Solid Mechanics
BETM 2303
Week 7 – Torsion

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Lesson Outcome

- ✓ To understand the deformation of elastic circular shaft by giving torque.
- ✓ To understand support reaction and calculate maximum torque that can be transmitted by a circular shaft

Torsion

Torque

"A moment that tends to twist a body about its axis of rotation"

Small twist angle results in unchanged radius and length of a shaft

If torque is applied to one free end of a shaft and the other end is fixed, the shaft will skew in from its original dimension

Torsion Formula

- Internal torque will be created within a body once external torque is applied to it

- Linear elastic material applies Hooke's Law, $\tau = G\gamma$

- Linear variation in shear stress is due to the fact that shear strain varies linearly along radial line of cross section

Torsion Formula

- Thus torque τ varies from 0 at the shaft axis to τ_{max} on the outer surface

- Shear stress distribution over the cross section in terms of the radial position ρ of an element can be defined as

$$\tau = \left(\frac{\rho}{c}\right)\tau_{\max}$$

Torsion Formula

- Resultant internal torque T along the cross section can be expressed as

$$T = \int_A \rho(\tau dA)$$

$$T = \int_A \rho\left(\frac{\rho}{c}\right)\tau_{\max} dA$$

Torsion Formula

- τ_{max}/c is constant. Thus,

$$T = \frac{\tau_{\max}}{c}\int_A \rho^2 dA$$

- Integral is based on shaft geometry that represents *polar moment of inertia, J*

- Equation above can be simplified as

$$\tau_{\max} = \frac{Tc}{J}$$

Torsion Formula

τ_{max} = max shear stress on the outer surface of the shaft

T = resultant internal torque acting at the cross section

J = polar moment of inertia of the cross-sectional area

c = outer radius of the shaft

Torsion Formula

At the middle point along the shaft ρ, shear stress can be calculated using

$$\tau = \frac{T\rho}{J}$$

- These equation is valid only when material is linear elastic and homogenous. It is specifically used for circular shaft

Solid Shaft

Polar moment of inertia *J* of a shaft with perfect circular cross section can be calculated using area element in the form a differential ring

$$J = \int_A \rho^2 dA = \int_0^c \rho^2 (2\pi\rho d\rho) = 2\pi \frac{1}{4} \rho^4 \Big|_0^c$$

$$J = \frac{\pi}{2} c^4$$

Tubular Shaft

- For shaft with tubular cross section, inner radius of C_1 and outer radius of C_2 the polar moment of inertia can be expressed as

$$J = \frac{\pi}{2}(C_2^{\ 4} - C_1^{\ 4})$$

- The distribution of shear stress along tubular cross sectional area varies linearly

Absolute Maximum Torsional Stress

- To determine absolute maximum torsional stress, the location where Tc/J is maximum must be find
- A torque diagram of internal torque T versus against its position x along the shaft longitudinal axis. Consequently, maximum ration of Tc/J can be determined

Key Points !

- When torque is subjected to circular shaft, the radial lines will rotate but cross section remains plane. As a result, shear strain will varies along any radial line

- In case of linear elastic homogenous material shear stress along the radial line also varies and does not exceed proportional limit

Power Transmission

- Shaft with circular cross section is always used for power transmission purpose
- In order to generate power, torque is given to the shaft
- Work generated by rotational motion of a shaft is equal to torque given times angle of rotation

Power Transmission

- Hence, the instant power generated at instant time can be expressed as

$$P = \frac{Td\theta}{dt}$$

$$P = T\omega$$

where ω is angular velocity

Power Transmission

- For a machine, frequency of shaft rotation is used as one of the parameter. It is equal to revolution of a shaft in one second

$$P = 2\pi f T$$

where 1 cycle = 2π rad

Power Transmission

- Size of a shaft cross section can be calculated using torsion formula, given the value of torque T and allowable shear stress τ_{allow}

$$\frac{J}{c} = \frac{T}{\tau_{allow}}$$

For a solid shaft, $J=(\pi/2)c^4$

Using substitution, shaft radius c can be determined

Angle of Twist

- A differential disk with thickness *dx* located at *x*, is taken out from the shaft as shown below

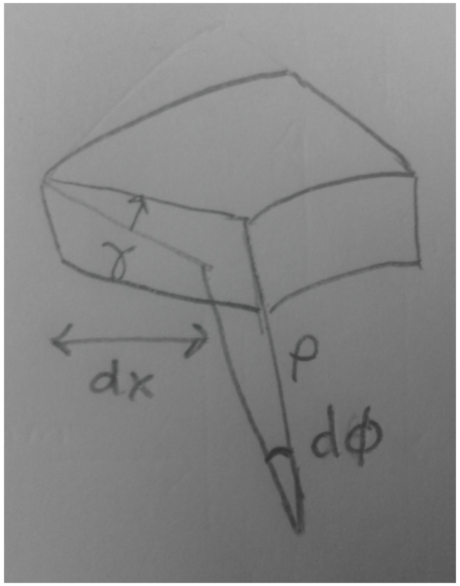

Figure 7.1

- The internal resultant torque is T(x)

Angle of Twist

- The twist will cause relative rotation of a face with respect to the other faces $d\Phi$

- This situation leads to shear strain γ of material element at arbitrary radius ρ

$$d\phi = \gamma \frac{dx}{\rho}$$

Angle of Twist

- According to Hooke's Law,

$$\gamma = \frac{\tau}{G}$$

- Using torsion formula,

$$\tau = T(x)\rho / J(x) \quad \text{and} \quad \tau = T(x)\rho / J(x)$$

Angle of Twist

- Using substitution,

$$d\phi = \frac{T(x)}{J(x)G(x)}dx$$

- Integration along shaft longitudinal axis produce angle of twist,

$$\phi = \int_0^L \frac{T(x)dx}{J(x)G(x)}$$

Angle of Twist

- Φ = angle of twist between two end of a shaft (radian)
- $T(x)$ = torque at arbitrary position x
- $J(x)$ = polar moment of inertia of a shaft
- $G(x)$ = shear modulus of elasticity

Constant Torque & Cross-Sectional Area

- In order for G to be constant, the material must be homogenous

- Constant external torque and cross sectional area integrate the angle of twist to produce

$$\phi = \frac{TL}{JG}$$

Multiple Torque

- In case there are more than one torque applied to the shaft or shaft has different area of cross section, vector addition must be used to calculate the angle of twist of each segment

$$\phi = \sum \frac{TL}{JG}$$

Internal Torque Analysis

- Method of section & equation of moment equilibrium will be used to calculate internal torque

- T(x) is used to calculate internal torque at arbitrary position if torque values differ along the shaft longitudinal length

Angle of Twist Analysis

- Polar moment of inertia must be calculated using J(x), where x represents different positions of each cross sectional area along the circular shaft

- If there are sudden changes of polar moment of inertia or internal torque between both ends of circular shaft, $\phi = \int (T(x) / J(x)G(x)dx$ or $\phi = TL / JG$ must be used whereby J,G, and T are constant

Statically Indeterminate Torque-Loaded Members

- A shaft with applied torque can be categorized as statically indeterminate if moment equation of equilibrium is unable to calculate unknown torques acting on longitudinal axis of the shaft

- For example,

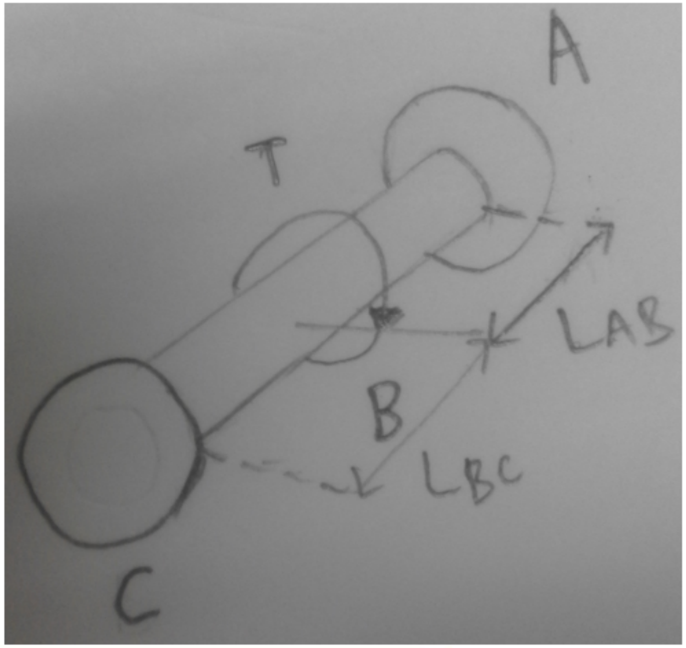

Figure 7.2

Statically Indeterminate Torque-Loaded Members

- If figure 7.2 is shown as free-body diagram,

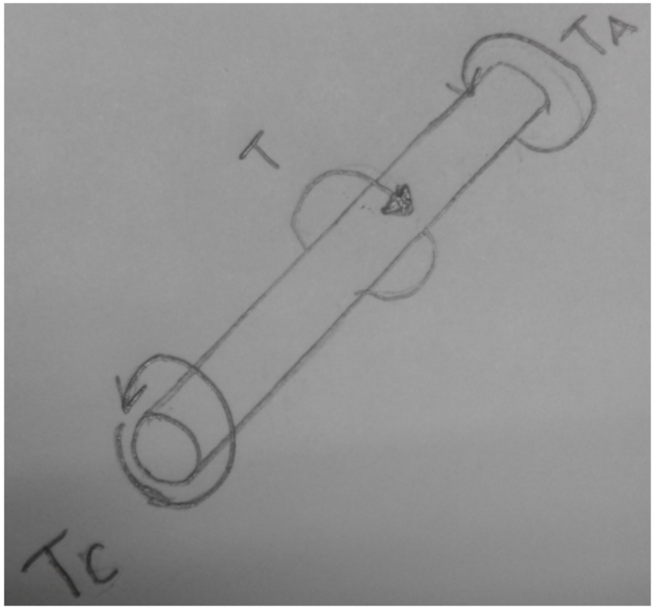

Figure 7.3

- Torque at points A and C are unknown

Statically Indeterminate Torque-Loaded Members

$$\sum M_x = 0$$

$$T - T_A - T_C = 0$$

- The problem can be solved using method of analysis, where angle of twist of one end with respect to the other end equal to zero

$$\phi_{A/C} = 0$$

Statically Indeterminate Torque-Loaded Members

- Load-displacement relation, Φ= TL/JG can be applied given that material is linear elastic

- Internal torque for AB is $+T_A$ and BC is $-T_C$. Thus,

$$\frac{T_A L_{AB}}{JG} - \frac{T_C L_{BC}}{JG} = 0$$

Statically Indeterminate Torque-Loaded Members

- It is known that $L = L_{AB} + L_{BC}$. Therefore,

$$T_A = T\frac{L_{BC}}{L} \qquad T_C = T\frac{L_{AB}}{L}$$

Unknown Torque Statically Indeterminate Shaft Analysis

Step 1

- State equilibrium condition by drawing free-body diagram & write the equation involved

Step 2

- Express compatibility equation between points along the shaft

Step 3

- Use torque-displacement relation to define angle of twist in terms of torque

Solid Noncircular Shaft

- Shear strain is different at different radius when torque is applied to a circular shaft

- For shaft that is not axisymmetric, the cross section will ***bulge*** or ***warp*** as shaft is twisted

- Therefore, torsional analysis of noncircular shaft is more complex

Solid Noncircular Shaft

- Shear-stress distribution for a shaft with square cross section can be determined using elasticity theory
- For all shapes, max shear stress, τ_{max} occurs at outer point of the cross section which is closest to the axis of a shaft
- It is clear that circular cross section is the most efficient shaft compare to other shapes like square, triangle or ellipse

Solid Noncircular Shaft

Shape of Cross Section	Max Shear Stress	Angle of Twist
Square	$\frac{4.81T}{a^3}$	$\frac{7.10TL}{a^4G}$
Triangle	$\frac{20T}{a^3}$	$\frac{46TL}{a^4G}$
Ellipse	$\frac{2T}{\pi ab^2}$	$\frac{(a^2+b^2)TL}{\pi a^3b^3G}$

Table 7.1

Example

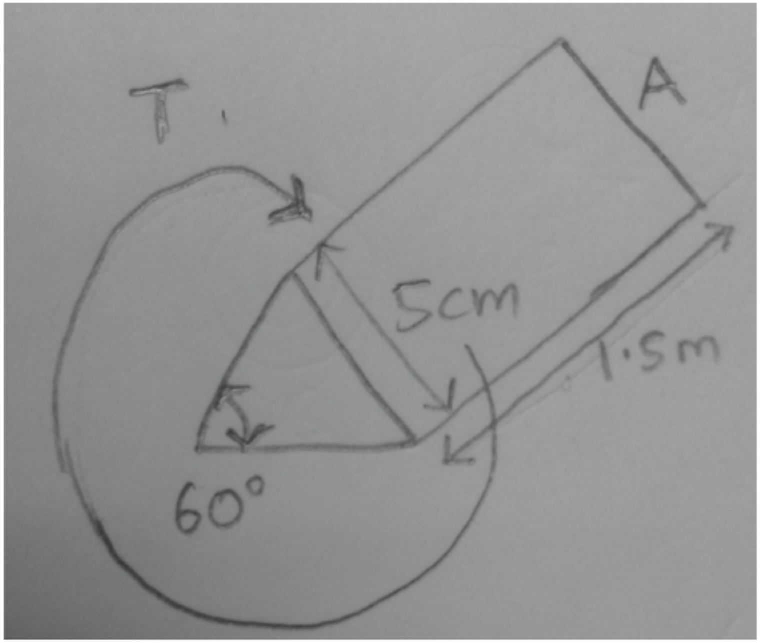

Figure 7.4

Aluminium shaft as in Figure 7.4 has cross section area of equilateral triangle. End A is fixed. Calculate the maximum torque that can be applied at the end of the shaft if allowable angle of twist, Φ_{allow} = 0.03 rad and allowable stress τ_{allow} = 52 MPa. Then calculate maximum torque that can be applied to a circular cross section with similar material

Example

For an equilateral triangle,

$$\tau_{allow} = \frac{20T}{a^3}$$

$$52(10^6)N/m^2 = \frac{20T}{(0.05m)^3}$$

$$T = 325Nm$$

Example

For an equilateral triangle,

$$\phi_{allow} = \frac{46TL}{a^4 G_{alu}}$$

$$0.03rad = \frac{46T(1.5m)}{(0.05m)^4[26(10^9)N/m^2]}$$

$$T = 70.65Nm$$

Example

In case of circular cross section, the area of the circle can be calculated as

$$A_{circle} = \pi c^2 = \frac{1}{2}(0.05m)(0.05m)\sin 60°$$

$$c = 0.02625m$$

Example

Using shear stress and angle of twist ,

$$\tau_{allow} = \frac{Tc}{J}$$

$$52(10^6)N/m^2 = \frac{T(0.02625m)}{(\pi/2)(0.02625)^4}$$

$$T = 1477.44Nm$$

Example

Using shear stress and angle of twist ,

$$\phi_{allow} = \frac{TL}{JG_{alu}}$$

$$0.03rad = \frac{T(1.5m)}{(\pi / 2)(0.02625m)^4[26(10^9)N / m^2]}$$

$$T = 387.83Nm$$

Stress Concentration

- At any position on a shaft where change in cross section happened, torsion formula, τ_{max} cannot be applied
- At this point, the distribution of shear-stress and shear strain can only be determined through experimental method or mathematical analysis (elasticity theory)
- Three types of discontinuities that usually occur for a cross section are couplings, keyways and a step shaft