CALCULUS FOR TECHNOLOGY (BETU 1023)

WEEK 5

APPLICATION OF DIFFERENTIATION

[1]KHAIRUM BIN HAMZAH, [2]IRIANTO, [3]ABDUL LATIFF BIN MD AHOOD, [4]MOHD FARIDUDDIN BIN MUKHTAR

[1]khairum@utem.edu.my, [2]irianto@utem.edu.my, [3]latiff@utem.edu.my, [4]fariduddin@utem.edu.my

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LEARNING OUTCOMES

By the end of this topic, students are able to:

- Solve the Related Rates Problems.
- Describe the behavior of a function (Analysis of Functions).
- Solve the applied maximum and minimum problems (Optimization problems).

Approximate Value and Error Rate of Change

Calculus was defined as the study of mathematically defined change. Science defined it is a changing situations in terns of several conditions or dimensions where one or more dimensions is changing with respect to another dimensions.

Let's begin with the definition of the derivative:

$$\frac{df}{dx} = f'(x)$$

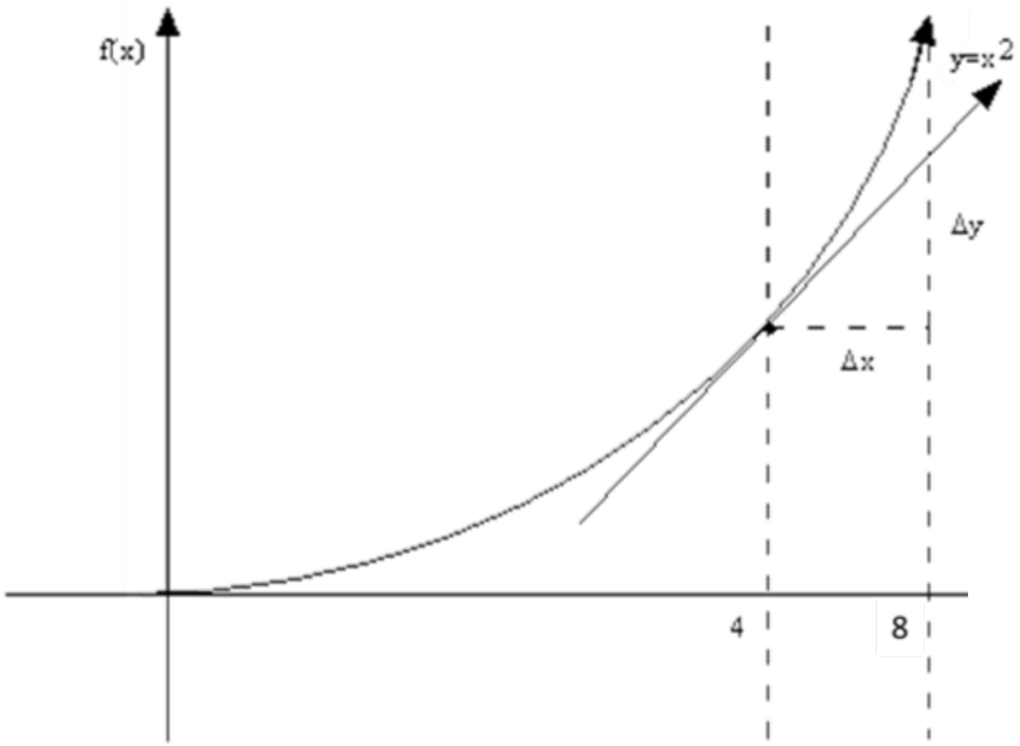
Previously, we have seen the first principle case which is:

$$\frac{df}{dx} = \lim_{x \to 0} \frac{f(x+h) - f(x)}{h}$$

where we can write as a changing situations:

$$\frac{df}{dx} = \lim_{x \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{x \to 0} \frac{\Delta f}{\Delta x}$$

Δf and Δx is known as the error rate of change for f and x respectively.



Graph shows a line with a function, $f(x) = x^2$ from $x = 4$ to $x = 8$.

First, noted that $\Delta x = 8 - 4 = 4$.

And the change of $f(x)$ over this interval is by definition

$$\Delta f = f(x_2) - f(x_2) \quad \text{or} \quad \Delta f = f(x + \Delta x) - f(x)$$

or from the previous definition, we will have:

$$\Delta f = f'(x)\Delta x$$

Hence, we have:

$$f(x) = x^2 \qquad \text{and}$$

$$f'(x) = 2x \qquad \Delta x = 4$$

$$\begin{aligned} \therefore\ \Delta f &= f'(x)\Delta x \\ &= (2x)(4) \quad \text{by refering at } x = 4 \\ &= 32 \end{aligned}$$

To approximate a value, considering the previous case we have:

$$\Delta f = f'(x)\Delta x$$

where it is also equal to

$$f(x+\Delta x) - f(x) = f'(x)\Delta x$$

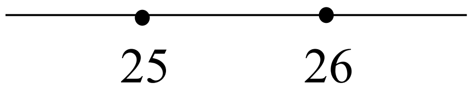
$$f(x+\Delta x) = f(x) + f'(x)\Delta x$$

Let's take an example on finding the approximate value of $\sqrt{26}$.

First, let a function be $f(x) = \sqrt{x}$ (where if we substitute $x = 26$, we got the $\sqrt{26}$)

Next, we consider a perfect square number that is closest to 26, which is 25 since 5^2.

Substituting into the equation, we have:



$$\Delta x = 26 - 25 = 1$$

$$f(x+\Delta x) = f(x) + f'(x)\Delta x$$

$$\rightarrow f(25+1) = f(25) + f'(25)(1)$$

$$f(26) = \sqrt{25} + \frac{1}{2\sqrt{25}}(1) \quad \text{where } f(x) = \sqrt{x}$$

$$= 5 + \frac{1}{10}$$

$$= 5.1$$

$\therefore$ The approximate value of $\sqrt{26} \approx 5.1$

Try Yourself

Find the approximate value for each given questions.

(a) $\sqrt{2}$

(b) 0.93^2

(c) $2\pi - 1$

(d) e^2

$Answer:$

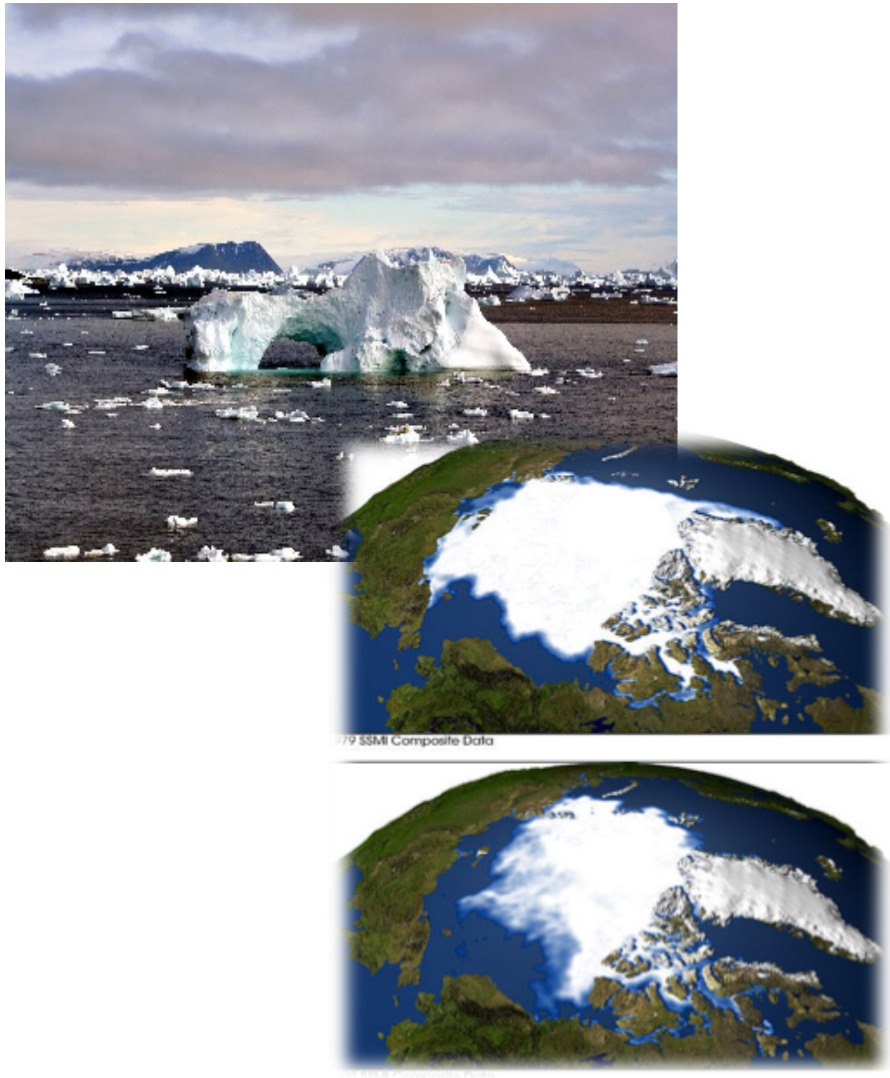
$(a) 1.414$

$(b) 0.86$

$(c) 5.3$

$(d) 7.4$

Related Rates Problems



A work done by Peter Wadhams, Professor of Ocean Physics and Head of the Polar Ocean Physics Group of University of Cambridge states the seriousness of Arctic meltdown as follows:

"As the ice area decreases you're replacing the area of white, which reflects about 90% of the solar radiation, by open water, which reflects less than 10%. As the sea ice retreats you're absorbing a lot more radiation and this increases the rate of warming at high latitudes."

A simple ice growth model involving differentiation and vectors were developed. It was used to predict that the growth of ice (thickness, h) over time is proportional to the square root of the elapsed time:

$$\frac{dh}{dt} = k\sqrt{t} \qquad \text{(where } k \text{ is a constant)}$$

This is one example where the importance of rates takes places in our daily life applications.

Related Rates Problems

In rates problems, we were determined to find the rate at which some quantity is changing by relating the quantity to the other quantities whose rates of change are known. Here are the strategy for solving related rates problems:

Step 1 Assign letters to all quantities that vary with time and any others that relevant to the problem.

Step 2 Identify the rates of change that are known and the rates of change that is to be found.

Step 3 Find an equation that relates the variables that involves in the case.

Step 4 Differentiate both side of the equation with respect to time (or others) to produce a relationship between the rates investigated.

Step 5 Finally, substitute all known values for the rates of change and the variable, then solve for the unknown rate of change.

Useful Formulas

Circle:

i) Area of circle : $A = \pi r^2$

ii) Circumference of a circle : $C = 2\pi r$

Sphere:

i) Volume : $V = \frac{4}{3}\pi r^3$

ii) Surface area: $A = 4\pi r^2$

Cylinder:

i) Volume : $V = \pi r^2 h$

ii) Surface area : $A = 2\pi rh + 2\pi r^2$

Cone:

i) Volume: $V = \frac{1}{3}\pi r^2 h$

Phytogorean Theorem : $a^2 + b^2 = c^2$

EXAMPLE 1

Consider a case where an oil spilled from a ruptured tank spreads in a circular pattern. The radius increases at a constant rate of 2 ft/s. How fast is the area of the spill increasing when the radius of the spill is 60 ft.

Solution

Step 1

$r = \text{radius}$

$A = \text{area}$

Step 2

$$\frac{dr}{dt} = 2$$

$$\frac{dA}{dt} = ? \quad \text{when } \ r = 60$$

Step 3

Given $\dfrac{dA}{dt}$ and $\dfrac{dr}{dt}$, thus we can relate them as:

$$\frac{dA}{dt} = \frac{dr}{dt} \times \frac{dA}{dr}$$

Since we need $\dfrac{dA}{dr}$, then the equation involves is $A = \pi r^2$.

Step 4

$$A = \pi r^2$$

$$\frac{dA}{dr} = 2\pi r \quad (r = 60)$$

$$= 120\pi$$

Step 5

$$\frac{dA}{dt} = \frac{dr}{dt} \times \frac{dA}{dr}$$

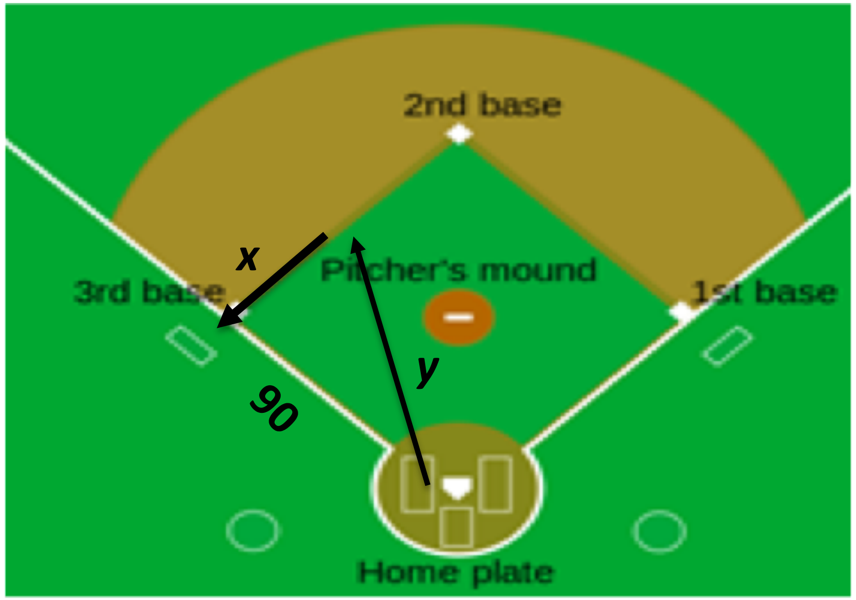
$$= 2 \times 120\pi$$

$$= 240\pi$$

The area of the spill increasing with 240π ft/s.

EXAMPLE 2

A baseball diamond is a square whose sides are 90 ft long. A baseball player is running to the third base from second base with a speed of 30 ft/s at the instant when he is 20 ft from third base. What is the rate of distance he is from home plate changing at the instant?



Solution

Step 1

$x =$ distance from 2^{nd} base to 3^{rd} base.

$y =$ distance from player's distance to the home plate.

Step 2

$$\frac{dx}{dt} = 30 \quad \text{when} \ \ x = 20$$

$$\frac{dy}{dt} = ?$$

Step 3

Given $\dfrac{dy}{dt}$ and $\dfrac{dx}{dt}$, thus we can relate them as:

$$\frac{dy}{dt} = \frac{dx}{dt} \times \frac{dy}{dx}$$

We use the property of Phy. Theorem based on the situation:

$$y^2 = x^2 + 90^2$$

Step 4

$$y^2 = x^2 + 90^2$$

$$\rightarrow y = \sqrt{x^2 + 90^2}$$

$$\frac{dy}{dx} = \frac{1}{2(\sqrt{x^2 + 90^2})} 2x \quad (x = 20)$$

$$= 0.2169$$

Step 5

$$\frac{dy}{dt} = \frac{dx}{dt} \times \frac{dy}{dx}$$

$$= 30 \times 0.2169$$

$$= 6.508$$

The distance from home plate is changing with 6.508 ft/s.

EXAMPLE 3

Figure 1 showed a camera placed 3000 ft from the base of a rocket launching pad. The rocket is 4000 ft above the launching pad when it is rising vertically with 880 ft/s. How fast must the camera elevation angle change at that instant to keep the camera aimed at the rocket?

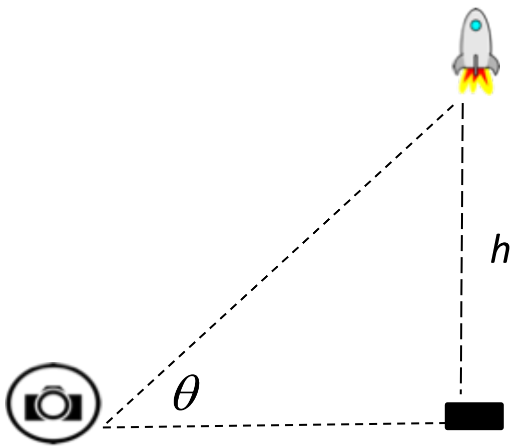


Figure 1

Solution

Step 1

$\theta =$ camera elevation angle in rad after t seconds.

$h =$ height of the rocket in feet after t seconds.

Step 2

$$\frac{dh}{dt} = 880 \quad \text{when} \quad h = 4000$$

$$\frac{d\theta}{dt} = ?$$

Step 3

Given $\frac{d\theta}{dt}$ and $\frac{dh}{dt}$, thus we can relate them as:

$$\frac{d\theta}{dt} = \frac{dh}{dt} \times \frac{d\theta}{dh}$$

To relate θ and h, use tangent relationship:

$$\tan\theta = \frac{h}{3000}$$

Step 4

$$\sec^2\theta \frac{d\theta}{dh} = \frac{1}{3000}$$

$$\frac{d\theta}{dh} = \frac{1}{\sec^2\theta(3000)}$$

Note that;

$$\sec\theta = \frac{5000}{3000} = \frac{5}{3}$$

$$\therefore \frac{d\theta}{dh} = \frac{1}{\left(5/3\right)^2 3000} = \frac{3}{25000}$$

Step 5

$$\frac{d\theta}{dt} = \frac{dh}{dt} \times \frac{d\theta}{dh}$$

$$= 880 \times 3/25000$$

$$= 66/625$$

$$= 0.11 \text{ rad}$$

The camera elevation angle is 0.11 rad/s or 6.05 deg/s.

EXAMPLE 4

A conical filter that is 16 cm high and radius of 4 cm at the top allows a liquid draining with a constant rate of 2 cm^3/min.

a) Find a formula that expresses the rate at which the depth of the liquid is changing in terms of depth.

b) At what rate is the depth of the liquid changing at the instant when the liquid in the cone is 8 cm deep?

Solution

Step 1

r = radius of the liquid in cm after t minutes.

y = height of the liquid in cm after t seconds.

V = volume of the liquid after t seconds.

Step 2

$$\frac{dV}{dt} = -2 \quad \text{(since } V \text{ decrease as } t \text{ increase)}$$

a) Formula that express the rate at which the depth of the liquid is changing in terms of the depth.

We want to get a formula that relate the rate of depth (y):

$$\frac{dy}{dt} = \frac{dV}{dt} \times \frac{dy}{dV}$$

Using the volume formula of a cone, we have

$$V = \frac{1}{3}\pi r^2 y$$

in terms of y, we have

$$V = \frac{1}{3}\pi\left(\frac{1}{4}y\right)^2 y$$

Since r and y changes :

$$\frac{r}{y} = \frac{4}{16}$$

$$r = \frac{1}{4}y$$

So simplify the equation:

$$V = \frac{1}{48}\pi y^3$$

$$\frac{dV}{dy} = \frac{1}{16}\pi y^2$$

$\therefore$

$$\frac{dy}{dt} = -2 \times \frac{16}{\pi y^2} = -\frac{32}{\pi y^2}$$

So, the formula we want to obtain is

$$\frac{dy}{dt} = -\frac{32}{\pi y^2}$$

b) Depth rate when *y = 8cm*

$$\frac{dy}{dt} = -\frac{32}{\pi y^2} \qquad \text{when } y = 8cm$$

$$= -\frac{32}{\pi (8)^2}$$

$$= -\frac{1}{2\pi}$$

$$\approx -0.16 \text{ cm/min}$$

Try Yourself

1. A spherical balloon is pumped with air at a rate of 5 cm^3/min. Determine how much the radius of the balloon is increasing when the diameter of the balloon is 20 cm. Ans : 3.9789×10^{-3} cm/min

2. A cylindrical water tank filled with water at a rate of 72 m^3/min. The tank has a radius of 6m with 10m height. How fast is the water level rising? Ans : 0.6367 m/min

3. A 15-foot ladder propped against a wall is sliding along the ground at a rate of 3 ft/sec. How fast is the ladder sliding down the wall when the base of the ladder is 12 feet from the wall? Ans : -4 ft/sec

REFERENCES

- James, S. (2012). *Calculus* (7th ed.). Cengage Learning.
- Bivens, I.C., Stephen, D., & Howard, A. (2012). Calculus Early Transcedentals (10th ed.). John Willey & Sons Inc.