+3$ the result will be close to 7.

EXAMPLE 2

If $f(x)=x^2-2$ the $\lim_{x\to 2} f(x)=?$

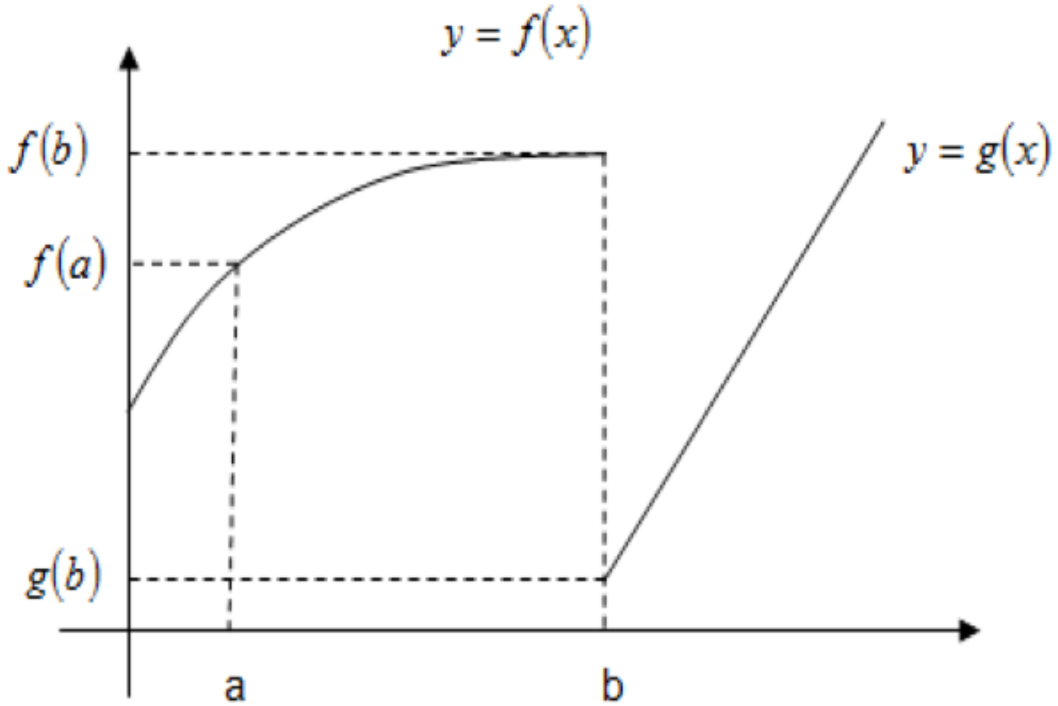
We substitute the value as x close to 2, then we can get

x	1	1.5	1.9	1.95	1.99	1.999
$f(x)$	-1	0.25	1.61	1.803	1.960	1.996

Therefore, $\lim_{x\to 2} f(x)=2$

THE EXISTENCE OF A LIMIT

Limits that exist as x approaches a certain value a



In the graph at $x = a$ is unbroken. The limit exist and is equal to the function value;

$$\lim_{x \to a} f(x) = f(a)$$

But at $x = b$, limit does not exists. It is because;

$$\lim_{x \to b^-} f(x) = f(b) \text{ and } \lim_{x \to b^+} g(x) = g(b)$$

where $f(b) \neq g(b)$

In general, we have the following theorem;

$$\lim_{x \to a} f(x) = L \text{ if and only if } \lim_{x \to a^+} f(x) = L = \lim_{x \to a^-} f(x)$$

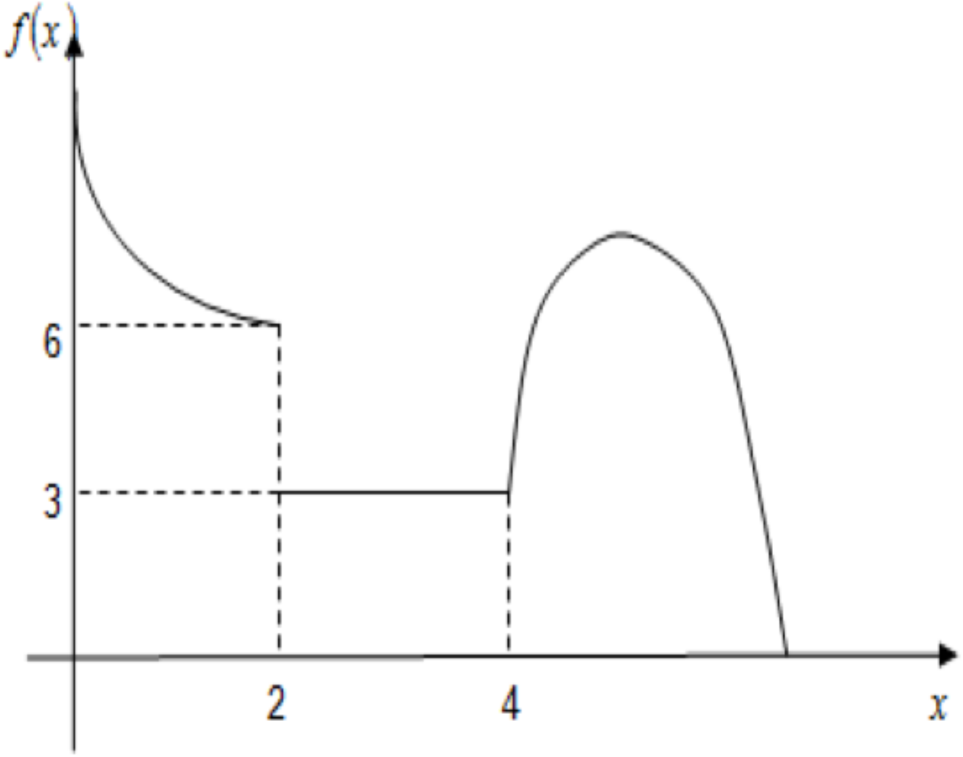
where by;

- the symbol $\lim_{x \to a^+} f(x)$ is called a right-hand limit. This expression refers to the limit of $f(x)$ for x near to a and greater than a.

- the symbol $\lim_{x \to a^-} f(x)$ is called a left-hand limit. This expression refers to the limit of $f(x)$ for x near to a and less than a.

EXAMPLE 3

From the given graph, find



a) $\lim_{x \to 4^+} f(x)$

b) $\lim_{x \to 4^-} f(x)$

c) $\lim_{x \to 4} f(x)$

d) $\lim_{x \to 2^-} f(x)$

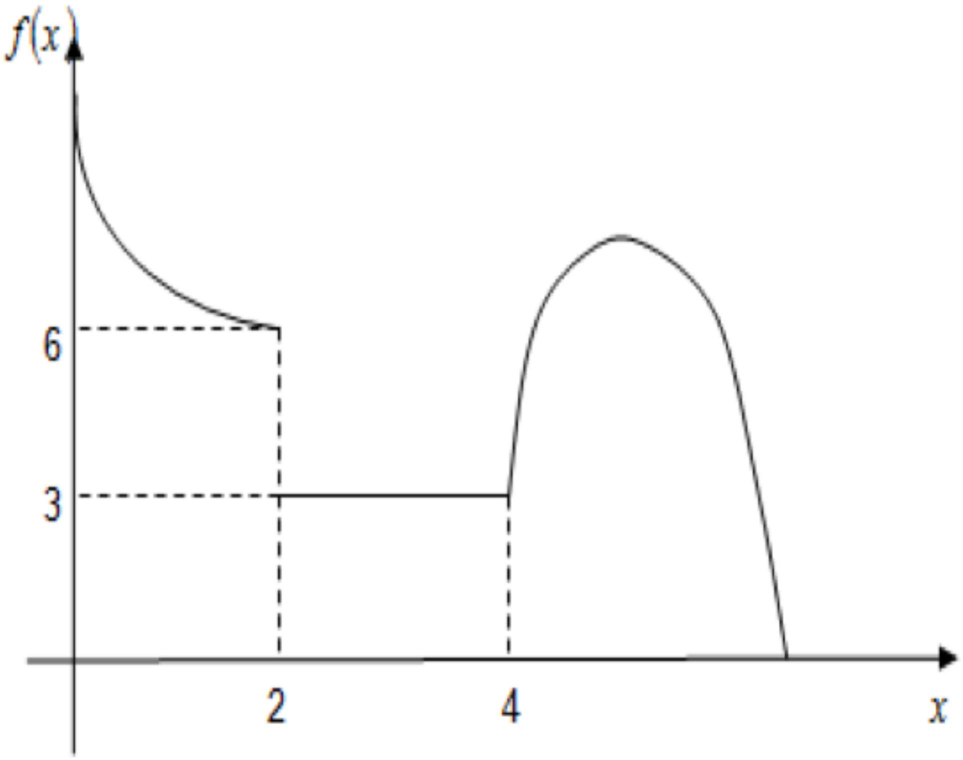
e) $\lim_{x \to 2^+} f(x)$

f) $\lim_{x \to 2} f(x)$

g) $\lim_{x \to 3} f(x)$

SOLUTION 3

From the given graph, find



a) $\lim_{x\to 4^+} f(x) = 3$

b) $\lim_{x\to 4^-} f(x) = 3$

c) $\lim_{x\to 4} f(x) = 3$

d) $\lim_{x\to 2^-} f(x) = 6$

e) $\lim_{x\to 2^+} f(x) = 3$

f) $\lim_{x\to 2} f(x)$ = does not exists

g) $\lim_{x\to 3} f(x) = 3$

LIMITS OF VARIOUS FUNCTIONS??

Basic limits

- $\lim_{x \to a} c = c$, where c is constant value
- $\lim_{x \to a} x = a$.

Limit Rules;

If $\lim_{x \to a} f(x) = L$, and $\lim_{x \to a} g(x) = K$, then;

- $$\lim_{x \to a}[f(x) \pm g(x)] = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x).$$
$$= L \pm K$$
- $$\lim_{x \to a}[f(x)g(x)] = \lim_{x \to a} f(x) \bullet \lim_{x \to a} g(x).$$
$$= LK$$

Limit Rules;

If $\lim_{x \to a} f(x) = L$, and $\lim_{x \to a} g(x) = K$, then;

- $$\lim_{x \to a}\left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}.$$

$$= \frac{L}{K}$$

- $$\lim_{x \to a} cf(x) = c \lim_{x \to a} f(x).$$

$$= cL$$

PROPERTIES OF NUMBER

Regarding to zero, 0

$0.k = 0$ $\frac{0}{0} = \text{indeterminate form}$ $0^{k} = 0$

$\frac{0}{k} = 0$ $0^{0} = 1$ $0 + 0 = 0$

$\frac{k}{0} = \infty$ $k^{0} = 1$ $0 - 0 = 0$

PROPERTIES OF NUMBER

Regarding to infinity,

$$\infty \pm k = \infty \qquad \infty - \infty = \text{indeterminate form} \qquad \infty.\infty = \infty$$

$$\infty + \infty = \infty \qquad 0.\infty = \text{indeterminate form} \qquad \frac{k}{\infty} = 0$$

$$\infty^{k} = \infty \qquad \frac{\infty}{\infty} = \text{indeterminate form} \qquad \frac{\infty}{k} = \infty$$

$$\infty^{\infty} = \infty \qquad \infty^{0} = \text{indeterminate form} \qquad \frac{\infty}{0} = \infty$$

$$\frac{0}{\infty} = 0 \qquad 0^{\infty} = \text{indeterminate form} \qquad 1^{\infty} = \text{indeterminate form}$$

SOLVING LIMIT

Solving limit problems requires a number of approaches as follows;

- Direct substitution provided a definite answer is obtained
- If direct substitutions result is indeterminate form, another approaches is needed such as:
  - Factorization technique
  - Multiplication with conjugate
  - Divide all the items with the highest power of denominator
  - Using trigonometric limit which is $\lim_{\theta \to 0}\left(\frac{\sin\theta}{\theta}\right) = 1$ and $\lim_{\theta \to 0}\left(\frac{1-\cos\theta}{\theta}\right) = 0$

EXAMPLE 4

Find the limit (if it exists) for each of the following;

a) $\lim_{x \to 3} \left(x^2 + 2x\right)$

b) $\lim_{x \to 2} \dfrac{1+x}{4+x}$

c) $\lim_{x \to 2} \dfrac{x^2 - 4}{x - 2}$

SOLUTION 4

a) $$\lim_{x\to 3}\left(x^2+2x\right)=\lim_{x\to 3}x^2+\lim_{x\to 3}2x$$
$$=9+6$$
$$=15$$

b) $$\lim_{x\to 2}\frac{1+x}{4+x}=\frac{1+2}{4+2}$$
$$=\frac{1}{2}$$

SOLUTION 4

c) $$\lim_{x\to 2}\frac{x^2-4}{x-2}=\frac{4-4}{2-2}$$

$$=\frac{0}{0},\text{indeterminate form}$$

Then we will evaluate the limit using factorization technique

$$\lim_{x\to 2}\frac{x^2-4}{x-2}=\lim_{x\to 2}\frac{(x-2)(x+2)}{(x-2)}$$

$$=\lim_{x\to 2}(x+2)$$

$$=4$$

EXAMPLE 5

Find the limit (if it exists) for each of the following;

a) $\lim_{x\to 3} \dfrac{\sqrt{x}-\sqrt{3}}{x-3}$

b) $\lim_{x\to 6} \dfrac{\sqrt{x-2}-2}{x-6}$

c) $\lim_{x\to 0} \dfrac{\dfrac{1}{4+x}-\dfrac{1}{4}}{x}$

SOLUTION 5

a) $\lim_{x\to 3}\frac{\sqrt{x}-\sqrt{3}}{x-3}=\frac{0}{0}$ Since indeterminate form, the conjugate multiplication technique is used when a radical appears in the function

$$\lim_{x\to 3}\frac{\sqrt{x}-\sqrt{3}}{x-3}=\lim_{x\to 3}\frac{\sqrt{x}-\sqrt{3}}{x-3}$$

$$=\lim_{x\to 3}\frac{\sqrt{x}-\sqrt{3}}{x-3}\times\frac{\sqrt{x}+\sqrt{3}}{\sqrt{x}+\sqrt{3}}$$

$$=\lim_{x\to 3}\frac{x-\sqrt{3}\sqrt{x}+\sqrt{x}\sqrt{3}-3}{(x-3)\left(\sqrt{x}+\sqrt{3}\right)}$$

$$=\lim_{x\to 3}\frac{x-3}{(x-3)\left(\sqrt{x}+\sqrt{3}\right)}$$

$$=\lim_{x\to 3}\frac{1}{\left(\sqrt{x}+\sqrt{3}\right)}$$

$$=\frac{1}{2\sqrt{3}}$$

SOLUTION 5

b) $\lim_{x \to 6} \frac{\sqrt{x-2}-2}{x-6} = \frac{0}{0}$

Since indeterminate form, the conjugate multiplication technique is used when a radical appears in the function

$$\lim_{x \to 6} \frac{\sqrt{x-2}-2}{x-6} = \lim_{x \to 6} \frac{\sqrt{x-2}-2}{x-6} \times \frac{\sqrt{x-2}+2}{\sqrt{x-2}+2}$$

$$= \lim_{x \to 6} \frac{(x-2)-4}{(x-6)\left(\sqrt{x-2}+2\right)}$$

$$= \lim_{x \to 6} \frac{x-6}{(x-6)\left(\sqrt{x-2}+2\right)}$$

$$= \lim_{x \to 6} \frac{1}{\sqrt{x-2}+2}$$

$$= \frac{1}{4}$$

SOLUTION 5

c) $\lim_{x\to 0}\dfrac{\dfrac{1}{4+x}-\dfrac{1}{4}}{x}=\dfrac{0}{0}$

Since indeterminate form, the factorization technique will be apply to solve this question

$$\lim_{x\to 0}\frac{\frac{1}{4+x}-\frac{1}{4}}{x}=\lim_{x\to 0}\frac{\frac{1\times 4}{(4+x)4}-\frac{(4+x)}{4(4+x)}}{x}$$

$$=\lim_{x\to 0}\frac{\frac{4-4-x}{16+4x}}{x}$$

$$=\lim_{x\to 0}\frac{-x}{x(16+4x)}$$

$$=\lim_{x\to 0}\frac{-1}{16+4x}$$

$$=-\frac{1}{16}$$

EXAMPLE 6

Find the limit of the following (if exists);

a) $\lim\limits_{x \to \infty} \dfrac{x^2}{7 - x^2}$

b) $\lim\limits_{x \to \infty} \dfrac{2x + 1}{\sqrt{x^2 + 3}}$

c) $\lim\limits_{x \to -\infty} \dfrac{2x + 1}{\sqrt{x^2 + 3}}$

d) $\lim\limits_{x \to -\infty} \dfrac{x^4 - x}{3 - 2x}$

SOLUTION 6

a) $\lim_{x\to\infty} \frac{x^2}{7-x^2} = \frac{\infty}{\infty}$, indeterminate form then

$$\lim_{x\to\infty} \frac{x^2}{7-x^2} = \lim_{x\to\infty} \frac{\frac{x^2}{x^2}}{\frac{7-x^2}{x^2}}$$

$$= \lim_{x\to\infty} \frac{1}{\frac{7}{x^2} - 1}$$

$$= \frac{1}{0-1}$$

$$= -1$$

SOLUTION 6

b) $$\lim_{x\to\infty}\frac{2x+1}{\sqrt{x^2+3}}=\lim_{x\to\infty}\frac{\dfrac{2x+1}{|x|}}{\dfrac{\sqrt{x^2+3}}{|x|}}$$

$$=\lim_{x\to\infty}\frac{2+\dfrac{1}{x}}{\sqrt{\dfrac{x^2+3}{x^2}}},\quad \text{since}\quad x=\sqrt{x^2}$$

$$=\frac{2+0}{\sqrt{1+0}}$$

$$=2$$

SOLUTION 6

c) $$\lim_{x\to-\infty}\frac{2x+1}{\sqrt{x^2+3}} = \lim_{x\to-\infty}\frac{\frac{2x+1}{|x|}}{\frac{\sqrt{x^2+3}}{|x|}}$$

$$= \lim_{x\to-\infty}\frac{\frac{2x+1}{-x}}{\sqrt{\frac{x^2+3}{x^2}}},$$

$$= \frac{-2-0}{\sqrt{1+0}}$$

$$= -2$$

SOLUTION 6

d) $$\lim_{x \to -\infty} \frac{x^4 - x}{3 - 2x} = \lim_{x \to -\infty} \frac{\dfrac{x^4 - x}{x}}{\dfrac{3 - 2x}{x}}$$

$$= \lim_{x \to -\infty} \frac{x^3 - 1}{\dfrac{3}{x} - 2}$$

$$= \frac{-\infty - 1}{0 - 2}$$

$$= +\infty$$

EXAMPLE 7

Find the limit of the following (if exists);

a) $\lim_{x\to 0} \frac{\sin 3x}{x}$

b) $\lim_{x\to 0} \frac{\sin 3x}{\sin 2x}$

c) $\lim_{x\to 0} \frac{\tan 2x}{x(x+3)^2}$

SOLUTION 7

a) $$\lim_{x\to 0}\frac{\sin 3x}{x} = \lim_{x\to 0}\frac{3\sin 3x}{3x}$$

$$= 3\lim_{x\to 0}\frac{\sin 3x}{3x}$$

$$= 3(1)$$

$$= 3$$

SOLUTION 7

b)

$$\lim_{x\to 0}\frac{\sin 3x}{\sin 2x}=\lim_{x\to 0}\frac{\dfrac{\sin 3x}{x}}{\dfrac{\sin 2x}{x}}$$

$$=\lim_{x\to 0}\frac{\dfrac{3\sin 3x}{3x}}{\dfrac{2\sin 2x}{2x}}$$

$$=\frac{3}{2}\lim_{x\to 0}\frac{\dfrac{\sin 3x}{3x}}{\dfrac{\sin 2x}{2x}}$$

$$=\frac{3}{2}\left(\frac{1}{1}\right)$$

$$=\frac{3}{2}$$

SOLUTION 7

c) $$\lim_{x\to 0}\frac{\tan 2x}{x(x+3)^2}=\lim_{x\to 0}\frac{\sin 2x / \cos 2x}{x(x+3)^2}$$

$$=\lim_{x\to 0}\frac{\sin 2x}{x(x+3)^2\cos 2x}$$

$$=\lim_{x\to 0}\frac{\sin 2x}{x}\times\frac{1}{(x+3)^2\cos 2x}$$

$$=2\lim_{x\to 0}\frac{\sin 2x}{2x}\times\frac{1}{(x+3)^2\cos 2x}$$

$$=2(1)\left(\frac{1}{9(1)}\right)$$

$$=\frac{2}{9}$$

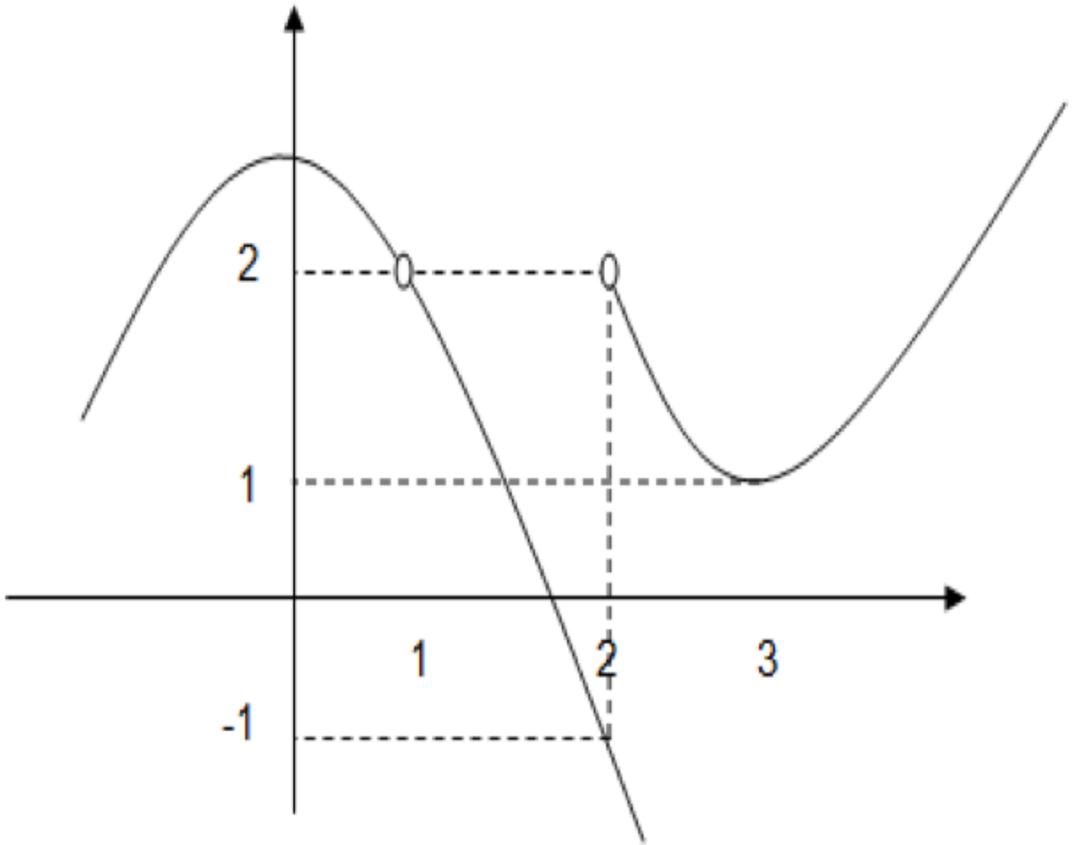
CONTINUITY OF A FUNCTION??

The basic idea of a continuous function is the graph function is connected, except where breaks in the domain occur. For a function to be continuous, the limit value must be equal to the function value at $x = a$. Three criteria must be follows;

- $f(a)$ is defined
- $\lim_{x \to a} f(x)$ exists
- $\lim_{x \to a} f(x) = f(a)$

EXAMPLE 8

From the graph below, determine whether $f(x)$ is continuous at;



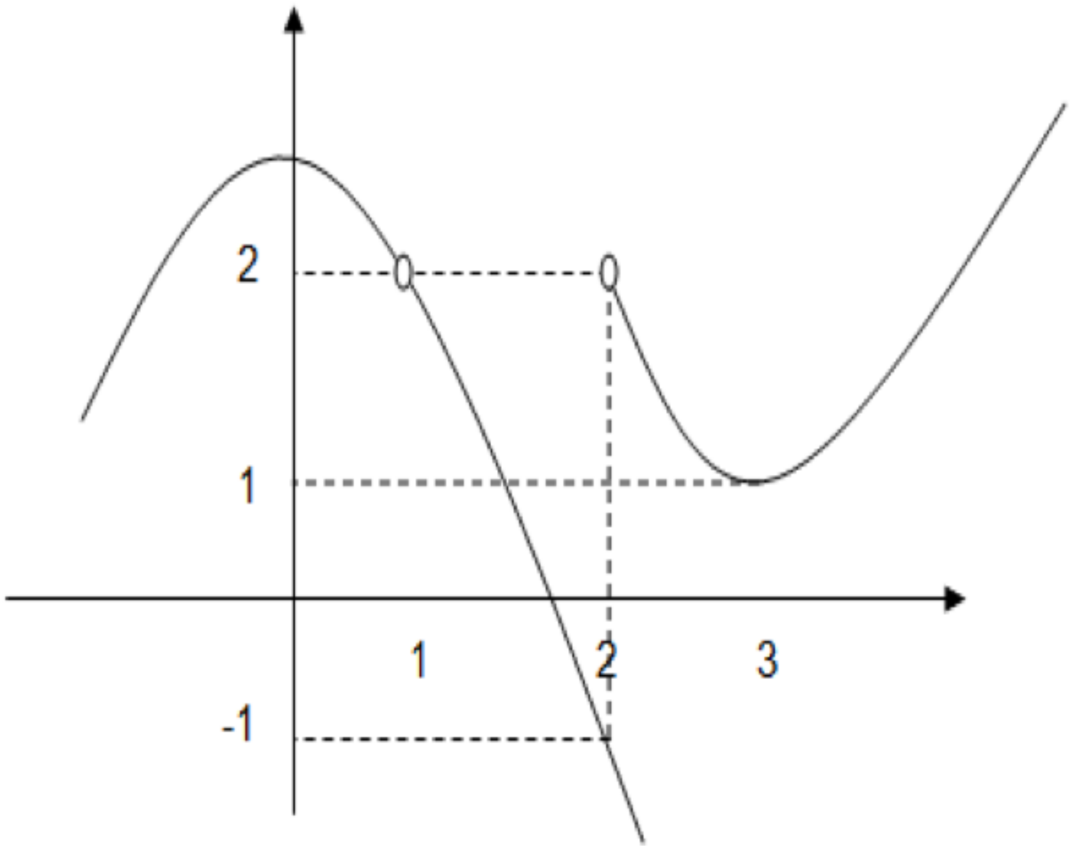
a) $x = 1$

b) $x = 2$

c) $x = 3$

SOLUTION 8

From the graph below, determine whether $f(x)$ is continuous at;



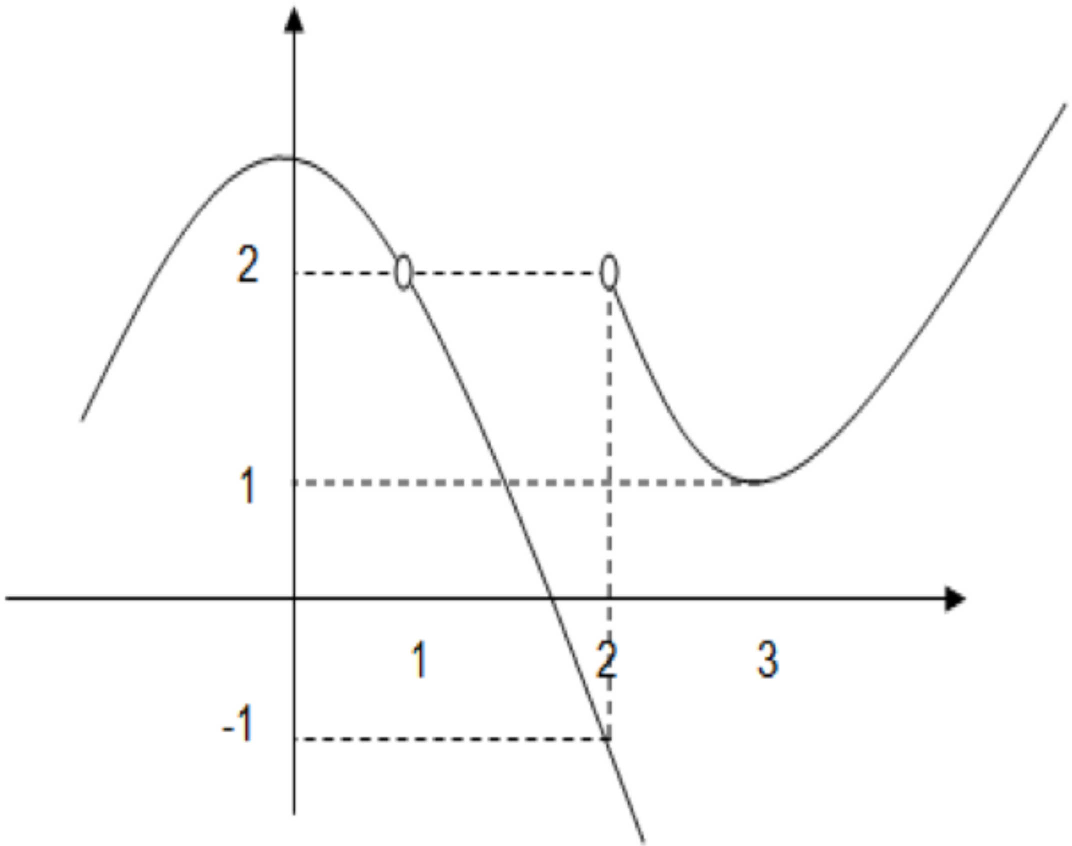
a) $x = 1$

$f(1) =$ not defined

Therefore $f(x)$ is not continuous at $x = 1$

SOLUTION 8

From the graph below, determine whether $f(x)$ is continuous at;



b) $x = 2$

$$f(2) = -1$$

$$\lim_{x \to 2^-} f(x) = -1$$

$$\lim_{x \to 2^+} f(x) = 2$$

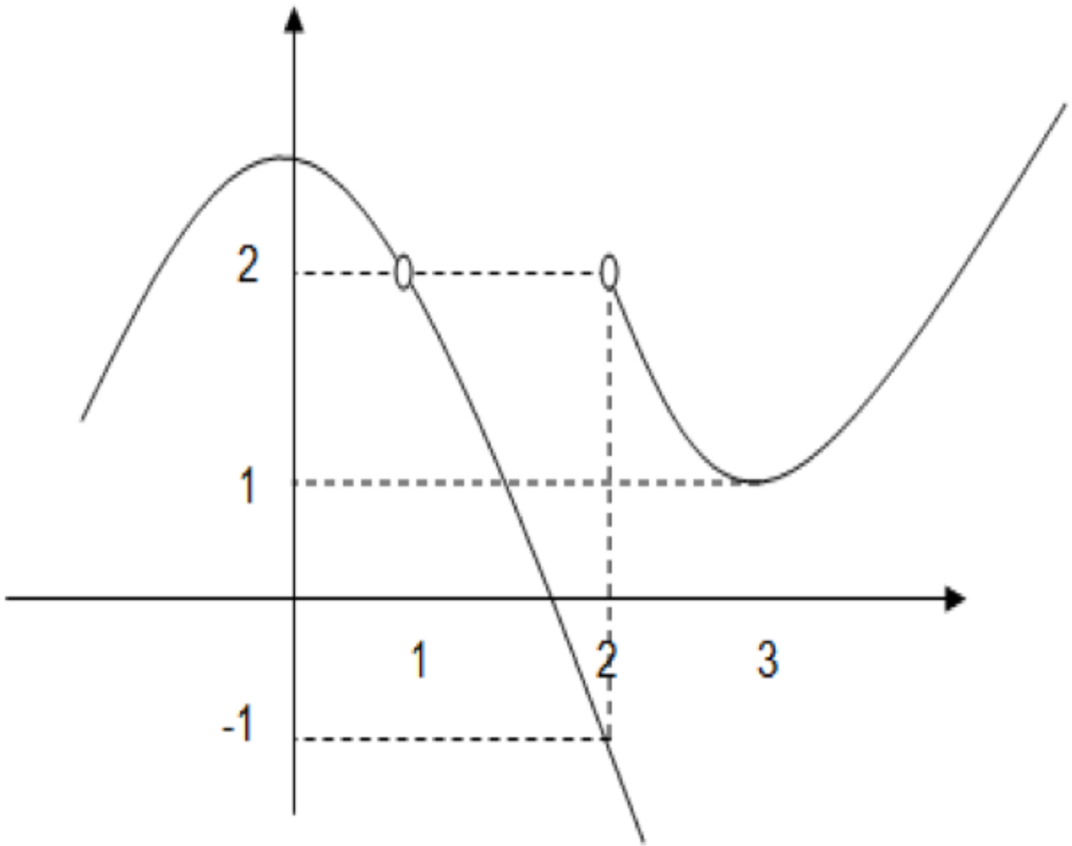
$$\lim_{x \to 2} f(x) = \text{does not exist}$$

$$\lim_{x \to 2} f(x) \neq f(2)$$

Therefore $f(x)$ is not continuous at $x = 2$

SOLUTION 8

From the graph below, determine whether $f(x)$ is continuous at;



c) $x = 3$

$$f(3) = 1$$

$$\lim_{x \to 3^-} f(x) = 1$$

$$\lim_{x \to 3^+} f(x) = 1$$

$$\lim_{x \to 3} f(x) = 1$$

$$\lim_{x \to 3} f(x) = f(3)$$

Therefore $f(x)$ is continuous at $x = 3$

EXAMPLE 9

Given $f(x)=\begin{cases} \dfrac{x}{2}+1, & x<0 \\ x^2, & 0\le x<2 \\ -2x+c, & x\ge 2 \end{cases}$

a) Determine whether $f(x)$ is continuous at $x=0$

b) Find the value of c such that $f(x)$ is continuous at $x=2$

SOLUTION 9

a) Determine whether $f(x)$ is continuous at $x=0$

$$f(0)=0$$

$$\lim_{x\to 0^-} f(x)=\frac{0}{2}+1$$

$$=1$$

$$\lim_{x\to 0^+} f(x)=0$$

$$\lim_{x\to 0} f(x)=\text{does not exist}$$

therefore $f(x)$ is not continuous at $x=0$

SOLUTION 9

b) Find the value of c such that $f(x)$ is continuous at $x = 2$

$$f(2) = -2(2) + c$$

$$= -4 + c$$

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x)$$

$$\lim_{x\to 2^-} x^2 = \lim_{x\to 2^+} -2x + c$$

$$4 = -4 + c$$

$$c = 8$$

TRY IT YOURSELF 1

Evaluate the following limits;

a) $$\lim_{x \to -6} \frac{x^2 - 36}{x + 6}$$

b) $$\lim_{x \to 4} \frac{\sqrt{2x - 7} - 1}{4 - x}$$

c) $$\lim_{x \to -\infty} \frac{\sqrt{x^2 + 2}}{2x + 4}$$

d) $$\lim_{x \to 0} \frac{(x + 1)^3 \sin x}{2x}$$

ANSWER

a) -12

b) $-\dfrac{2}{\sqrt{15}+1}$

c) $-\dfrac{1}{2}$

d) $\dfrac{1}{2}$

TRY IT YOURSELF 2

Given $f(x) = \begin{cases} \dfrac{\sqrt{x}-2}{x-4} & , x < 4 \\ \dfrac{k}{2x} & , 4 \le x < 6 \\ \dfrac{x^2-1}{x^2-x-2} & , x \ge 6 \end{cases}$

a) Find the value of k such that $f(x)$ is continuous at $x = 4$

b) Determine whether $f(x)$ is continuous at $x = 6$

ANSWER

a) $k = 2$

b) Not continuous

REFERENCES

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- Bivens, I.C., Stephen, D., & Howard, A. (2012). Calculus Early Transcedentals (10th ed.). John Willey & Sons Inc.